The Ergodic Distribution of Wealth with Random Shocks

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Abstract

A convergence model in which wealth accumulation is subject to i.i.d. random shocks is examined. The accumulation function shows what k_{t+1} - wealth at $t+1$ - would be given k_t and with no shock. It has a positive slope, but its concavity or convexity is indeterminate. The focus is the ergodic distribution of wealth. This distribution satisfies a Fredholm integral equation. The ergodic distribution can be characterized in some respects by direct analysis of the stochastic process governing wealth accumulation and by use of the Fredholm equation without solution. Multiple local maxima in the ergodic distribution cannot be ruled out.

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0.1 Introduction

The Neoclassical convergence model has been influential in recent years. In its basic form the model says that all units - which might be countries, regions, even individual families, as desired - tend to converge to a common level of capital and output per head. The theory leads to a relationship similar to:

$$k_{t+1} = h\left[k_t\right] \tag{1}$$

where k_t is the logarithm of wealth (or income), and there is a unique stable value of $k = k*$, such that $k* = h\left[k*\right]$. This model refers to wealth, that is capital including human capital. And the economic theory to which the leading writers appeal deals with wealth accumulation. Empirical studies, however, typically use income rather than wealth, because income is far better measured. In what follows I shall always refer to wealth, even when discussing studies which use income. When it is precisely an income measure which is used, income may be interpreted as a proxy for wealth. Nothing essential in what follows is affected by the income-wealth distinction.

An econometric model based on (1) has to include an error term. This may be interpreted as random departures from the strict model, or as showing the effect of missing variables, or as both. In any case, if random errors are important in their magnitudes they affect the process of convergence. Consider an empirical study based on a linearized and re-arranged version of the relation (1), viz:

$$k_{t+1} - k_t = f\left[k_t\right] - k_t = \alpha - \beta \cdot k_t + \epsilon_t \tag{2}$$

With the normal finding $0 < \beta < 1$, equation (2) says that on average poor units (countries) grow faster than rich units. This has been called

$\beta-$*convergence*. This concept of convergence is not the same as $\sigma-$*convergence*, which means that the variance of the population of k values declines over time[1]. Friedman (1992) claims that interpreting a negative coefficient on k_t in a regression like (2) as convergence exhibits "Galton's Fallacy", on the ground that a negative coefficient is consistent with no tendency for the variance of k_t to decrease with time: it may even increase.

The point fits well with the argument of this paper. If k_t is distributed according to its ergodic distribution, which is one which reproduces itself[2], there is plainly no $\sigma-$convergence. Take a value of k_t far from $k*$. The expected value of k_{t+1} conditional on such a value of k_t will be closer to $k*$. It is $\alpha - (\beta - 1) \cdot k_t$ when (2) applies. Here, while individual dispersed units tend to converge, their density is made good by units, including the less-dispersed, pushed outwards by random shocks. An ergodic distribution of k values is invariant over time in the sense that it reproduces itself next period, although individual values will vary, partly systematically, showing β-convergence, and partly randomly, due to stochastic realisations of ϵ_t.

In an important contribution Quah (1993) noted independently that a Galton's Fallacy problem exists. Quah considers income generation as a pure Markov process. See also Quah (1996a) and (1996b). His empirical investigation looks at observed transition patterns, without considering the ergodic distribution. Also he does not derive his Markov transitions from economic theory. His aim is to confront theoretically derived convergence models with the hard facts shown in the data. He finds a tendency for

[1] For a clear exposition of the two concepts of covergence, and empirical discussion, see Sala-i-Martin (1996).

[2] To talk of k_t as being continuously distributed does not imply that k takes an uncountable infinity of values. An interval in the density of the distribution may measure the probability density of finding k within that interval of values. Then an ergodic distribution is one for which the probability density of finding k within any interval is the same next period as this period.

convergence within two groups - high and low income. He names this finding "twin peaks". There are non-negligible probabilities that a country will shift from one group to the other, but these transition probabilities are too low to iron out the twin peaks in the distribution.

0.2 Convergence and the Accumulation Function

It is convenient to work with the logarithm of wealth because it is not bounded below by zero, which makes possible infinite-tail distributions, such as the normal. Obviously, were k to be normally distributed, the wealth itself would be distributed as the log-normal distribution. Starting with the model (1), we add i.i.d. errors ϵ with mean zero to obtain:

$$k_{t+1} = h\left[k_t\right] + \epsilon_t \tag{3}$$

The function $h\left[\cdot\right]$ will be called the accumulation function, as it shows how much capital would be held one period later, starting from a level k_t, were no random shock to arrive to throw the accumulation process off its intended path. The agent starts with k_t, turns that into $h\left[k_{t+1}\right]$ by saving, and ends up with k_{t+1} after the shock has taken effect. One could assume:

$$k_{t+1} = h\left[k_t + \epsilon_t\right] \tag{4}$$

meaning that shocks affect wealth before the accumulation decision is made. However (3) fits best with existing econometric approaches. It is important in interpreting (3) to understand what is implied by the i.i.d. assumption, and what is not implied by it. The additive i.i.d. shock entails that the value of $h\left[\cdot\right]$ is unaffected by the particular value taken by ϵ. That does not imply that $h\left[\cdot\right]$ is unaffected by the distributional properties of ϵ, in particular by the fact that ϵ does not always take the value zero. To put it simply, as $h[\cdot]$ shows an optimal saving rule, that rule may be influenced by the existence of uncertainty. Computing the properties of optimal saving rules under uncertainty is formidably difficult and will not be attempted below. As will be seen, there is ambiguity for an important property of $h[\cdot]$ even when the saving decision takes no account of uncertainty. When $h[\cdot]$ shows an optimal saving rule which reflects the existence of uncertainty, it does not use information on the current value of ϵ.

For stability one must have:

$$\left[\frac{\partial h[k]}{\partial k}\right]_{h=h*} < 1 \tag{5}$$

Because accumulation cannot be supposed to be an entirely determinate mechanical process, random influences have to be added in. Obviously any econometric estimation of the model has to allow for random departures from the model. The aim of the present paper is to derive properties of an ergodic distribution of k_t generated by the stochastic process (3).

0.3 A Hydraulic Model

For understanding the ergodic distributional properties of a variable generated by the stochastic process (3), the following strange, yet understandable, hydraulic model is helpful. In the centre is a rift valley, running due north-south, and viewed in cross section. Rivers flow down from highlands on the east side and from the west. Position is measured by a variable k which runs from $-\infty$ (indefinitely far west) to $+\infty$ (indefinitely far east).

These are not normal rivers, fed by springs, and rainfall originating outside the river system. The system is completely closed. While there is rainfall, it all originates from water in the rivers themselves. Evaporation constantly redistributes water within the system. The amount of water evaporated depends on the volume at a point. One molecule of water may travel any distance, east or west. The probability of any such journey depends upon the absolute distance travelled, and it decreases monotonically with absolute distance. Elevation as such has no effect in precipitation. Indeed the high highlands will tend to be dry, because they will be far from the great mass of water. Finally water runs down hill and it runs faster the steeper the absolute gradient.

The bottom of the rift valley is at $k*$. The flow of river water towards the valley represents non-stochastic transformation of values of k through the function $h[k]$; which is to say that it represents neoclassical convergence. Evaporation and the random redistribution of water represent the effect of i.i.d. shocks, which will be called scattering below. The depth of water at any point k represents the density of wealth at that point. When this hydraulic system is an ergodic equilibrium state, depth is constant at any point. The rivers flow always towards $k*$. However evaporation and the random redistribution of water frustrate that process. A deep lake may build up around $k*$. Yet if redistribution is significant, the lake can never contain all the water in the system, because redistribution will always throw some water back into the highlands.

0.4 Concavity of the Accumulation Function

In deriving the properties of an ergodic distribution, the more that is known about the function $h[\cdot]$ the better. For instance, can $h[\cdot]$ be assumed to be strictly concave?

Unfortunately concavity of $h[\cdot]$ does not hold in general. Thus suppose that (1) shows the outcome of the optimal wealth accumulation of a Ramsey-style agent, who maximizes:

$$\sum_{t=1}^{\infty} \delta^{t-1} U \left[f\left(K_t\right) + K_t - K_{t+1} \right] \tag{6}$$

where K is wealth itself, not its logarithm k. K_1 is given, $\delta < 1$, $U[\cdot]$ is a strictly concave utility function, $f(\cdot)$, which is the production function in per capita terms, is strictly concave, and the argument $f(K_t) + K_t - K_{t+1}$ shows consumption at t. Given a value for K_t, there is a unique optimal value for consumption at t, denote it $C[K_t]$. Then K_{t+1} may be expressed as a function of K_t as:

$$K_{t+1} = f\left(K_t\right) + K_t - C\left[K_t\right] \tag{7}$$

Differentiating (7) twice with respect to K_t gives:

$$\frac{d^2 K_{t+1}}{d K_t^2} = \frac{d^2 f}{d K_t^2} - \frac{d^2 C}{d K_t^2} \tag{8}$$

and the concavity/convexity of the function from K_t to K_{t+1} is given by the right-hand side of (8). For a concave production function, the first term is negative. What can be said of the second term?

The maximized value of (6) depends uniquely on the initial level of K. Denote that maximized value by:

$$V[K] \tag{9}$$

Theorem 1 *$V[\cdot]$ is a strictly concave function of K.*
Proof: *Take two distinct values of K, K^1 and K^2. These deliver the values $V[K^1]$ and $V[K^2]$ by means of two time paths of K starting at respectively K^1 and K^2. Because the production function is strictly concave, a convex combination of these two paths is feasible starting from the same con-*

vex combination of K^1 *and* K^2*. This path will deliver utility strictly greater than Therefore* $\frac{d^2V}{dK^2} < 0.\square$

Denote by Y the total output available for consumption or investment. Thus:

$$Y = f(K) + K \tag{10}$$

where time subscripts have been omitted. A necessary, dynamic programming, condition for an optimum is that C should maximize:

$$U[C] + \delta V[Y - C] \tag{11}$$

which requires:

$$U'[C] - \delta V'[Y - C] = 0 \tag{12}$$

where primes denote differentiation. As (12) is an identity in Y, further differentiation gives:

$$\{U''[C] + \delta V''[Y - C]\} \frac{dC}{dY} - V''[Y - C] = 0 \tag{13}$$

Equation (13) shows that $\frac{dC}{dY}$ is positive, as would be expected, because both U'' and V'' are negative. Differentiating once again gives:

$$\{U'''[C] - \delta V'''[Y - C]\} \frac{dC}{dY} + V'''[Y - C] + \delta V'''[Y - C] \frac{dC}{dY}$$

$$-V'''[Y - C] + \{U''[C] + \delta V''[Y - C]\} \frac{d^2C}{dY^2} = 0 \tag{14}$$

This equation simplifies to:

$$\frac{d^2C}{dY^2} = \frac{U'''[C]\frac{dC}{dY}}{U''[C] + \delta V''[Y-C]} \tag{15}$$

If $U(C) = C^\alpha$ for $0 < \alpha < 1$, then $U'''[C] > 0$, and $\frac{d^2C}{dY^2} < 0$.

Notice that the sign of $\frac{d^2K_{t+1}}{dK_t^2}$ in (8) depends in part not on $\frac{d^2C}{dY^2}$ but on $\frac{d^2C}{dK^2}$. However there is a straightforward connection, because:

$$\frac{d^2C}{dK^2} = \frac{d^2C}{dY^2}\left(\frac{dY}{dK}\right)^2 + \frac{dC}{dY}\frac{d^2Y}{dK^2} \tag{16}$$

Therefore any sign-ambiguity for$\frac{d^2C}{dY^2}$ translates to some sign-ambiguity for $\frac{d^2C}{dK^2}$.

Finally note that ambiguity of the concavity/convexity of the mapping from K_t to K_{t+1} has to be translated to the concavity/convexity of the mapping from k_t to k_{t+1}, that is the accumulation relation in logarithms. However the translation is simple. We have:

$$\frac{d\log K_{t+1}}{d\log K_t} = \frac{dK_{t+1}}{dK_t}\frac{K_t}{K_{t+1}} \tag{17}$$

Differentiating once more with respect to $\log K_t$ gives:

$$\frac{d^2\log K_{t+1}}{d\log K_t^2} = \frac{d^2K_{t+1}}{dK_t^2}\frac{(K_t)^2}{K_{t+1}} + \frac{dK_{t+1}}{dK_t}\frac{K_tK_{t+1} - (K_t)^2\frac{dK_{t+1}}{dK_t}}{(K_{t+1})^2} \tag{18}$$

which does not resolve any sign-ambiguity.

0.5 Conditions for an Ergodic Process

With economic theory silent concerning the precise form of the accumulation function, we may turn to the conditions which will have to be assumed if the process:

$$k_{t+1} = h[k_t] + \epsilon_t \tag{19}$$

is to be ergodic. The process is ergodic if the probability of k_t taking a particular value becomes independent of k_1 as $t \to \infty$. A condition for (19) to be ergodic is given in Granger and Tersvirta (1993), p.10, who derive it from a result due to Doukhan and Ghindés (1980). The process (19) is ergodic if $h[\cdot]$ is continuous and:

$$\left[\frac{|h[k]|}{|k|}\right] < 1 \text{ for } |k| \text{ large.} \tag{20}$$

If the production function $f(k)$ is bounded above, then $h[k]$ is bounded above, in which case (20) is satisfied for large positive k. Let $k \to -\infty$. Because k is the logarithm of capital, this is equivalent to capital going to zero. If the unit has a source of income other than capital, gathering berries for example, then $h[k]$ will be bounded away from $-\infty$ as $k \to -\infty$, and (20) will be satisfied. If there is no extraneous source of income, and if wage income goes to zero when capital goes to zero, as would happen with a Cobb-Douglas production function, the limit of:

$$\frac{h[k]}{k} \text{ as } k \to -\infty \tag{21}$$

will be given by L'Hospital's rule as the limit of $\frac{dh[k]}{dk}$ as $k \to -\infty$. For that limit to be < 1, the elasticity of k_{t+1} with respect to k_t must be < 1 for k sufficiently small. While that is not an implausible condition, it is not guaranteed.

In summary, conditions for ergodicity can be derived and while they are not unacceptable, they are not without force. For large positive k, the condition bounds $\frac{h[k]}{k}$ above, as would happen were $h[k]$ to be strongly concave. However no concavity/convexity as such is implied by bounding conditions. As $k \to -\infty$, $\frac{h[k]}{k}$ must be bounded above, and that would happen with a strongly convex function. Again, no concavity/convexity as such is implied by the bounding condition. What is certain, in any case, is that for central values of k; those close to the mean and mode of the distribution, there is no restriction concerning the sign of the second derivative $\frac{d^2 h[k]}{dk^2}$. All the derivations which follow have to take that fact into account.

0.6 A Fredholm Equation for an Ergodic Solution

The ergodic density of values generated by the stochastic process:

$$k_{t+1} = h\left[k_t\right] + \epsilon_t \tag{22}$$

is described by a *Fredholm Equation* of the second kind[3]:

$$\Lambda\left[k\right] = A \int_{-\infty}^{+\infty} \pi\left[k - h\left[\kappa\right]\right] \cdot \Lambda\left[\kappa\right] d\kappa \tag{23}$$

where $\pi\left[\cdot\right]$ is the density of the random effect ϵ_t, and A is a constant chosen so that $\Lambda\left[k\right]$ integrates to 1. The integral on the right-hand side of (23) is the sum of all transitions from κ to k weighted by the probability that the initial value is κ, which is $\Lambda\left[\kappa\right]$, and the probability of a transition to k, which is the probability that ϵ_t takes the value $k - h\left[\kappa\right]$. Placing the same function $\Lambda\left[\cdot\right]$ on both sides of (23) identifies the ergodic fixed point outcome.

This derivation is somewhat similar to the so-called *Theory of Breakage* which leads to the equation:

$$F_j\left(x\right) = \int_u H_j\left[\frac{x}{u}\right] dF_{j-1}\left[u\right] \tag{24}$$

for which see Aitchison and Brown (1957), pp.26-7.

The process:

[3] See Hildebrand (1961) p. 381-2. In section 4.5 of the same chapter the author explains the connection between this type of equation and the joint effect of many causes.

$$k_{t+1} = h\left[k_t + \epsilon_t\right] \tag{25}$$

generates another *Fredholm Equation, viz:*

$$\Lambda\left[k\right] = \int_{-\infty}^{+\infty} \pi\left[h^{-1}\left[k\right] - \kappa\right] \cdot \Lambda\left[\kappa\right] d\kappa \tag{26}$$

which is quite similar.

To keep things simple, we concentrate on the Fredholm Equation (23).

To be able to write out an equation showing the ergodic solution as (23) is encouraging. Unfortunately this equation cannot be solved for $\Lambda\left[k\right]$. However it yields three useful results.

Theorem 2 *The set of functions satisfying (23) is convex*[4]*.*

Proof: Is immediate. If $\Lambda^1\left[k\right]$ and $\Lambda^2\left[k\right]$ both satisfy (23), then:

$$\lambda \cdot \Lambda^i\left[k\right] = A^i \int_{-\infty}^{+\infty} \pi\left[k - h\left[\kappa\right]\right] \cdot \lambda \cdot \Lambda^i\left[\kappa\right] d\kappa \tag{27}$$

for $i = 1$ or 2, and for any value of λ. Hence:

$$\lambda \cdot \Lambda^1\left[k\right] + (1-\lambda) \cdot \Lambda^2\left[k\right]$$
$$= \left[\lambda A^1 + (1-\lambda)A^2\right] \int_{-\infty}^{+\infty} \pi\left[k - h\left[\kappa\right]\right] \cdot \left\{\lambda \cdot \Lambda^1\left[\kappa\right] + (1-\lambda) \cdot \Lambda^1\left[\kappa\right]\right\} d\kappa \tag{28}$$

□

The next theorem uses the Fredholm equation to establish continuity of $\Lambda\left[k\right]$ with respect to k. An essential assumption is that $\pi\left[\cdot\right]$ should be everywhere continuous.

[4] To say that the set of functions is convex is not, of course, to say that the functions are convex functions.

Theorem 3 *If $\pi[\cdot]$ is everywhere continuous, a Fredholm equation value for $\Lambda[k]$ is continuous in k.*

Proof: If the integral on the right-hand side of (23) is well defined it must equal the limit as $L \to \infty$ of:

$$A\int_{-L}^{+L} \pi[k - h[\kappa]]\,\Lambda^i[\kappa]\,d\kappa \tag{29}$$

Which entails that for any $\delta > 0$ $\exists$ L_0 such that $L \geq L_0$ implies:

$$A\int_{-\infty}^{+\infty} \pi[k - h[\kappa]]\,\Lambda^i[\kappa]\,d\kappa - A\int_{-L}^{+L} \pi[k - h[\kappa]]\,\Lambda^i[\kappa]\,d\kappa < \delta \tag{30}$$

The equation (30) says that the Fredholm integral for any k may be approximated to any desired degree of accuracy by an integral with a finite - although possibly extremely long - range. Take any value of k, denoted k_0. Choose $\gamma > 0$. Let L satisfy (30) for $\delta = \frac{\gamma}{3}$. For each value of γ there must exist $\varepsilon > 0$ such that:

$$|\pi[k - h[\kappa]] - \pi[k_0 - h[\kappa]]| < \frac{\gamma}{3} \tag{31}$$

holds for all $|k_0 - k| < \varepsilon$ and $h^{-1}[-L - \gamma] \leq \kappa \leq h^{-1}[+L + \gamma]$. It must be possible to satisfy these conditions because both the continuous functions $h[\cdot]$ and $\pi[\cdot]$ are uniformly continuous on the compact interval $h^{-1}[-L - \gamma] \leq \kappa \leq h^{-1}[+L + \gamma]$. Then:

$$\begin{aligned} & A\int_{-L}^{+L} \{\pi[k_0 - h[\kappa]] - \pi[k_0 - h[\kappa]]\}\,\Lambda[\kappa]\,d\kappa \\ & \leq A\int_{-L}^{+L} |\pi[k_0 - h[\kappa]] - \pi[k_0 - h[\kappa]]|\,\Lambda[\kappa]\,d\kappa \\ & \leq \frac{\gamma}{3}A\int_{-L}^{+L} \Lambda[\kappa]\,d\kappa \leq \frac{\gamma}{3}A\int_{-\infty}^{+\infty} \Lambda[\kappa]\,d\kappa = \frac{\gamma}{3} \end{aligned} \tag{32}$$

Therefore for any γ however small we have been able to find an open interval centred on k_0 such that (29) differs from its value at k_0 by at most $\frac{\gamma}{3}$ for any

k in the interval.

Denote the value of (29) according to the values taken by k and L as $I(k, L)$. Then:

$$|I(k_0, \infty) - I(k, \infty)| \leq |I(k_0, L) - I(k, L)| + \frac{2\gamma}{3} \leq \gamma \tag{33}$$

for all $|k_0 - k| < \varepsilon$. This is continuity, as required.□

Continuity of $\Lambda[k]$ does not rule out the possibility that an equilibrium density might split into two or more disjoint segments: say a high wealth segment with positive density; a low wealth segment with positive density; and a region between these two where $\Lambda[k] = 0$. Suppose, for instance, that $k*$ lies in the high wealth region. Then agents cannot escape from the low wealth region because random shocks always push them back down. We call such a distribution *disjoint*. The next theorem will rule out such an extreme separation of agents who are identical except for their initial wealth cannot occur in equilibrium. It demands a tighter specification of the properties of the distribution of ϵ than has been required so far.

Definition 1 *The distribution of ϵ will be said to be weakly-regular if $\pi(\cdot)$ is continuous and:*

$$\pi(x) > 0 \tag{34}$$

for all x in the interval $[-a, +a]$ for a suitably small value of a. Obviously a weakly-regular distribution could exhibit strange features in comparison with familiar statistical distributions, and a stronger regularity requirement will be introduced below.

Theorem 4 *If the distribution of ϵ is weakly-regular, an ergodic density cannot be disjoint.*

<u>Proof</u>: *If the distribution is disjoint there will exist at least one open range*

of values of k, (k^-, k^+) such that $\Lambda[k] = 0$ in that interval and $\Lambda[k] > 0$ for values of k as close to k^- or k^+ as required. Suppose $k^- < k*$. Similar arguments take care of the opposite inequality and the case $k^- = k*$. Consider the closed range of k values $[h^{-1}[k^-], k^-]$. We have:

$$\int_{h^{-1}[k^-]}^{k^-} \Lambda[\kappa]\, d\kappa > 0 \tag{35}$$

for otherwise we choose a lower value for k^-. Define $k_0 = \frac{h[k^-]-k^-}{2} > k^-$.

$$\Lambda[k_0] \leq A \int_{h^{-1}[k^-]}^{k^-} \pi[k_0 - h[\kappa]] \cdot \Lambda[\kappa]\, d\kappa \tag{36}$$

It will be seen that $k_0 - h[\kappa]$ ranges from:

$$k_0 - h\left[h^{-1}\left[k^-\right]\right] = k_0 - k^- > 0 \tag{37}$$

and:

$$k_0 - h\left[k^-\right] = -\frac{k^- + h[k^-]}{2} < 0 \tag{38}$$

The fact that $\pi[k_0 - h[\kappa]]$ must be continuously positive over at least part of the range between the limits of the integral in (35), and (35) itself, imply $\Lambda[k_0] > 0$, contrary to the assumption that the disjoint distribution assumed is ergodic.□

0.7 Cumulative Distribution, h-Transformation and Scattering

In analysing the distribution of k values, it is sometimes convenient to work in terms of the cumulative distribution. Hence $\Delta(k)$ is the proportion of the population with wealth not greater than k. Clearly $\Delta(-\infty) = 0$ and $\Delta(\infty) = 1$.

Notice that the effect on the distribution of wealth in moving from one period to the next is the sum of two separate transformations. First each k

value maps to $h[k]$. We call this *h-transformation*. Next all values are scattered by the addition of random shocks ϵ_t. We call this *scattering*. Consider the first step. Before h-transformation:

$$\Lambda[k] = \frac{d\Delta[k]}{dk} \tag{39}$$

Whereas after h-transformation:

$$\Gamma[k] = \Delta\left[h^{-1}[k]\right] \tag{40}$$

where $\Gamma[k]$ is the cumulative distribution of k after transformation. Then:

$$\frac{d\Gamma[k]}{dk} = \frac{d\Delta[h^{-1}[k]]}{d[h^{-1}[k]]}\frac{d[h^{-1}[k]]}{dk} = \frac{\Lambda[h^{-1}[k]]}{\frac{dh[k]}{dk}} \tag{41}$$

is the density of wealth distribution after $h-$transformation. Equation (41) defines how the accumulation function affects the distribution of wealth in the absence of random effects.

Denote the transformed distribution by $\Phi(k)$. So:

$$\Phi(k) = \frac{\Lambda[h^{-1}[k]]}{\frac{dh[k]}{dk}} \tag{42}$$

Consider a maximum of $\Phi(k)$ at $k = k^0$. Then:

$$\left\{\frac{d\Lambda[h^{-1}[k]]}{dk} - \Lambda\left[h^{-1}[k]\right]\frac{d^2h[k]}{dk^2}\right\}_{k=k^0} = 0 \tag{43}$$

Equation (43) is useful when locating a maximum, including a mode, of a wealth distribution after $h-$transformation when the location of a maximum of $\Lambda[k]$ is known. Suppose, for instance, that $h[k]$ is so nearly linear in the relevant range that $\frac{d^2h[k]}{dk^2}$ may be replaced by zero. Then (43) says that one should look for a maximum of $\Phi(k)$ to the left (right) of a maximum of $\Lambda[k]$ according as k is less than (greater than) $k*$.

The sequential effects of $h-$transformation and scattering in an ergodic case can be exhibited mathematically as follows. Take any value of k. Suppose $k < k*$. A symmetrical argument works for the other side. For any level of wealth between $h^{-1}[k]$ and k, $h-$transformation will carry wealth across the border marked by k from lower to higher values of wealth. Next, after $h-$transformation, scattering will carry a certain mass of wealth across the same border, travelling in the same direction, while scattering will carry another mass of wealth across the border in the opposite direction. It is an evident equilibrium condition for an ergodic distribution that the net movement of wealth across the border shall be zero. That condition is expressed in the following equation.

$$\int_{h^{-1}[k]}^{k} \Lambda[\kappa]\, d\kappa + \int_{-\infty}^{k} \{1 - \Pi(k-\kappa)\} \Phi(\kappa) d\kappa = \int_{k}^{+\infty} \Pi(k-\kappa) \Phi(\kappa) d\kappa \quad (44)$$

where $\Pi(\cdot)$ is the cumulative distribution of i.i.d. shocks; that is the probability that ϵ_t will be $\leq$ the argument of $\Pi(\cdot)$.

As (44) holds as an identity in k, we may differentiate (44) with respect to k to obtain:

$$\Lambda[k] - \frac{\Lambda[h^{-1}[k]]}{\frac{dh[k]}{dk}} + \{1 - \Pi(0)\} \Phi(k) - \int_{-\infty}^{k} \pi(k-\kappa) \Phi(\kappa) d\kappa$$

$$= -\Pi(0) \Phi(k) + \int_{k}^{+\infty} \pi(k-\kappa) \Phi(\kappa) d\kappa \quad (45)$$

Notice that is the distribution of shocks is symmetrical about 0, then $\Pi(0) = \frac{1}{2}$. However the argument does not use that property. Simplifying (45) taking into account (42) gives:

$$\Lambda[k] = \int_{-\infty}^{+\infty} \pi(k-\kappa) \frac{\Lambda[h^{-1}[\kappa]]}{\frac{dh[\kappa]}{d\kappa}} d\kappa \quad (46)$$

which provides another integral equation description of ergodic equilibrium. The right-hand side of (46) is the density given at k when the distribution $\Lambda[\kappa]$ is first subject to $h-$transformation and then to scattering. Then (46) says that these twin processes shall map density at each k into itself. Notice that one can go directly from the original Fredholm equation (23) to (46) by a simple change to the variable of integration, from κ to $h[k]$.

Analysis of a Fredholm distribution can proceed either from the integral equation (23), or from the equation defining the process itself (22). The next section adopts the latter approach.

0.8 Direct Analysis from the Stochastic Process Equation

Applying the mathematical expectation operator E to (3) gives:

$$Ek = Eh[k] \tag{47}$$

The relation of $Eh[k]$ to $h[Ek]$ depends upon the concavity/convexity of $h[\cdot]$, which we have seen to be ambiguous. Subtracting (47) from (3) and rearranging gives:

$$\begin{aligned} E[k_t - Ek]^2 &= E\left\{[h[k_t] - Eh[k] + \epsilon_t]^2\right\} \\ &= E\left\{h[k]^2\right\} + E\left\langle\{Eh[k]\}^2\right\rangle + E\{\epsilon_t\}^2 - 2E\{h[k]\,Eh[k]\} \\ &= E\{[\epsilon]\}^2 + E\left\{h[k]^2\right\} - \{Eh[k]\}^2 \end{aligned} \tag{48}$$

where time subscripts have been dropped, because they are irrelevant when an ergodic distribution is under consideration. On account of ϵ_t being i.i.d., expectations of products involving ϵ_t have been equated to zero.

Notice that the variance of $h(k)$ is given by:

$$E\left\{h(k)-Eh(k)\right\}^2 = E\left\{h(k)^2+\left[Eh(k)\right]^2-2h(k)Eh(k)\right\}$$

$$= E\left\{h(k)^2\right\}-\left\{Eh(k)\right\}^2 \tag{49}$$

Now equations (48) and (49) together can be interpreted in a very natural result.

Theorem 5 *An $h-$transformation always subtracts variance from the distribution of k. For an ergodic distribution it subtracts precisely the amount of variance that is added by scattering.*

Proof: Notice that the result is not trivial. While $h-$transformation obviously moves every k closer to $k*$, there is no immediate guarantee that it moves every k closer to the mean of the k values. However from (45) the second moment of the distribution of k in general, and hence the same moment in an ergodic distribution, is the sum of the variances of $h(k)$ and of ϵ. In that case, k itself must have a larger variance than $h(k)$. Evidently scattering restores equality, as required.□

Similar calculations for the third moment of the distribution of k, assuming $E\left\{\epsilon^3\right\}=0$, produce:

$$E\left\{k-Ek\right\}^3 = E\left\{h(k)^3\right\}-3E\left\{h(k)\right\}E\left\{h(k)^2\right\}+2\left[E\left\{h(k)\right\}\right]^3 \tag{50}$$

Theorem 6 *If the distribution of shocks ϵ_t has a third moment about its mean equal to zero; hence in particular if it is symmetrical about zero; an $h-$transformation does not affect the third moment of $h(k)$ about its mean.*

Proof: The right-hand side of (50) is the third moment $h(k)$ about its mean. Therefore the result follows immediately.□

0.9 Bell-Shaped Distributions

The following arguments touch particularly on the question of whether an ergodic equilibrium wealth distribution can be normal. As k is the logarithm of wealth, that is equivalent to asking whether wealth can be log-normally distributed in the limit.

Definition 2 *A twice-differentiable function $q(x)$ on the range $[\underline{a}, \overline{a}]$, where these end points may be $-\infty$ or $+\infty$, will be said to be Symmetric-Bell-shaped (SBS) if:*

1. There exists a unique value x_0 such that $q(x)$ takes its maximum value.

2. The value of $q(x)$ is uniquely determined by $|x - x_0|$; and

3. The value of $q(x)$ decreases monotonically with $|x - x_0|$.

In addition to the above, a distribution will be called regular symmetric-bell-shaped if, starting from $x = \underline{a}$,

$$\frac{d^2 q}{dx^2} \tag{51}$$

is first positive, then negative until a point at which $x > x_0$, and finally positive to $\overline{a}$.

The definition describes a normal distribution, but is not confined to that particular case.

Definition 3 *The stochastic process (22) or the Fredholm equation (23) will be said to be standard if the density function of errors $\pi[\cdot]$ is SBS with its maximum at zero.*

Recall that the mapping of an ergodic distribution solution into itself consists of the sum of two separate steps. The first is the effect of the $h-$transformation; the second is scattering.

The statement of the next theorem is most easily understood in terms of the cumulative distribution $\Gamma[k]$. The wealth distribution has an infinite tail if there exist no finite value of k such that $\Gamma[k]$ takes either the value 0 or 1. In the former case the distribution will be said to have an infinite left-hand tail; in the latter case an infinite right-hand tail.

Theorem 7 *If the distribution $\pi[\cdot]$ has an infinite left- or right-hand tail, a Fredholm equation solution will have respectively an infinite left- or right-hand tail. A Fredholm equation solution may have an infinite tail without $\pi[\cdot]$ having an infinite tail if:*

$$|k - h(k)| \tag{52}$$

is bounded above for all large $|k|$ by A, where A is within the range of the absolute value of one of the tails of $\pi[\cdot]$.

Proof: The first statement is obvious. Whatever value $h[k_t]$ may take, if the distribution $\pi[\cdot]$ has an infinite tail, positive probability attaches to ϵ_t taking any value whatsoever in that direction; hence positive probability, and therefore distribution density, attaches to k_{t+1} taking any value in that direction. The second statement is also plain. However far out k may be, as long as positive probability attaches to the movement towards the centre $h(k) - k$ being undone by a random shock, some positive density of k can survive the tendency to convergence brought about by $h-$ transformation.□

The next result is most useful in characterizing the shape of an ergodic distribution because it shows exactly what scattering by itself does, and particularly how it influences the SBS property.

Theorem 8 *If the stochastic process for wealth is standard, a wealth distribution, which may or may not be ergodic, is SBS after scattering if and only* *if it is SBS before scattering.*

Proof: Let $\Lambda^2(k)$ be the distribution resulting when $\Lambda^1(k)$ is modified by a process of pure scattering:

$$k_{t+1} = k_t + \epsilon_t \qquad (53)$$

A Fredholm equation, similar to (23) but simpler, describes the relation between the two distributions:

$$\Lambda^2[k] = A\int_{-\infty}^{+\infty} \pi[k-\kappa]\cdot\Lambda^1[\kappa]\,d\kappa \qquad (54)$$

Sufficiency: Suppose that $\Lambda^1[k]$is SBS with its mode at $k = k^0$. The right-hand side of the equation (54) may be written:

$$A\int_{-\infty}^{k^0} \pi[k-\kappa]\cdot\Lambda^1[\kappa]\,d\kappa + A\int_{k^0}^{+\infty} \pi[k-\kappa]\cdot\Lambda^1[\kappa]\,d\kappa \qquad (55)$$

Let $k - k^0 = d$. Then if $k' = 2k^0 - k$, the absolute value of the distance of k' from k^0 is again d. If k is to the right of k^0, the integral equation (54) shows that the value $\Lambda^2[k]$ is obtained by integrating for κ from $-\infty$ to k^0, and then adding the integral all the way to $+\infty$, with k being interior to the second integration. When the value of $\Lambda^2\left[k'\right]$ is obtained, the calculation is completely mirror-image symmetric. When k is equi-distant to the left of k^0, the value $\Lambda^2\left[k'\right]$ is obtained by integrating for κ from $-\infty$ to k^0, and then adding the integral all the way to $+\infty$, with k now being interior to the first integration. The net effect of both integrations is the same, as equal levels of $\Lambda^1[\cdot]$ and $\pi[\cdot]$ appear, and are multiplied together, during the two stages. We have the SBS property for $\Lambda^2[k]$ around k^0.

Necessity: Suppose that $\Lambda^2[k]$ is SBS around k^0 but that $\Lambda^1[k]$ satisfying (54) is not SBS. If $\Lambda^1[k]$ is not SBS around k^0, it can always be expressed as:

$$\Lambda^1[k] = \Lambda^{\cap}[k] + \Lambda^{\rightarrow}[k] \qquad (56)$$

where $\Lambda^{\cap}[k]$ is SBS around k^0, and $\Lambda^{\rightarrow}[k] = 0$ for $k \leq k_0$. It is important to take into account the fact that $\Lambda^{\rightarrow}[k]$ is not itself a density function, and may take negative values over some ranges of k. Then:

$$\Lambda^2[k] = A\int_{-\infty}^{+\infty} \pi[k-\kappa]\cdot\Lambda^{\cap}[\kappa]\,d\kappa + A\int_{k^0}^{+\infty} \pi[k-\kappa]\cdot\Lambda^{\rightarrow}[\kappa]\,d\kappa \qquad (57)$$

The distribution given by the first integral will be SBS by the sufficiency argument above. Suppose that the second integral gives an SBS distribution, in which case a non-SBS distribution would have been transformed by (54) into an SBS distribution, contrary to the theorem. Take a point $k + k^0$ and a value of $\kappa \geq k^0$. Starting from κ, the probability of getting to k is $\pi[k-\kappa]$; and the probability of getting to $-k$ is $\pi[k+\kappa]$. If the second integral of (57) gives an SBS distribution:

$$A\int_{k^0}^{+\infty} \pi[k-\kappa]\cdot\Lambda^{\rightarrow}[\kappa]\,d\kappa = A\int_{k^0}^{+\infty} \pi[k+\kappa]\cdot\Lambda^{\rightarrow}[\kappa]\,d\kappa \qquad (58)$$

identically in k. However:

$$|k-\kappa| \leq |k+\kappa| \qquad (59)$$

and $\pi[x]$ decreases monotonically with $|x|$. Therefore (58) can not be satisfied, and sufficiency is established.□

From Theorem 8 it may be seen that the shape of an ergodic distribution, and particularly question of whether it could be SBS, turns on the effect of $h-$transformation. This is because the effect of scattering is neutral where the SBS property is concerned. Hence suppose that $\Lambda[k]$ is SBS with its maximum at k^0. From (42) the density of k after $h-$transformation is given by:

$$\Phi(k) = \frac{\Lambda\left[h^{-1}[k]\right]}{\frac{dh[k]}{dk}} \qquad (60)$$

Then $\Phi(k)$ must also have a maximum at k^0 because, by symmetry, scattering applied to an SBS distribution cannot affect the location of a maximum. Take two values of k, $k^0 - \Delta$ and $k^0 + \Delta$. Because $\Phi(k)$ is SBS, we must have:

$$\frac{\Lambda\left[h^{-1}\left[k^0-\Delta\right]\right]}{\frac{dh[k-\Delta]}{dk}} = \frac{\Lambda\left[h^{-1}\left[k^0+\Delta\right]\right]}{\frac{dh[k+\Delta]}{dk}} \tag{61}$$

Theorem 9 *If $h\left[k\right]$ is linear, a distribution is ergodic only if it is SBS.*
Proof: Take a distribution, SBS around k^0, $\Lambda\left[k\right]$. After $h-$transformation it will be as (57). If:

$$h\left[k\right] = \alpha + \beta k \tag{62}$$

$\frac{dh[k]}{dk} = \beta$ *is constant, and* $h^{-1}\left[k\right] = \frac{k-\alpha}{\beta}$. *Further:*

$$\Lambda\left[h^{-1}\left[k^0-\Delta\right]\right] = \Lambda\left[h^{-1}\left[k^0+\Delta\right]\right] \tag{63}$$

by the SBS property. Therefore (60) is SBS. After scattering the distribution is SBS. Therefore an SBS distribution maps to an SBS distribution.
Now suppose that $\Lambda\left[k\right]$ is not SBS around k^0. However because it is ergodic, it maps to an SBS distribution around k^0 with the mapping (23). Then it must be mapped to an SBS distribution by $h-$transformation, because scattering will not give the SBS property unless is already present. When $h\left[k\right]$ is linear, there exists a value of Δ such that (60) is not satisfied, because $\Lambda\left[k\right]$ is not SBS around k^0. A non-SBS distribution cannot map to an SBS distribution.
□

If $h\left[k\right]$ is approximately linear, an ergodic distribution could be approximately SBS. Theorem 9 shows a strong connection between linearity of $h\left[k\right]$ and the SBS property. It does not show however that the $h-$transformation of an SBS distribution could not be SBS while $h\left[k\right]$ is non-linear. Differentiating (60) with respect to k gives:

$$\frac{d\Phi(k)}{dk} = \frac{\frac{d\Lambda\left[h^{-1}[k]\right]}{dh^{-1}[k]} - \Lambda\left[h^{-1}\left[k\right]\right]\frac{d^2h[k]}{dk^2}}{\left(\frac{dh[k]}{dk}\right)^2} \tag{64}$$

For a given SBS function $\frac{d\Phi(k)}{dk}$ (64) can be viewed as a non-linear differential equation in $h\left[k\right]$ and it may be possible to prove that it has no non-linear solutions. I have not been able to obtain such a proof. Even so, the equation itself shows at least that a very specific functional form for $h\left[k\right]$ will always be required if non-linear $h-$transformation of an SBS distribution is to be SBS. Therefore the linear case may be the only important one for practical purposes. As there is no theoretical reason to expect linearity, or any other specific form, the SBS property appears to be an improbable outcome.

0.10 Single Peakedness

Given that the SBS property for the wealth distribution has been shown to be improbable, what about looking for a weaker regularity condition? The following condition is implied by SBS but is certainly much weaker.

Definition 4 *The wealth distribution will be said to be single-peaked if all its local maxima are attained on one convex set of values of k.*

The definition allows a "table mountain" case in which the maximum value is attained over a connected range of values of k. That case apart, the definition rules out multiple local maxima as distinct peaks. We know already that a linear $h\left[k\right]$ function produces an SBS outcome. It follows that multiple local maxima can only arise if $h\left[k\right]$ exhibits significant non-linearity.

The hydraulic system described in Section 3 above can also throw light on single-peakedness. Suppose that the steepness of the rift valley walls on both sides increases monotonically with absolute distance from $k*$, and is symmetrical on the two sides. Far from $k*$ water is moved quickly towards $k*$. All the water that far out has been transported a long distance. There cannot be much of it, and swift running rivers must be shallow. As one moves closer to the floor of the valley, the absolute gradient becomes lower and rainfall rises, because the total water not too far away increases. Now rivers flow slowly and are deep. Therefore the depth of water rises until it reaches its maximum at $k*$. The density of water is SBS around $k*$.

Now modify the model just described. On the west side insert a range of values of k along which the gradient is quite flat. Follow it by a very steep interval closer to $k*$, and then return to a similar gradient to that prevailing on the opposite valley wall. The amount of water above these ranges will hardly be affected if redistribution is strong and the ranges described cover short intervals. Therefore water will move through the intervals first slowly, next rapidly, then it will slow down. Depth will be high, then lower, then high again. The depth of water, which is to say the density of wealth, will exhibit twin peaks.

For the accumulation of wealth the model just described corresponds to the following state of affairs. For a range of low levels of wealth, is accumulated towards $k*$, but at a slow rate. Then, when wealth gets a bit higher, the pace of accumulation picks up sharply. Later it moderates. Economic theory cannot exclude such a case. The only way to avoid a twin-peak outcome in such a case is to have a high density in the steep (fast-flowing) section. That will never be an equilibrium because that high density would be rapidly dissipated by flow towards $k*$ which rainfall could not replace.

An informal mathematical version of this pictorial argument runs as follows. Take a regular model with no twin-peaks in its ergodic wealth distribution. Over a range of values of $k < k*$ which is small relative to a range which contains much of the density of $\pi[\cdot]$, distort the $h[\cdot]$ function so as to make its derivative large. Figure 1 sketches the $h[\cdot]$ function so distorted. From (60) it may be seen that either the density in the distorted range will be relatively low, or density in that particular section of the distribution which maps into the range under $h-$transformation will be relatively high. In the latter case, density to the left of the last section must again be high, and so on. Therefore, unless density in the range over which the $h[\cdot]$ function has been distorted is relatively low in ergodic equilibrium, density will have to be relatively high for extremely low values of k. This is not possible. The effect of $h-$transformation will always be to pull that density closer to $k*$. And scattering by itself cannot sustain high density far from $k*$, because it is neutral as to direction.

An example of this type depends upon the magnitude of $\frac{dh[k]}{dk}$ varying considerably over a narrow range: first rising then falling. One could rule that out by assumption. Particularly if $h[\cdot]$ is everywhere either weakly concave or weakly convex, a case such as the example cannot occur.

When Quah[5] published his empirical evidence showing the twin-peak pattern in international cross section per capita income data, I read it as evidence against the simple convergence model, as no doubt did many other readers. It is interesting to note that Quah himself advances no such claim. First he is very clear that he is describing the development of income distribution over a short period of time. Secondly, Quah is aware of the possibility that even the apparently disconnected distributions he observes may be generated by a process which in the long run is ergodic. Now the theoretical investigation of an ergodic wealth distribution has shown that it may have twin (indeed

[5]See Quah (1996a) and (1996b).

multiple) local peaks. So it seems that even such a surprising feature may be completely consistent with a convergence model.

That is not a good way of looking at matters. To be worthy of study, an economic model has not only to be true in some high abstract sense; it has to be useful. A twin peak case can only arise when $h[k]$ is severely non-linear. The estimation of such a model presents many difficulties. In a way the strength of the Baumol-Barro convergence model was its crude simplicity. If it has to be rescued by refined mathematical argument, it loses its appeal. Also, in the example, and more generally, twin peaks in an ergodic distribution can only happen over a range within the reach of a single-period realization of the random shock. Therefore if twin peaks are an important feature of the distribution, it must be the case that shocks are large in absolute value. This is another way of saying that the explanatory power of the model is weak.

0.11 Concluding Remarks

The long history of the analysis of income or wealth distributions, going back to Pareto, includes different approaches. One is purely empirical. The shape of the distribution is examined and the fitness of simple mathematical specifications is investigated. Another approach is to start with postulates concerning the process which generates the distribution and then to investigate mathematically what is the limiting distribution which results. Yet the limiting distribution does not have to be the object of concern. The shorter term conditional transfer process can itself be the focus of investigation. Indeed the neoclassical convergence theorists can only do that, because for them the limiting distribution is trivial, being a state in which all countries - or individuals in the case of a personal distribution - are at the common limit point $k*$. When the transfer process is taken to be random there are wider possibilities than when it is modelled using economic theory.

The present paper marries two different traditions. They are the neoclassical approach, according to which wealth accumulation is systematic and deliberate; and the random shocks approach, according to which wealth accumulation is purely haphazard. As would be expected, such a model is complicated, and direct mathematical solution is hardly possible. Even so, we have been able to obtain a series of results which together effectively characterize the limiting distribution of the logarithms of wealth values.

It is a continuous connected distribution which may or may not have infinite tails. It is unlikely to be symmetric-bell-shaped, although that case is possible. While it may not be single-peaked, mild assumptions on the accumulation function can assure that is However these assumptions are not derivable from economic theory.

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FIGURE 1
A Twin-Peak Case