# Concepts and Properties of Substitute Goods *

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**Abstract**

We distinguish two notions of substitutes for discrete inputs of a firm. Class substitutes are defined assuming that units of a given input have the same price while unitary substitutes treat each unit as a distinct input with its own price. Unitary substitutes is necessary and sufficient for such results as the robust existence of equilibrium, the robust inclusion of the Vickrey outcome in the core, and the *law of aggregate demand*, while the class substitutes condition is necessary and sufficient for robust monotonicity of certain auction/tâtonnement processes. We analyze the concept of pseudo-equilibrium which extends, and in some sense approximates, the concept of equilibrium when no equilibrium exists. We characterize unitary substitutes as class substitutes plus two other properties. We extend the analysis to divisible inputs, with a particular focus on robustness of the concepts and their relation to the *generalized law of aggregate demand*.

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# 1 Introduction

The notion of substitute inputs expresses the idea that when the price of one type of input rises, the number of units demanded of the other inputs cannot fall. But what are "types" of inputs? If electricity generated at locations A and B are perfectly substitutable in production, should we regard these as one class of input or two? It turns out that important results of price theory and multi-unit auction theory hinge on the way such questions are answered. When substitute comparisons only apply across distinct classes of goods, we will say that the firm has a *class-substitute valuation.* If even units of the same class of good are substitutes when priced independently, we will say that the firm has a *unitary-substitute valuation.* The biggest surprises in our analysis are that even in very ordinary-looking problems, identical inputs may fail to be substitutes for one another and that this failure is consequential for standard economic analysis.

We illustrate this point with simple examples. Suppose that the price of output is one and that the amount $f$ of output produced by a firm as a function of two types of discrete inputs $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$ is:

| $f$ | $y = 0$ | $y = 1$ | $y = 2$ |
|---|---|---|---|
| $x = 0$ | 0 | 1 | $\sqrt{2}$ |
| $x = 1$ | 1 | 1 | $\sqrt{2}$ |

$f$ is submodular in its two arguments and has nonincreasing marginal returns.[1] The firm chooses $x$ and $y$ to maximize $f(x, y) - rx - wy$. Since the function $f$ is submodular, the inputs $x$ and $y$ are substitutes. Substitutes means that when comparing any two price vectors $p$ and $p'$ for which the firm's optimum is unique, if $p \geq p'$ and $p_i = p'_i$, then the demand for good $i$ is weakly higher at prices $p$.

Next, consider a formulation in which the two units of input $y$ are treated as distinct. Let $y = y_1 + y_2$ and suppose $y_1, y_2 \in \{0, 1\}$. In this formulation, the prices are also

[1]It is easy to see that functionally identical inputs can fail to be substitutes for one another in the usual sense of price theory when there are increasing marginal returns to that type of input. In order to be clear that this is not what underlies our example, we chose $f$ with nonincreasing marginal returns.

potentially distinct, so the firm maximizes $f(x, y_1 + y_2) - rx - w_1 y_1 - w_2 y_2$. It is as if we had distinguished blue and red versions of the input, where the color is devoid of any consequences for production. It is easy to check that if the input prices are $(r, w_1, w_2) = (0.2, 0.3, 0.2)$, then the firm's unique profit-maximizing input vector is (0,1,1), but if $(r, w_1, w_2) = (0.2, 0.3, 0.7)$, then the profit-maximizing choice is (1,0,0). This demonstrates that an increase in the price of input $y_2$ reduces the demand for input $y_1$: different units of the same type of good may fail to be substitutes.

Examples of this sort are hardly rare. For instance, an airline that is acquiring landing slots at a hub airport may wish to have some number N of slots, for illustration N=2, within a particular period, say from 2:00pm to 2:15pm or from 3:00pm to 3:15pm. The two periods define class substitutes: if slots at 2-2:15 are expensive, the airline may substitute slots at 3-3:15. Slots within a given time period, however, are not substitutes: the airline wants both or neither.

One important distinction between class and unitary substitutes arises when studying the question of whether market-clearing prices exist. Using models in which goods are priced individually, Gul and Stacchetti (1999) and Milgrom (2000), establish that when goods are substitutes, market-clearing prices always exist. Moreover, they display monotonic auction processes that converge to these market-clearing prices. In those formulations, substitutes means unitary substitutes: the results do not extend to the case of class substitutes. For suppose that good $y$ is treated as a single class and that the available supply for the two classes of goods is given by the vector $(1, 2)$. Suppose that firm 1 has valuation $f$ as before, and that there is a second firm with unit valuation $g(x, y) = 1_{y \geq 1}$. At the efficient allocation, firm 2 uses one unit of $y$ and firm 1 uses one unit of $x$. To induce firm 1 to make this choice, the price of input $y$ must be strictly positive, but then firm 1 will strictly prefer not to buy any units of input $y$ and firm 2 will strictly prefer to buy exactly one unit. Hence, there will be a strict excess supply of $y$: no market clearing prices exist.

If the supply vector is anything else besides $(1, 2)$ in this example with only class substitutes, then not only does a market clearing price vector exist, but more is true. First, the set of market clearing price vectors is a sublattice. Second, a continuous *tâtonnement* or *clock auction* process beginning with low prices converges monotonically upward to the minimum market clearing price vector. A similar process beginning with high prices converges monotonically downward to the maximum market clearing price vector.

How does the clock auction perform when there are no market clearing prices? Suppose that firm 1 has valuation $f$ as above, firm 2 has valuation $v(x, y) = .05 \times 1_{y \geq 1}$, supply is $(1, 1)$ and we initially set the input price vector to $(0, 0)$. At that price there is strict excess demand for good $y$ but not for good $x$. The price of good $y$ is gradually increased. When $p_y$ becomes greater than .05, firm 2's demand drops to 0 units of good $y$. Eventually, the price reaches a level $p_y$ at which the firm 1 is indifferent between buying one unit of $x$ or two units of $y$ and firm 2 is indifferent between 0 and 1 unit of good $x$ (since $p_x = 0$). The indifference equation is $1 = \sqrt{2} - 2p_y$, so $p_y = (\sqrt{2} - 1)/2$. When the clock reaches $p_y$, firm 1 demands either 1 unit of $x$ or two units of $y$. Consequently demand is strictly less or strictly more than supply for good $y$. At price $(0, p_y)$, aggregate demand consists of the bundles $(2, 1)$, $(2, 0)$, $(0, 1)$ and $(0, 2)$, thus contains the supply $(1, 1)$ in its convex hull. We define[2] such a situation where supply is in the convex hull of aggregate demand as a *pseudo-equilibrium*. In this example, there is no equilibrium and the clock auction terminates at the minimum pseudo-equilibrium price vector.

Examples of this sort are potentially significant for the design of activity rules in auctions. At prices $(p_x, p_1, p_2) = (.4, .4, .41)$, firm 1 demands $(x, y_1, y_2) = (0, 1, 1)$ while at prices $(p_x, p_1, p_2) = (.4, .5, .41)$, firm 1 demands $(x, y_1, y_2) = (1, 0, 0)$. Suppose these two price vectors represent successive prices in an ascending auction and that the next price vector is $(.5, .5, .41)$. The firm's demand now shifts to $(0, 1, 1)$: its total demand rises from 1 unit to 2 units. Hatfield and Milgrom (2004) had shown that the unitary substitutes property implies that a profit-maximizing firm satisfies the *law of aggregate demand*: as prices rise, total demand (i.e. the sum of quantities demanded across all goods) does not increase. Activity rules for ascending auctions with or without clocks typically require that the demand expressed during an auction must satisfy that property,[3] and our example shows that such rules can block straightforward bidding when goods are class substitutes (but not when they are unitary substitutes).

These observations herald more general results, which are the subject of this paper.

The remainder of this paper is organized as follows. Section 2 defines class-substitute valuations, based on a multi-unit formulation of the economy, and unitary-substitute valuations, based on a binary formulation. Section 3 characterizes class-substitute and

[2]In fact, such property is equivalent to our definition of pseudo-equilibrium. See Definition 11.

[3]An exception is the revealed-preference activity rule of Ausubel and Milgrom (2002).

unitary-substitute valuations in terms of the firm's dual profit function. The dual characterization adds transparency to some of our central results. Section 4 further analyzes the concepts of substitutes and their relations. Gul and Stacchetti had shown that unitary substitutes is equivalent to a certain *single-improvement property* defined using nonlinear prices. We show that it is also equivalent to a similar property defined using only linear prices. We demonstrate that unitary substitutes is equivalent to class substitutes plus two additional conditions. We also show that, while the law of aggregate demand may fail with class substitutes, it always holds when an additional assumption is made, which we call the *consecutive-integer property*. Section 5 considers the implications of class and unitary substitutes for aggregate demand. We show that the unitary substitutes condition is sufficient and necessary (in a quantified sense) for the robust existence of market-clearing prices. We show that the class substitutes condition implies that the set of pseudo-equilibrium price vectors is a non-empty sublattice and that this set coincides with the set of equilibrium prices whenever an equilibrium exists. We show that the unitary-substitutes is a sufficient and, in a similar quantified sense, necessary condition for Vickrey payoffs to be in the core.

Section 6 presents our analysis of clock auctions when bidders have class-substitute valuations. We introduce a continuous model to represent clock auctions with small bid increments. We first show that class substitutes is necessary and sufficient for the monotonicity of a certain tâtonnement-like clock auction and that continuous descending or ascending clock auctions always terminate at a pseudo-equilibrium. In one version of the clock auction model, the auction terminates at the smallest pseudo-equilibrium price.

Section 7 presents further properties of pseudo-equilibria, elaborating on the idea that they are approximate equilibria when equilibria do not exist.

Section 8 extends the analysis to divisible goods. We show that for the case of divisible goods and concave valuations, a natural extension of unitary substitutes coincides with class substitutes. In that case, the law of aggregate demand and its unit-free extensions generally fail. Thus, for concave valuations, the law of aggregate demand characterizes the difference between the cases of discrete goods and divisible goods.

Section 9 concludes.

## 2 Definitions

Consider an economy with $K$ goods, in which good $k$ is available in $N_k$ units for $k \in \mathcal{K} = \{1, \ldots, K\}$. Let

$$\mathcal{X} = \Pi_{k \in \mathcal{K}} \{0, 1, \ldots, N_k\}$$

and

$$\tilde{\mathcal{X}} = \Pi_{k \in \mathcal{K}} \{0, 1\}^{N_k}$$

represent the space of possible bundles of the exchange economy in its *multi-unit* and *binary* formulations. The obvious correspondence between these formulations is represented by the function $\phi : \tilde{\mathcal{X}}$ into $\mathcal{X}$. Formally,

$$x_k = \varphi_k(\tilde{x}) = \sum_{j=1}^{N_k} \tilde{x}_{kj}.$$

Definition 1 (Multi-Unit Valuation) *A multi-unit valuation $v$ is a mapping from $\mathcal{X}$ into $\mathbb{R}$.*

Definition 2 (Binary Valuation) *A binary valuation $\tilde{v}$ is a mapping from $\tilde{\mathcal{X}}$ into $\mathbb{R}$.*

The binary valuation $\tilde{v}$ *corresponds to* the multi-unit valuation $v$, if for every $\tilde{x}$, $\tilde{v}(\tilde{x}) = v(\varphi(\tilde{x}))$. We denote by $\mathcal{V}$ the space of multi-unit valuations and $\tilde{\mathcal{V}}$ the space of corresponding binary valuations. Similarly, $\mathcal{P} = \mathbb{R}_+^K$ and $\tilde{\mathcal{P}} = \Pi_{k \in \mathcal{K}} \mathbb{R}_+^{N_k}$ denote the respective price spaces of the multi-unit and binary economies. The first space formulation permits only linear prices for each category of goods, while the second space effectively allows non-linear prices for each type of good separately, with the marginal price for each good weakly increasing. Throughout the paper, we assume that agents have quasi-linear utilities.

Assumption 1 (Quasi-linearity) *The utility of an agent with multi-unit valuation $v$ acquiring a bundle $x$ at price $p$ is*

$$u(x, p) = v(x) - px.$$

*Similarly, the utility of an agent with binary valuation $\tilde{v}$ acquiring a bundle $\tilde{x}$ at price $\tilde{p}$ is*

$$\tilde{u}(\tilde{x}, \tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}.$$

Given a binary valuation $\tilde{v}$ and a price vector $\tilde{p} \in \tilde{\mathcal{P}}$, define the demand function of the agent at price $\tilde{p}$ by

$$\tilde{D}(\tilde{p}) = \arg\max_{\tilde{x} \in \tilde{\mathcal{X}}} \{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}.$$

Similarly, we define the multi-unit demand $D$ of an agent with valuation $v$ as

$$D(p) = \arg\max_{x \in \mathcal{X}} \{v(x) - px\}.$$

With quasi-linear preferences, there is no distinction to be made between gross and net substitutes, so we drop the modifier and make the following definitions.

DEFINITION 3 (UNITARY-SUBSTITUTE VALUATION) *A multi-unit valuation $v$ is a* unitary-substitute valuation *if its binary form $\tilde{v}$ satisfies the* binary substitutes property*: for any prices $\tilde{p}$ and $\tilde{q}$ in $\tilde{\mathcal{P}}$ such that $\tilde{p} \leq \tilde{q}$, and $x \in \tilde{D}(\tilde{p})$, there exists a bundle $\tilde{x}'$ in $\tilde{D}(\tilde{q})$ such that*

$$\tilde{x}'_{kj} \geq \tilde{x}_{kj}$$

*for all $(k, j)$ such that $\tilde{p}_{kj} = \tilde{q}_{kj}$.*

DEFINITION 4 (CLASS-SUBSTITUTE VALUATION) *A multi-unit valuation $v$ is a* class-substitute valuation *if it satisfies the* multi-unit substitutes property*: for all prices $p$ and $q$ such that $p \leq q$ and $x \in D(p)$, there exists a bundle $x'$ in $D(q)$ such that*

$$x'_k \geq x_k$$

*for all $k$ in $\mathcal{K} = \{\kappa \in \mathcal{K} : p_\kappa = q_\kappa\}$.*

The unitary substitutes condition is at least weakly more restrictive than the class substitutes condition, because the latter applies only for linear prices while the former applies also for nonlinear prices. Moreover, the class substitutes condition only compares units of distinct goods, while the unitary substitutes condition requires that units of the *same* good be substitutes. Section 1 illustrates the limits of class-substitute valuations. In particular, class-substitute valuations may violate the law of aggregate demand.

# 3 Duality Results

To any multi-unit valuation $v$ we associate the dual profit function $\pi : \mathcal{P} \to \mathbb{R}$ such that

$$\pi(p) = \max_{x \in \mathcal{X}} \{u(x, p) = v(x) - px\}.$$

Similarly, to any binary valuation $\tilde{v}$ we associate the dual profit function

$$\tilde{\pi}(\tilde{p}) = \max_{\tilde{x} \in \tilde{\mathcal{X}}} \{\tilde{u}(\tilde{x}, \tilde{p}) = \tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\}.$$

We will use the following result.

Definition 5 (Multi-Unit Concavity) *A multi-unit valuation is concave if it can be extended to a concave function on* $\mathbb{R}^K$.

Theorem 1 *Let* $v$ *be a multi-unit valuation and* $\pi$ *be its dual profit function. Then, for all* $x \in \mathcal{X}$,

$$v(x) \leq \min_{p \in \mathcal{P}} \{\pi(p) + px\}.$$

*Moreover,* $v$ *is concave if and only if*

$$v(x) = \min_{p \in \mathcal{P}} \{\pi(p) + px\} \tag{1}$$

*for all* $x \in \mathcal{X}$.

*Proof.* The first claim follows from the definition of $\pi$. The second claim is proved by applying the separating-hyperplane theorem. ■

Ausubel's and Milgrom's dual characterization of unitary substitute valuations extends straightforwardly to the cases treated here.

Theorem 2 (Ausubel and Milgrom (2002)) $v$ *is a class-substitute valuation if and only if* $\pi$ *is submodular, and these hold if and only if the dual profit function* $\tilde{\pi}$ *of its binary form* $\tilde{v} = \phi(v)$ *is submodular on the restricted domain where goods of the same type have equal prices. In addition,* $v$ *is a unitary-substitute valuation if and only if the dual profit function* $\tilde{\pi}$ *of its binary form* $\tilde{v} = \phi(v)$ *is submodular.*

*Proof.* The proofs of the two statements follow the proof of Theorem 10 in Ausubel and Milgrom (2002). ■

One can alternatively characterize class substitutes using the larger price space $\tilde{\mathcal{P}}$ of the binary formulation. The relevant multi-unit prices are expressed in that formulation by the subset $\mathcal{P}_L$ of $\tilde{\mathcal{P}}$ in which goods of the same type have the same price. This subset is isomorphic to the set $\mathcal{P}$ of linear prices used in the multi-unit economy. The class-substitute property then corresponds to the requirement that the dual profit function is submodular on $\mathcal{P}_L$, while the unitary-substitute property requires submodularity on the whole price space. An immediate consequence of this alternative formulation is the following:

Theorem 3 *Any unitary-substitute valuation is also a class-substitute valuation.*

The converse is not true. For example, suppose there is only one type of good, so that every valuation $v$ is a class-substitute valuation. Let $v(0) = 0$, $v(1) = 1$ and $v(2) = 3$ and suppose prices are $(p_1, p_2) = (1.4, 1.4)$, at which both units are demanded. Increasing $p_1$ to 1.7 would reduce demand to 0, thus violating the unitary-substitute property. The same example establishes that a multi-unit valuation can be submodular even when the related binary valuation is not.

We have seen than class-substitute valuations need not be submodular. The following result shows that adding the requirement that $v$ is concave does yield submodularity.

THEOREM 4 *Any concave class-substitute valuation is submodular.*

*Proof.* From Theorem 1, we have

$$v(x) = \min_{p \in \mathcal{P}} \{\pi(p) + px\} = \max_{p} \{-\pi(p) - px\}.$$

From Theorem 2, $\pi$ is submodular. Therefore, $v$ is the maximum over $p$ of a function that is supermodular in $p$ and $-x$, which implies that $v$ is supermodular in $-x$ or, equivalently, submodular in $x$. ■

THEOREM 5 *Let $\tilde{v}$ be a unitary-substitute valuation. Then,*

$$\tilde{v}(\tilde{x}) = \min_{\tilde{p} \in \tilde{\mathcal{P}}} \{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}$$

*Proof.* Given $\tilde{x}$, define $\tilde{p}$ as $\tilde{p}_a = 0$ if $\tilde{x}_a = 1$ and $\tilde{p}_a = \infty$ if $\tilde{x}_a = 0$. Clearly, $\tilde{x} \in \tilde{D}(\tilde{p})$. The rest of the proof is identical to the proof of Theorem 1. ■

Underlying Theorem 4 is the fact that concavity allows $v$ to be expressed by formula (1). As Theorem 5 shows, concavity is not required in the binary form to obtain that equation, which offers a way to understand why unitary substitutes implies submodularity.

# 4 Relations between Concepts of Substitutes

Gul and Stacchetti (1999) introduce the single-improvement property for binary valuations and show that it is equivalent to the substitutes property, as follows.

Definition 6 (Binary Single-Improvement Property) *A binary valuation $\tilde{v}$ satisfies the single-improvement property if for any price vector $\tilde{p}$ and $\tilde{x} \notin \tilde{D}(\tilde{p})$, there exists $\tilde{y}$ such that $u(\tilde{y}, \tilde{p}) > u(\tilde{x}, \tilde{p})$,*

$$\|(\tilde{y} - \tilde{x})_+\|_1 \leq 1,$$

*and*

$$\|(\tilde{x} - \tilde{y})_+\|_1 \leq 1.$$

Theorem 6 (Gul and Stacchetti (1999)) *A monotonic valuation is a unitary-substitute valuation if and only if it satisfies the binary single-improvement property.*

We now extend these results to multi-unit economies.

Definition 7 (Multi-Unit Single-Improvement Property) *A valuation $v$ satisfies the multi-unit single-improvement property if for any $p$ and $x \notin D(p)$, there exists $x'$ such that $u(x', p) > u(x, p)$,*

$$\|(x' - x)_+\|_1 \leq 1$$

*and*[4]

$$\|(x - x')_+\|_1 \leq 1.$$

The only difference in the definitions of binary and multi-unit single-improvement properties therefore resides in the price domain where the property has to hold.

Throughout the paper, we will denote by $e_k$ the vector of $\mathbb{R}^K$ whose $k^{th}$ component equals one and whose other components equal zero.

Theorem 7 *If $v$ satisfies the multi-unit single-improvement property then it is a class-substitute valuation.*

*Proof.* Suppose by contradiction that the class-substitute property is violated: there exist $p$, $k$, a small positive constant $\varepsilon$, and a bundle $x$ such that $x \in D(p)$ and for all $y \in D(p + \varepsilon e_k)$, there exists $j \neq k$ such that $y_j < x_j$. Set $\hat{p} = p + \varepsilon e_k$. We have $x \notin D(\hat{p})$ and $y_k < x_k$ for all $y \in D(\hat{p})$ (since D(p) clearly contains bundles with strictly less than $x_k$ units of good $k$). Therefore $x$ is only dominated by bundles $y$ that have strictly less units of at least two goods, implying that $\|(x - y)_+\|_1 \geq 2$, which violates the single-improvement property. ■

The converse in not true, as we now illustrate.

[4]Here the norm is defined on $\mathbb{R}^K$, whereas it was defined on $\mathbb{R}^{\sum_k N_k}$ in the binary setting.

Counter-Example 1 *There exist class-substitute valuation that do not satisfy the multi-unit single-improvement property.*

*Proof.* In the first example of Section 1, the valuation is submodular in a two-good economy, thus satisfies the class substitutes property. However, for $r = 0.2$ and $w = 0.3$, the bundle $(1, 0)$ is only dominated by the bundle $(0, 2)$, which violates the single-improvement property. ■

Definition 8 (Multi-Unit Submodularity) *A multi-unit valuation $v$ is submodular if for any vectors $x$ and $x'$ of $\mathcal{X}$*

$$v(x) + v(x') \geq v(x \wedge x') + v(x \vee x')$$

For completeness, we record as a theorem that submodularity of a binary valuation implies the same property in the corresponding multi-unit valuation. In fact, by an argument similar to the one in the duality section, one can show that submodularity of the multi-unit valuation is equivalent to submodularity of the binary valuation on a restricted subspace of the valuation space.

Theorem 8 *Let $v$ be a multi-unit valuation. If the binary valuation $\tilde{v} = \phi(v)$ is submodular, then $v$ is submodular.*

We have seen earlier that the reverse implication is not true. It is well known that when the economy only has two divisible goods and the valuation is concave, submodularity of $v$ is equivalent to the substitutes property. It is also well known that the extreme points of a demand set are unchanged when one replaces $v$ with its concave hull (i.e. the smallest concave function above $v$). The following theorem takes these observations one step farther.

Theorem 9 *Suppose that $v$ is a 2-good valuation and $\hat{v}$ is its concave hull. Then $v$ is a class-substitute valuation if and only if $\hat{v}$ is submodular.*

*Proof.* We have

$$\hat{v}(x) = \min_p \{\pi(p) + px\} = -\max_p g(p, x)$$

where $g(p, x) = -\pi(p) - px$. $v$ is a class-substitute valuation if and only if $\pi$ is submodular. In that case, $g$ is supermodular in $(p_1, p_2, -x_1, -x_2)$. A theorem by Topkis (1998) then implies that $\max_p g$ is supermodular in $x$ so $\hat{v}$ is submodular in $x$. For the reverse direction, we have

$$\pi(p) = \max_x \{\hat{v}(x) + px\} = \max_x h(x, p).$$

If $\hat{v}$ is submodular, $h$ is supermodular in $(x_1, -x_2, p_1, -p_2)$. The same theorem then implies that $\pi$ is supermodular in $(p_1, -p_2)$ or, equivalently, submodular in $(p_1, p_2)$. ∎

The next theorem contains a key result for the existence of Walrasian equilibria in multi-unit economies. The proof, as well as other theorems whose conclusions involve concavity of $v$, uses Gul and Stacchetti's characterization theorem (Theorem 6) and thus requires monotonicity of $v$. Throughout the rest of the paper, we assume that $v$ is nondecreasing.

Assumption 2 *Agent valuations are nondecreasing.*

Theorem 10 *If $v$ is a unitary-substitute valuation, then any bundle $x$ is optimal at some linear price.*

*Proof.* Let $x$ be any bundle, and $\tilde{x}$ be a binary representation of this bundle. From[5] Theorem 5, we have

$$v(x) = \tilde{v}(\tilde{x}) = \min_{\tilde{p}} \{\tilde{\pi}(\tilde{p}) + \tilde{p}\tilde{x}\}. \tag{2}$$

Moreover, from Theorem 17, the set $M$ of minimizers of the right-hand side of (2) is a complete lattice. In particular, it has a largest element $\tilde{p}$. We claim that this element is a linear price on the support of $\tilde{x}$. That is, for any good $k$ such that $x_k \geq 1$, $\tilde{p}_{ki} = \tilde{p}_{kj}$ whenever $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$. Suppose by contradiction that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units $i$, $j$ of some good $k$ such that $\tilde{x}_{ki} = \tilde{x}_{kj} = 1$. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units $i$ and $j$ of good $k$, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a minimizer of (2). Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also in $M$, which contradicts maximality of $\tilde{p}$. We have thus shown that $\tilde{p}$ is linear on the support of $\tilde{x}$: for each good $k$ there exists a price $p_k$ such that $\tilde{p}_{ki} = p_k$ for all $i$ such that $\tilde{x}_{ki} = 1$. Obviously, $\tilde{p}_{kl} = +\infty$ whenever $\tilde{x}_{kl} = 0$. For any good $k$ such that $x_k \in \{1, N_k - 1\}$, the firm is indifferent, at $\tilde{p}$, between $x$ and some bundle $y^k$ such that $y^k_k < x_k$, otherwise it would be possible to increase $p_k$, which would contradict maximality of $\tilde{p}$. We can choose $y^k$ so that it is optimal if we slightly increase the price of some particular unit of good $k$. Since $\tilde{v}$ is a unitary substitute valuation, we can choose $y$ such that $y^k_k = x_k - 1$, and $y^k_j \geq x_j$ for all $j$. Since $\tilde{p}_{kl} = +\infty$ outside of the support of $\tilde{x}$, we necessarily have $y^k_j = x_j$ for $j \neq k$. Therefore we exactly have

$$y^k = x - e_k.$$

Such indifference bundles exist for all goods $k$ such that $1 \leq x_k \leq N_k - 1$.

[5] As the reader can easily verify, Theorems 5 and 17 are independent of this proof.

We now prove that $x$ is optimal for the linear price vector $p = (p_k)_{k \in \mathcal{K}}$, where $p_k = +\infty$ when $x_k = 0$, $p_k = 0$ when $x_k = N_k$, and $p_k$ is defined as above when $1 \leq x_k \leq N_k - 1$. That is, we can impose $\tilde{p}_{kl} = p_k$ for all units, including those for which $\tilde{x}_{kl} = 0$, and preserve optimality of $x$. For all goods such that $x_k \in [1, N_k - 1]$, reset all unit prices outside the support of $\tilde{x}$ from $+\infty$ to $p_k$. This change does not affect optimality of $x$ among bundles $z$ such that $z \leq x$, and it does not affect indifference between $x$ and the bundles $y^k$. For any good $k$, consider the bundle $z^k = x + e_k$. Since $\tilde{v}$ is submodular, Theorem 13 implies that $v$ is component-wise concave (see p. 16). Therefore,

$$v(z^k) - v(x) \leq v(x) - v(y^k) = p_k,$$

which implies that $z^k$ is weakly dominated by $x$. Now for two goods $k \neq j$ such that $x_k \geq 1$ and $x_j < N_j$, consider the bundle $z^{kj} = x - e_k + e_j$. We claim that $z$ is also weakly dominated by $x$. To see this, we use the following Lemma, whose proof is in the Appendix.[6]

Lemma 1 *If $v$ is a unitary-substitute valuation, $k$ and $j$ are two goods and $x$ is a bundle such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$, then*

$$v(x + e_k + e_j) - v(x + e_k) \geq v(x + 2e_j) - v(x + e_j).$$

Applying Lemma 1 to the bundle $x - e^j - e_k$ yields

$$v(x) - v(y^j) \geq v(z^{kj}) - v(y^k),$$

which implies, since $v(x) = v(y^j) + p_j = v(y^k) + p_k$, that

$$v(x) - p_k \geq v(z^{kj}) - p_j,$$

and thus that $x$ weakly dominates $z$. We have thus proved that $\tilde{x}$ has no single improvement. From Theorem 6, $\tilde{v}$ satisfies the single-improvement property. Therefore, $\tilde{x}$ must be optimal at the linear price $\tilde{p}$ such that $\tilde{p}_{kl} = p_k$ for all $l \in \{1, \ldots, N_k\}$. Equivalently, the bundle $x$ is optimal at price $p = (p_k)$, which concludes the proof. ■

We can now state the properties of unitary-substitute valuations in linear-pricing economies.

Theorem 11 *Suppose that $v$ is a unitary-substitute valuation. Then it satisfies the following properties:*

[6] As can be easily checked, the proof of Lemma 1 is independent of the proof of the present theorem.

[Concavity] *$v$ is concave.*

[Class-Substitute Property.] *For any $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, and $x \in D(p)$, there exists $x' \in D(p + \varepsilon e_k)$ such that*

$$x'_j \geq x_j \text{ for all } j \neq k.$$

[Law of Aggregate Demand.] *For any $p \in \mathcal{P}$, $k \in \mathcal{K}$, $\varepsilon > 0$, and $x \in D(p)$, there exists $x' \in D(p + \varepsilon e_k)$ such that*

$$\|x'\|_1 \leq \|x\|_1.$$

[Consecutive-Integer Property.] *For any $p \in \mathcal{P}$ and $k \in \mathcal{K}$, the set*

$$D_k(p) = \{z_k : z \in D(p)\}$$

*consists of consecutive integers.*

*Proof.* Theorem 3 implies that $v$ satisfies the class-substitute property, and Hatfield and Milgrom (2004) show that $v$ must satisfy the law of aggregate demand. Therefore, it remains to show that $v$ is concave and satisfies the consecutive-integer property.

We first show that $v$ is concave. Theorem 10 implies that for any $x$ there exists $p$ such that

$$\pi(p) = v(x) - px,$$

where $\pi$ is the dual profit function defined in Section 3. From the first part of Theorem 1,

$$v(x) \leq \min_p \pi(p) + px.$$

Combining the two equations above yields

$$v(x) = \min_p \pi(p) + px,$$

for all $x$. Applying the second part of Theorem 1 then proves that $v$ is concave.[7]

Last, we show the consecutive-integer property. Suppose by contradiction that there exist $p$, $k$, and two bundles $x$ and $y$ in $D(p)$ such that $x_k \geq y_k + 2$ and $z \in D(p) \Rightarrow z_k \notin (y_k, x_k)$. Consider the binary price vector $\tilde{p}$ that is linear and equal to $p_j$ for all good $j \neq k$, and that equals $p_k$ for the first $x_k$ units of good $k$ and $+\infty$ for the remaining units of good $k$. Clearly, there exist binary forms $\tilde{x}$ and $\tilde{y}$ of $x$ and $y$ that belong to $\tilde{D}(\tilde{p})$, and there is

[7] As can be easily verified, the proof of Theorem 1 is independent of the present proof.

no bundle $\tilde{z}$ in $\tilde{D}(\tilde{p})$ such that $z_k \in (y_k, x_k)$. If the price of one unit of good $k$ is slightly increased, the demand for good $k$ thus falls directly below $z_k$, implying that the demand of another unit of good $k$, whose price had not increased, has strictly decreased, which violates the unitary-substitute property for $\tilde{v}$. ■

The consecutive-integer property is not implied by concavity of $v$. For example, in a (multi-unit) two-good economy, concavity is compatible with the demand set $D(p) = \{(1,0),(0,2)\}$. However, this demand set violates the consecutive-integer property: the set $D_2(p) = \{0,2\}$ does not consist of consecutive integers. The consecutive-integer property rules out valuations causing a sudden decrease in the consumption of a good (independently of the consumption of other goods). For example, there are no prices at which the firm is indifferent between bundles containing, say, 5 and 10 units of a good, but strictly prefers these bundles to any bundle containing between 6 and 9 units of that good. In that sense, there are no "holes" in the demand set with respect to any good. In terms of demand, the property implies a progressive reaction to price movements: as the price of a good increases, the optimal demand of that good decreases unit by unit.

By contrast, concavity is not required for the law of aggregate demand.

Theorem 12 *If $v$ is a class-substitute valuation that satisfies the consecutive-integer property, then it satisfies the law of aggregate demand.*

*Proof.* See the Appendix.

The class-substitute property and the law of aggregate demand do not imply the consecutive-integer property. For example, in an economy with one good available in two units, consider the non-concave valuation $v(0) = 0$, $v(1) = 1$, and $v(2) = 4$. $v$ is trivially a substitutes valuation, and satisfies the law of aggregate demand. However, at price $p = 2$, the demand set is $\{0,2\}$, which violates the consecutive-integer property. This is also an example of a class-substitute valuation that is not concave.

To obtain sharp results, we consider the concept of component-wise concavity, which is weaker than concavity and entails diminishing marginal returns in each component separately.

Definition 9 (Component-wise Concavity) *A multi-unit valuation $v$ is component-*

*wise concave if for all $x$ and $k$,*

$$v(x_k + 1, x_{-k}) - v(x) \geq v(x_k + 2, x_{-k}) - v(x_k + 1, x_{-k}).$$

THEOREM 13 *A multi-unit valuation $v$ is submodular and component-wise concave if and only if its binary form $\tilde{v} = \phi(v)$ is submodular.*

*Proof.* By a theorem of Topkis (1998), it is sufficient to consider binary bundles $x$ and $y$ that differ in just two components. If the two components represent the same good, then submodularity of the binary form is the same as component-wise concavity. If the two components represent different goods, then submodularity of the binary form is implied by submodularity of the multi-unit form (and conversely). ■

The last three properties listed in Theorem 11 describe the demands corresponding to a unitary-substitute valuation in linear-pricing economies. Even though unitary-substitute valuations are defined by their demands in response to nonlinear prices, the identified properties turn out to be sufficient to characterize unitary substitutes. That is the essential content of Theorem 14 below.

Before proving this theorem, we state a new "minimax" result, in which one of the choice set is a lattice and the other choice set consists of nonlinear prices. The proof of this result is in the Appendix.

If $x$ is a multi-unit bundle and $\tilde{p}$ is a nonlinear price vector, let $(\tilde{p}, x)$ denote the cost of acquiring bundle $x$ under $\tilde{p}$. That is,

$$(\tilde{p}, x) = \sum_{k \in \mathcal{K}} \sum_{i=1}^{x_k} \tilde{p}_{k(i)},$$

where $\tilde{p}_{k(i)}$ is the price of the $i^{th}$ cheapest unit of good $k$.

PROPOSITION 1 (MINIMAX) *Suppose that $v$ is a concave class-substitute valuation satisfying the consecutive-integer property, and let $\tilde{p}$ be a nonlinear price vector. Then,*

$$\max_x \min_p \{\pi(p) + px - (\tilde{p}, x)\} = \min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}$$

THEOREM 14 *Let $v$ be a multi-unit valuation. The following properties are equivalent.*

*(i) $v$ is a unitary-substitute valuation.*

*(ii) $v$ is a concave class-substitute valuation, and satisfies the consecutive-integer property.*

*Proof.* We know from Theorem 11 that (i) implies (ii). We now show that (ii) implies (i). From Theorem 2, it is enough to show that $\tilde{\pi}$ is submodular. Consider any nonlinear price vector $\tilde{p}$. We have

$$\tilde{\pi}(\tilde{p}) = \max_{\tilde{x}}\{\tilde{v}(\tilde{x}) - \tilde{p}\tilde{x}\} = \max_{x}\{v(x) - (\tilde{p}, x)\}.$$

Since $v$ is concave, Theorem 1 implies that

$$\tilde{\pi}(\tilde{p}) = \max_{x}\{\min_{p}\{\pi(p) + px\} - (\tilde{p}, x)\} = \max_{x}\{\min_{p}\{\pi(p) + px - (\tilde{p}, x)\}\}.$$

From Proposition 1, the max and min operators can be swapped:

$$\tilde{\pi}(\tilde{p}) = \min_{p}\{\max_{x}\{\pi(p) + px - (\tilde{p}, x)\}\} = \min_{p}\{\pi(p) + \max_{x}\{px - (\tilde{p}, x)\}\}.$$

As can be easily verified, the inner maximum equals

$$\sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+.$$

Therefore,

$$\tilde{\pi}(\tilde{p}) = \min_{p}\left\{\pi(p) + \sum_{k\in\mathcal{K}}\sum_{i=1}^{N_k}(p_k - \tilde{p}_{ki})_+\right\}.$$

Since $v$ is a class-substitute valuation, $\pi$ is submodular by Theorem 2. Moreover, the function $(x, y) \to (x - y)_+$ is submodular as a convex function of the difference $x - y$. Therefore, $\tilde{\pi}(\tilde{p})$ is the minimum over $p$ of an objective function that is submodular in $p$ and $\tilde{p}$, which shows that it is submodular in $\tilde{p}$.[8] ■

It turns out that, *given* concavity and the class-substitute property, the law of aggregate demand is equivalent to the consecutive integer property. Some of the main results above are combined and extended in the following theorem.

Theorem 15 (Equivalence of Substitute Concepts) *Let $v$ be a multi-unit valuation. The following statements are equivalent.*

*(i) $v$ satisfies the binary single-improvement property.*

*(ii) $v$ is a unitary-substitute valuation.*

[8]See Topkis (1968).

*(iii) $v$ is a concave class-substitute valuation and satisfies the consecutive-integer property.*

*(iv) $v$ is a concave class-substitute valuation and satisfies the law of aggregate demand.*

*(v) $v$ is concave and satisfies the multi-unit single-improvement property.*

*Proof.* $(i) \Leftrightarrow (ii)$ is Gul and Stacchetti's theorem (see Theorem 6). $(ii) \Leftrightarrow (iii)$ is a restatement of Theorem 14. Theorem 12 shows that $(iii)$ implies $(iv)$. For the converse, the class-substitute property implies[9] for all $p$ that any edge $E$ of $D(p)$ has direction $e_i$ or $e_i - \alpha e_j$ for some goods $i, j$. In the first case, concavity implies that all integral bundles on the edge belong to the demand. In the second case, $\alpha = 1$. Otherwise, slightly modifying the price would reduce demand to that edge, and increasing $p_i$ if $\alpha > 1$ or $p_j$ if $\alpha < 1$ would violate the law of aggregate demand. This, along with concavity, implies that the consecutive-integer property holds along all edges, and thus for $D(p)$. $(i)-(iv)$ implies $(v)$: $(i)$ clearly implies the multi-unit single-improvement property, and $(iii)$ implies concavity. We conclude by showing that $(v)$ implies $(iii)$. We already know from Theorem 7 that if $v$ satisfies $(v)$, then it is a class-substitute valuation. Therefore, there only remains to show that $v$ satisfies the consecutive-integer property. Suppose it doesn't. There exists a price vector $p$, a good $k$, and a unit number $d$ such that $D_k = \{z_k : z \in D(p)\}$ is split by $d$: the sets $D_k^- = D_k \cap [0, d-1]$ and $D_k^+ = D_k \cap [d+1, N_k]$ are disjoint and cover $D_k$. Now slightly increase $p_k$. The new demand set $D'$ is such that $D'_k \subset D_k^-$. Pick any bundle $y$ that is optimal under the new price within the set $\{x \in \mathcal{X} : x_k \geq d\}$. Then $y_k > d$, because $p_k$ has only been slightly increased and any bundle with $d$ units of good $k$ was strictly dominated by $D_k^+$. At the new price, $y$ is dominated but cannot be strictly improved upon with reducing the amount of good $k$ by at least two units, which violates the single-improvement property. ■

The multi-unit single-improvement property alone is not equivalent to unitary substitutes. For example, in an economy with two goods available in two units, consider the valuation $v$ defined by $v(x) = \|x\|_1 - .1r(x)$, where $r(x)$ equals 1 if $x$ contains exactly one unit of each good, and 0 otherwise. The valuation is not concave, and therefore cannot be a unitary-substitute valuation. However, one can easily verify that $v$ satisfies the multi-unit single-improvement property.

[9] See the proof of Proposition 1.

We conclude this section with a property of concave, class-substitute valuations. For any (multi-unit) bundle $x$, let $\mathcal{P}(x)$ denote the set of price vectors such that $x \in D(p)$.

THEOREM 16 *If $v$ is a class-substitute valuation, then for all $x$, $\mathcal{P}(x)$ is either the empty set or the complete sublattice of $\mathcal{P}$ given by $\mathcal{P}(x) = \arg\min\{\pi(p) + px\}$.*

*Proof.* Fix $x \in \mathcal{X}$. From Theorem 5,

$$v(x) \leq \min_p \{\pi(p) + px\}.$$

Suppose that the inequality is strict. Then $v(x) - px < \pi(p)$ for all $p$, so $\mathcal{P}(x)$ is the empty set. Now suppose that

$$v(x) = \min_p \{\pi(p) + px\}.$$

Then, for all $p \in \arg\min\{\pi(p) + px\}$,

$$v(x) - px = \pi(p),$$

so $x \in D(p)$. Conversely, if $x \in D(\bar{p})$ for some price $\bar{p}$, we have

$$\arg\min\{\pi(p) + px\} = v(x) = \pi(\bar{p}) + \bar{p}x.$$

Therefore,

$$\mathcal{P}(x) = \arg\min\{\pi(p) + px\}.$$

From Theorem 2, $\pi(p)$ is submodular. Therefore $\mathcal{P}(x)$ is the set of minimizers of a submodular function over a sublattice $\mathcal{P}$; hence, it is a sublattice of $\mathcal{P}$. Completeness is obtained by a standard limit argument. ■

In the binary formulation, all bundles can be achieved through nonlinear pricing, by setting some unit prices to zero and others to infinity. Therefore, Theorem 16 takes a simpler form. For any binary bundle $\tilde{x}$, let $\tilde{\mathcal{P}}(\tilde{x})$ denote the set of price vectors such that $\tilde{x} \in \tilde{D}(\tilde{p})$.

THEOREM 17 *If $\tilde{v}$ is a binary valuation satisfying the unitary substitutes, then $\tilde{\mathcal{P}}(\tilde{x})$ is a complete, non-empty lattice for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

*Proof.* For any bundle $\tilde{x}$, there exists a price $\tilde{p}$ such that $\tilde{x} \in \tilde{D}(\tilde{p})$. Therefore, $\tilde{\mathcal{P}}(\tilde{x})$ is nonempty. The rest of the proof is similar to the proof of Theorem 16. ■

## 5 Aggregate Demand and Equilibrium Analysis

The first theorem of this section extends results by Gul and Stacchetti and by Milgrom asserting *necessary* conditions for the existence of Walrasian equilibrium in the binary formulation. These theorems assume that individual valuations are drawn from a set that includes all unit-demand valuations (Gul and Stacchetti), which are defined next, or all additive valuations (Milgrom).[10] They establish that if the set of valuations includes any that are not unitary substitutes, then there is a profile of valuations to be drawn from the set such that no competitive equilibrium exists.

These results are unsatisfactory for our multi-unit context, because they allow preferences to vary among identical items and the constructions used in those papers hinge on that freedom. The next theorem extends the earlier results by including the restriction that firms' binary valuations are consistent with some multi-unit valuation, that is, that firms treat all units of the same good symmetrically.

Definition 10 *A unit-demand valuation is such that for all price $p$ and $x \in D(p)$, $\|x\|_1 \leq 1$.*

Let $N = \sum_k N_k$ denote the total number of units in the economy.

Theorem 18 *Consider a multi-unit endowment $\mathcal{X}$ and a firm having a concave, class-substitute valuation $v_1$ on $\mathcal{X}$ that is not a unitary-substitute valuation. Then there exist $I$ firms, $I \leq N$, with unit-demand valuations $\{v_i\}_{i \in I}$, such that the economy $E = (\mathcal{X}, v_1, \ldots, v_{I+1})$ has no Walrasian equilibrium.*

*Proof.* See the Appendix.

Since preferences are assumed to be quasi-linear, one can conveniently analyze equilibrium prices and allocations in terms of the solutions to certain optimization problems. With that objective in mind, consider an economy consisting of $n$ firms with valuations $\{v_i\}_{1 \leq i \leq n}$. The valuations $v_i$ are defined for $\{x \in \mathbb{N}^K : x_k \leq N_k \ \forall k \in \mathcal{K}\}$. It is convenient to extend the domain of $v_i$ by setting $v(x) = v(x \wedge (N_1, \ldots N_K))$ for all $x$ in $\mathbb{N}^K$. We now define the market-valuation $v$ of the economy by

$$v(x) = \max\left\{\sum v_i(x_i) : \sum x_i = x \text{ and } x_i \in \mathbb{N}^K\right\}.$$

[10] An *additive* valuation is a valuation with the property that the value of any set is equal to the sum of the values of the singletons in the set.

and the market dual profit function of the economy by

$$\pi(p) = \max_{x \in \mathbb{R}^K} \{v(x) - px\}.$$

The function $\pi$ is convex, as can be checked easily.

THEOREM 19 *For all $p \in \mathcal{P}$,*

$$\pi(p) = \sum_{1 \le i \le n} \pi_i(p).$$

*Proof.*

$$\begin{aligned} \pi(p) &= \max_x \{\max\{\textstyle\sum_i v_i(x_i) : \sum_i x_i = x\} - px\} \\ &= \max_{x_1,\ldots,x_n} \textstyle\sum_i \{v_i(x_i) - px_i\} \\ &= \textstyle\sum_i \pi_i(x_i), \end{aligned}$$

which concludes the proof. ■

Theorem 19 cannot be extended to nonlinear prices. To see this we observe, for example, that the cheapest unit of a given good can only be allocated to a single firm when computing the market dual profit function, whereas it is included in all individual dual profit functions involving at least one unit of this good. It is thus easy to construct examples where the market dual profit function is strictly lower than the sum of individual dual profit functions, the latter allowing each firm to use the cheapest units.

COROLLARY 1 *If all firms have class-substitute valuations, then the market valuation $v$ is also a class-substitute valuation.*

*Proof.* If individual firms have substitute valuations, Theorem 2 implies that individual profit functions are submodular. By Theorem 19, the market dual profit function is therefore a sum of submodular functions, and so itself submodular. Theorem 2 then allows us to conclude that $v$ is a substitute valuation. ■

DEFINITION 11 *A price vector $p$ is a* pseudo-equilibrium price *of the economy with endowment $\bar{x}$ if*

$$p \in \arg\min\{\pi(p) + p\bar{x}\}.$$

Sections 6 and 7 use the following characterization of pseudo-equilibrium prices.

PROPOSITION 2 *$p$ is a pseudo-equilibrium price if and only if $\bar{x}$ is in the convex hull of $D(p)$.*

*Proof.* By definition $p$ minimizes the convex function $f : p \to \pi(p) + p\bar{x}$. Therefore, 0 is in the subdifferential of $f$ at $p$.[11] That is, $0 \in \partial\pi(p) + \bar{x}$. The extreme points of $\partial\pi(p)$ are the opposite of bundles that are demanded at price $p$. Moreover, $-D(p) \subset \partial\pi(p)$. Therefore $-Co(D(p)) = \partial\pi(p)$. Combining these results yields $\bar{x} \in Co(D(p))$. ■

Let $\mathcal{P}(\bar{x})$ denote the set of pseudo-equilibrium prices.

PROPOSITION 3 *If all firms have class-substitute valuations, then $\mathcal{P}(\bar{x})$ is a complete sublattice of $\mathcal{P}$.*

*Proof.* Individual class-substitute valuations imply that $\pi_i$ is submodular for all $i$ by Theorem 2. Therefore, $\pi$ is submodular. The proof is then identical to the proof of Theorem 16. ■

THEOREM 20 *The economy with endowment $\bar{x}$ has a Walrasian equilibrium if and only if*

$$v(\bar{x}) = \min_p \{\pi(p) + p\bar{x}\}.$$

*Moreover, if the economy with endowment $\bar{x}$ has a Walrasian equilibrium, then the set of Walrasian equilibrium prices is exactly the set $P(\bar{x})$ of pseudo-equilibrium prices.*

*Proof.* Theorem 1 implies that $v(\bar{x}) \leq \min_p\{\pi(x)+p\bar{x}\}$. Suppose that $v(\bar{x}) = \pi(p)+px$ for some $p$. Let $\bar{x}_i$ denote the bundle received by firm $i$ for some fixed allocation maximizing the objective in the definition of $v$. For all $i$ we have

$$v_i(\bar{x}_i) - p\bar{x}_i \leq \pi_i(p).$$

Summing these inequalities yields $v(\bar{x}) \leq \pi(p) - p\bar{x}$. By assumption, the last inequality holds as an equality, which can only occur if

$$v_i(\bar{x}_i) - p\bar{x}_i = \pi(p)$$

for all $i$, implying that $(p, \bar{x}_1, ..., \bar{x}_n)$ is a Walrasian equilibrium.

To prove the second claim, suppose that $(\{\bar{x}_i\}_{1\leq i\leq n}, p)$ is a Walrasian equilibrium. Then, $v_i(\bar{x}_i) = \pi_i(p) + p\bar{x}_i$ for all $i$. Summing these equalities yields $v(\bar{x}) = \pi(p) + p\bar{x}$, which implies that $v(\bar{x}) = \min_p\{\pi(p)+p\bar{x}\}$ (since the minimum is always above $v(\bar{x})$). It is clear from the first part of the proof that if the economy has a Walrasian equilibrium, the set

[11] See for example Rockafellar (1970).

of Walrasian prices is exactly the set of pseudo-equilibrium prices. ■

Theorem 20 shows that whenever a Walrasian equilibrium exists, the concepts of pseudo-equilibrium and equilibrium coincide. In binary economies, where nonlinear pricing is available, the question of the existence of a Walrasian equilibrium have been solved by Gul and Stacchetti (1999) and Milgrom (2000), who both show that equilibrium exists in the binary formulation when goods are unitary substitutes and establish the two partial converses described above.

For the multi-unit formulation, we have already established the partial converse in Theorem 18. We now consider the other direction: we prove that unitary substitutes implies the existence of a Walrasian equilibrium with *linear* pricing. This result is then used to prove the stronger theorem that unitary-substitute valuations are closed under aggregation: if all valuations satisfy unitary-substitutes, then so does the market valuation.

Theorem 21 (Linear-Pricing Walrasian Equilibrium) *In a multi-unit exchange economy with individual unitary-substitute valuations, there exists a Walrasian equilibrium with linear prices.*

*Proof.* Considering the binary form of the economy, Gul and Stacchetti (1999, Corollary 1) have shown that the set of (nonlinear pricing) Walrasian equilibria is a complete lattice. In particular, it has smallest and largest elements. We now prove that these two elements consist of linear prices, which proves the result. Suppose by contradiction that the largest element $\tilde{p}$ is such that $\tilde{p}_{ki} \neq \tilde{p}_{kj}$ for some units $i$, $j$ of some good $k$. Then the price vector $\tilde{p}'$ equal to $\tilde{p}$ except for units $i$ and $j$ of good $k$, where $\tilde{p}_{ki}$ and $\tilde{p}_{kj}$ are swapped, is also a Walrasian equilibrium. Therefore $\tilde{p} \vee \tilde{p}' > \tilde{p}$ is also a Walrasian equilibrium, which contradicts maximality of $\tilde{p}$. Linearity of the smallest element is proved similarly. ■

Corollary 2 (Concavity of Aggregate Demand) *In a multi-unit exchange economy with individual unitary-substitute valuations, the market valuation is concave.*

*Proof.* Denote by $x$ the total endowment of the economy, and $n$ the number of firms. We show that for all $y$ such that $0 \leq y \leq x$, there exists a linear price vector $p$ such that $y$ is in the demand set of the market valuation. From Theorem 21, we already know that the result is true if $y = x$. Thus suppose that $y < x$. Consider an additional firm with valuation $v_{n+1}(z) = Kz \wedge (x - y)$, where $K$ is a large constant, greater than the total value of other firms for the whole endowment $x$. One can easily check that $v_{n+1}$ is an assignment valuation, and therefore a unitary-substitute valuation (see Hatfield and Milgrom (2004)). Applying Theorem 21 to the economy with $(n+1)$ firms, there exists a Walrasian equilibrium with linear price vector $p$. At this price, the additional firm obtains the bundle $x-y$ since its marginal utility dominates all other firms' for any unit up to this bundle, and vanishes beyond this bundle. This implies that the remaining firms ask for $y$ at price $p$, or equivalently, that $y$ belongs to the demand set of $n$ firms' market valuation at price $p$. Concavity is then obtained as in the proof of Theorem 11. ■

THEOREM 22 (AGGREGATION) *If individual firms have unitary-substitutes valuations, then the market valuation $v$ is a unitary-substitute valuation.*

*Proof.* Let $\{v_i\}_{1\leq i\leq n}$ denote the family of individual valuations and $v$ denote the market valuation, defined by

$$v(x) = \max\left\{\sum_i v_i(x_i) : \sum x_i = x, x_i \in \mathbb{N}\right\}.$$

From Theorem 14, we will prove the result if we show that $v$ is a concave class-substitute valuation that satisfies the consecutive-integer property. Corollary 2 states that $v$ is concave. From Corollary 1, $v$ is a class-substitute valuation. It thus remains to show that $v$ satisfies the consecutive-integer property. For any price $p$, the demand set of $v$ is the solution of

$$\max_x\{v(x)-px\} = \max_x\left\{\max\left\{\sum_i v_i(x_i) : \sum_i x_i = x\right\} - px\right\} = \sum_i \max_{x_i} v_i(x_i) - px_i.$$

Therefore, $D(p) = \sum_i D_i(p)$. In particular, the projection of $D$ on the $k^{th}$ coordinate satisfies $D_k = \sum_i D_{i,k}$. The sets $D_{i,k}$ consist of consecutive integers by Theorem 11, implying that $D_k$ also consists of consecutive integers. ■

Finally, we examine the connections between unitary-substitute valuations and the structure of the core of the associated cooperative game. The setting considered in this section is the same as Ausubel and Milgrom (2002), but with the multi-unit formulation replacing their binary formulation. We first recall the definitions of coalitional value functions, the core, and Vickrey payoffs.

Suppose that, in addition to bidders, there exists a single owner (labeled "0") of all units of all goods, who has zero utility for her endowment.

DEFINITION 12 *The coalitional value function of a set $S$ of bidders is*

$$w(S) = \max\left\{\sum_{i\in S} v_i(x_i) : \sum x_i \in \mathcal{X}\right\}$$

*if* $0 \in S$, *and* $w(S) = 0$ *otherwise.*

Denote $L$ the set consisting of all bidders and the owner of the good.

DEFINITION 13 *The core of the economy is the set*

$$Core(L, w) = \left\{\pi : w(L) = \sum_{l\in L} \pi_l, w(S) \leq \sum_{l\in S} \pi_l \text{ for all } S \subset L\right\}.$$

DEFINITION 14 *The Vickrey payoff vector (the payoff at the dominant-strategy equilibrium of the generalized Vickrey auction) is*

$$\bar{\pi}_l = w(L) - w(L \setminus l)$$

*for* $l \in L \setminus 0$, *and*

$$\bar{\pi}_0 = w(L) - \sum_{l\in L\setminus 0} \bar{\pi}_l.$$

DEFINITION 15 *The coalitional value function $w$ is bidder-submodular if for all $l \in L \setminus 0$ and sets $S$ and $S'$ such that $0 \in S \subset S'$,*

$$w(S) - w(S \setminus l) \geq w(S') - w(S' \setminus l).$$

THEOREM 23 *Suppose that there are at least $2 + \max_k N_k$ bidders. If any bidder has a concave, class-substitute valuation that is not a unitary-substitute valuation, then there exist linear or unit-demand valuations for remaining bidders such that the coalitional value function is not bidder-submodular and the Vickrey payoff vector is not in the core.*

*Proof.* See the Appendix.

# 6 Walrasian Tâtonnement and Clock Auctions

This section analyzes auctions where goods are available in multiple units and prices are linear. The goods are summarized by a vector $\bar{x} \in \mathcal{X} = \mathbb{N}^K_{++}$. We propose a class of algorithms guaranteeing monotonic convergence of the auction to a pseudo-equilibrium whenever bidders have class-substitute valuations. Combining that with the the results of Section 5 leads to the conclusion that if bidders have unitary-substitute valuations, the auctions converge to a Walrasian equilibrium.

For the present analysis, we define a clock auction as a price adjustment process in which the path of prices is monotonic—either increasing or decreasing. In practice this monotonicity and other features, especially activity rules for bidders (see Milgrom (2000)), differentiate clock auctions from a Walrasian tâtonnement. In order to understand the relation between substitute valuations and clock auctions, it is useful to start the analysis with Walrasian tâtonnement, and then draw implications of the corresponding results in terms of clock auctions. We first derive general results for an idealized economy with prices changing continuously through time and where bidders submit their entire demand set. We then show how the results apply to economies with a discrete price and time, and where bidders only ask for one bundle at each stage of the auction.

## 6.1 Continuous time and price

There are $n$ bidders with valuations $\{v_1, ..., v_n\}$ and a corresponding market valuation $v$. At any time $t$, a price vector $p(t)$ is posted. We limit attention to linear pricing. Each bidder submits his demand set,[12] resulting in an aggregate demand $x(t)$ in the demand set $D(p(t))$ of $v$.

The goal of this section is to construct algorithms that are monotonic and converge to a pseudo-equilibrium. We focus on algorithms for which initial price is low, then increases and converges to the smallest pseudo-equilibrium price $\underline{p}$. Reverse algorithms, where price decreases and converges to the largest pseudo-equilibrium price can be constructed in a similar way.

We have seen that pseudo-equilibrium prices are the minimizers of the convex function $f : p \to \pi(p) + \bar{x}p$. The most natural algorithms to find such minimizers are steepest-descent algorithms. At any time, price changes are determined by the gradient of $f$ whenever $f$ is differentiable, and by the vector of smallest norm of its subdifferential otherwise.[13] Such algorithms are particular Walrasian tâtonnement, as they adjust prices

[12] Later, we consider the case where the bidder only asks for a single bundle in his demand set.

[13] By definition, the subdifferential $\partial f(p)$ at $p$ of a convex function $f$ is the set of vectors $x$ such that $f(q) - f(p) \geq x(q - p)$ for all $q$. The subdifferential is always a nonempty convex set, and coincides with

so as to eliminate excess demand. Moreover, they follow the steepest descent and are therefore particularly efficient. For any price vector $p$, we denote by $z(p)$ the point of smallest norm in the opposite of the differential of $f$ at $p$. When $f$ is differentiable, $z$ corresponds to the excess (aggregate) demand $D(p) - \bar{x}$. In general, $z$ is the vector of smallest norm in the convex hull of the set of excess demand. Intuitively, an algorithm is a procedure that determines the evolution of price through time as a function of excess demand $D(p) - \bar{x}$ and of time itself. In continuous time, an algorithm should thus be defined by a function $F$ such that

$$\dot{p}(t) = F(D - \bar{x}, t).$$

However, this definition is not appropriate in our setting, because $F$ need not be continuous. The steepest-descent algorithm, in particular, follows discontinuous changes of direction. In general, we will say that an algorithm is *well-defined* if, from any initial price, it generates a unique trajectory in the price space. The previous considerations lead to the following definition.

DEFINITION 16 *A continuous, correspondence-based, steepest descent algorithm is defined by*

$$\dot{p}_r(t) = \alpha(t, p(t))z(p(t)), \tag{3}$$

*where the subscript $r$ denotes right derivative, the function $\alpha : (t, p) \to \alpha(t, p)$ is real-valued and continuous, and takes values in $[\underline{\alpha}, \bar{\alpha}]$ for some positive constants $\underline{\alpha} < \bar{\alpha}$.*

Using right derivatives addresses discontinuities of $z(p)$. The lower bound $\underline{\alpha}$ ensures that the algorithm does not stall at a suboptimal price, and the upper bound ensures that that the equation is integrable. The following theorem states that, starting from any sufficiently low price, the algorithm is well defined, monotonic and converges to the lowest pseudo-equilibrium price, $\underline{p}$. Let $\mathcal{L} = \{p : p \leq \underline{p} \text{ and } z(p) \geq 0\}$.

THEOREM 24 *Any continuous, correspondence-based, steepest-descent algorithm is well defined. Suppose that bidders have class-substitutes valuations. For any such algorithm, if $p(0) \in \mathcal{L}$, then $p(t) \in \mathcal{L}$ for all $t$, $p(t)$ is increasing and converges to $\underline{p}$ in finite time.*

The proof is in the Appendix. Theorem 24 implies that, when bidders have class-substitute valuations, any steepest-descent algorithm starting from low prices is an ascending clock auction and converges to the smallest pseudo-equilibrium price. This result is important

$f$'s gradient whenever it is differentiable.

in practice, and can be reformulated as follows. We define a continuous, correspondence-based, ascending clock auction as a continuous, correspondence-based steepest-descent algorithm, except that (3) is replaced by $\dot{p}_r(t) = \max\{\alpha(t, p(t))z(p(t)), 0\}$.

COROLLARY 3 *If bidders have class-substitute valuations, any continuous, correspondence-based clock auction starting from a price in $\mathcal{L}$ converges to the smallest pseudo-equilibrium price.*

In particular, if goods are class substitutes, ascending clock auctions will find an equilibrium whenever there exists one. By contrast, it is easy to build examples of valuations violating class-substitutes such that a Walrasian equilibrium exists but ascending clock auctions fail to find it.

Our result extends Ausubel (2005), which proposes a similar algorithm when goods are only available in one unit each. When goods are available in multiple units, the strategy in earlier research has typically been to consider each unit as a distinct good. This implies that the price of each unit evolves separately, which increases the complexity of the auction proportionally to the number of units, compared to our algorithm. Moreover, previous auction algorithms assumed what we earlier defined as unitary-substitute valuations. Theorem 24 and its corollary show that class-substitute valuations are in fact enough to guarantee monotonicity and convergence to a pseudo-equilibrium.

In theory, $\mathcal{L}$ depends on bidder valuations, which may see problematic, given that the auctioneer does know them. In practice, the assumption $p_0 \in \mathcal{L}$ means that the clock auction can start at any price low enough to guarantee that there is excess demand in all goods. This obviously includes zero initial prices, but also "reasonably low" reserve prices.

## 6.2 Discrete time and price

We now consider the case in which prices evolve on a grid. In such setting, it is natural to consider discrete-time models, as nothing happens in any interval of time during which prices remain constant. We thus consider a discrete time scale, where prices are adjusted at each period.[14] The first goal of this section is to show that the results derived in the previous section are approximately true, in the sense that trajectories obtained with dis-

[14]The lapse between two periods has no importance, and in fact could in principle vary during the auction, possibly stochastically.

cretized algorithms are very close to those generated by continuous algorithms, provided that the price grid is thin enough. The second goal of the section is to show that the algorithm still works if bidders only announce one desired bundle at each period, rather than their entire demand set, consistent with what is observed in practice.

A price grid is a lattice $\mathcal{P}_\eta = (\eta\mathbb{N})^K$, where $\eta$ is a small positive constant. A discrete algorithm generates a sequence of prices $\{p_t : t = 0, 1, \ldots\}$ in $\mathcal{P}_\eta$, whose evolution is determined by excess demand at any period. In a discrete setting, algorithms are always well-defined. A new issue is that price changes, which are restricted to a grid, may not be able to follow exactly the gradient $z$. In general, vector directions can be approximated up to the thinness of the grid, which can be arbitrarily small. The following lemma goes further by showing that, provided the grid is thin enough, even the *exact* direction is feasible. Following the previous section, we let $z(p)$ denote the vector of smallest norm in the convex hull of the excess (aggregate) demand set $D(p) - \bar{x}$. The proof of the lemma is in the Appendix.

Lemma 2 (Feasible directions of descent) *Suppose that the number of bidders is less than some constant $N > 0$, and that no bidder can demand more than overall supply $\bar{x}$. Then, for any grid $\mathcal{P}_\eta$, there exists $\alpha(\eta) > 0$ such that $\alpha(\eta)z(p) \in \mathcal{P}_\eta$ for all $p$ and all bidder valuations. Moreover, $\alpha$ can be chosen such that $\alpha(\eta) \to 0$ as $\eta \to 0$.*

In the rest of this section, we may therefore assume that the price grid is thin enough for price changes to exactly follow steepest-descent directions and be arbitrarily small. In order to stay exactly on the grid, we assume from now on that step sizes are integer multiples of $\alpha(\eta)$. Another issue is that discrete algorithms sometimes "overshoot", meaning that the discrete price sequence crosses a region boundary while the continuous algorithm follows the boundary, causing the discrete algorithm to enter regions where some goods are in excess supply, and where the algorithm gradient $z$, which is not continuous, takes very different values from the gradient of the continuous algorithm. The purpose of the following lemma is to show that such overshoots are not important, as nearby trajectories of any discrete steepest-descent algorithm stay close to each other. Let $\{p(t)\}_{t\in\mathbb{N}}$ and $\{q(t)\}_{t\in\mathbb{N}}$ denote the trajectories generated by a given steepest-descent algorithm, starting from respective initial prices $p(0)$ and $q(0)$.

Lemma 3 (Nearness Lemma) *Suppose that the number of bidders is less than some constant $N > 0$, that no bidder can demand more than aggregate supply $\bar{x}$, and that there exists a vector $M \in \mathbb{R}_+^K$ such that bidders demand none of good $i$ whenever $p_i > M_i$.*

*Then, for any $\varepsilon > 0$, there exists $\eta > 0$ and $\bar{\alpha} > 0$ such that for all grids thinner than $\eta$ and step size less than $\bar{\alpha}$, $\|p(0) - q(0)\| < \varepsilon$ implies $\|p(t) - q(t)\| < \varepsilon$ for all periods and all bidder valuations.*

*Proof.* See the Appendix.

The nearness lemma states that overshooting is not going to affect the trajectory by more than some arbitrarily small constant. This leads to the following theorem, which states that the discrete algorithm essentially follows the continuous one. For any price $p_0$, denote by $T(p_0) = \{p(t) : t \in \mathbb{R}_+, p(0) = p_0\}$ the trajectory generated by the continuous, correspondence-based steepest-descent algorithm of the previous section, and let $T(p_0, \varepsilon) = \cup_{p \in T(p_0)} B(p, \varepsilon)$ denote the tube[15] of radius $\varepsilon$ around $T(p_0)$.

Theorem 25 (Discrete Steepest-Descent Algorithm) *For any $\varepsilon > 0$, there exists $\eta > 0$ and $\bar{\alpha} > 0$ such that for any grid thinner than $\eta$, step size less than $\bar{\alpha}$, and initial price $p_0$, the trajectory generated by the discrete steepest descent algorithm is contained in $T(p_0, \varepsilon)$.*

*Proof.* Starting in the same region, trajectories of both algorithms are undistinguishable, since they follow the same direction. Let $t_0$ denote the first time that the trajectory $T$ of the discrete algorithm overshoots, causing the two paths to have distinct vectors. Let $\epsilon > 0$ be a positive constant (to be chosen later), and denote by $p_{t_0}$ the price of the discrete algorithm, and by $q_{t_0}$ a price on $T(p_0)$ such that $\|p_{t_0} - q_{t_0}\| < \epsilon$. Such a price exists if the step size $\bar{\alpha}(\epsilon)$, which gives an upper bound on the overshoot, is small enough. Let $T_1$ denote the trajectory that the discretized algorithm would generate if it were starting from $q_{t_0}$. By construction $T_1$ coincides with $T(p_0)$ until there is a second overshoot. By the nearness lemma, $T$ and $T_1$ are within $\varepsilon$ from each other. Therefore, when $T_1$ overshoots, at time $t_1$, there is a price $q_{t_1}$ of $T(p_0)$ such that $\|p(t_1) - q_{t_1}\| < 2\epsilon$. Iterating the process, we thus prove that, up to the $k^{th}$ overshoot, we have $T \subset T(p_0, k\epsilon)$ when $T$ is truncated at $t = t_k$. The number of overshoots is bounded above by the number $R$ of regions (since any region is visited at most once by the continuous algorithm, see proof of Theorem 24). Therefore, the result obtains by setting $\epsilon = \varepsilon / R$. ■

As a by-product of Theorem 25, we can get rid at little cost of the assumption that bidders submit their entire demand set. Bidder valuations can be seen as vectors of the finite-dimensional space $\mathcal{V} = \mathbb{R}^{\bar{x}}$. A property of an algorithm holds "for almost all economies"

[15] $B(p, \varepsilon)$ is the open ball centered at $p$ and radius $\varepsilon$.

if it holds for all bidder valuations, except for a subset of Lebesgue measure zero of $\mathcal{V}^n$, where $n$ is the number of bidders. A singleton-based steepest-descent algorithm, is the same as the discrete steepest-descent algorithm, except that bidders ask only one bundle at each period. Concretely, this means that instead of using the vector of smallest norm of the excess demand set, the algorithm may follow any vector of that set. The following result shows that this information loss does not affect Theorem 25 except possibly on a set of economies with Lebesgue measure zero.

Theorem 26 (Singleton-Based Algorithm) *Under the assumptions of Theorem 25, let $p_0$ be any initial price of the algorithm. The trajectory of a singleton-based steepest-descent algorithm is contained in $T(p_0, \varepsilon)$ for almost all economies.*

The proof is based on the following proposition.

Proposition 4 *For all $(v_1, \dots, v_n) \in \mathcal{V}^n$, the demand correspondence $p \to D(p)$ is single-valued almost everywhere in $\mathcal{P}$ with respect to the Lebesgue measure on this set.*

*Proof.* We suppose first that there is a unique bidder. For any two bundles $x$ and $x'$, the subset $P(x, x')$ of $\mathcal{P}$ defined by $P(x, x') = \{p : p(x - x') = v(x) - v(x')\}$, is the intersection of a hyperplane with the positive orthant $\mathcal{P}$, and has therefore zero Lebesgue measure. Since the number of possible bundles is finite, the set

$$Q = \bigcup_{x \neq x'} P(x, x'),$$

which contains all prices at which the bidder's demand is multi-valued, also has zero Lebesgue measure. For a countable (in particular, finite) number of bidders, the set of prices where aggregate demand is multi-valued is contained in $Q^a = \cup Q_i$, which has zero Lebesgue measure. ■

Proposition 4 implies that the set of economies such that $Q^a \cap \mathcal{P}_\eta \neq \emptyset$ has Lebesgue measure zero. Therefore, singleton-based and correspondence-based algorithms are identical in almost all economies.

In practice, the auctioneer does not know bidder valuations. Theorem 26 implies that for any belief that is absolutely continuous with respect to the Lebesgue measure, the algorithm is arbitrarily close to the continuous, correspondence-based steepest descent algorithm of the ideal economy. In particular, the algorithm completely ignores bidders' indifference sets. This feature contrasts with Gul and Stacchetti (2000), whose algorithm gives much importance to indifference sets.

# 7 Pseudo-equilibrium as Approximate Equilibrium

In this section we derive some key properties of pseudo-equilibria, showing the importance of the concept as an approximation of Walrasian equilibrium. For any good, bidder and price, define

$$g_{i,k}(p) = \max\{d : \exists(x,y) \in D_i(p)^2 \text{ and } y_k - x_k = d \text{ and } \nexists z \in D_i(p) : z_k \in (x_k, y_k)\}.$$

Thus, $g_{i,k}(p)$ is the maximal gap in the bidder $i$'s demand for good $k$ at price $p$. The maximal gap in aggregate demand for good $k$ at price $p$ is defined similarly and denoted by $g_k(p)$. Finally, denote by $\gamma_k(p) = \max_i g_{i,k}(p)$ the largest gap for good $k$ for any bidder $i$.

LEMMA 4 *For all $k$ and $p$,*

$$g_k(p) \leq \gamma_k(p).$$

*Proof.* Let $D_k(p)$ denote the projection of aggregate demand on good $k$, and $\bar{d}_k(p)$ denote its maximum. We need to show that for all $d \in D_k(p)$, such that $d < \bar{d}_k(p)$ there exists $d' \in D_k(p)$ such that $0 < d' - d \leq \gamma_k(p)$. By construction, there exists $x$ and $y$ in $D(p)$ such that $x_k = \sum_i x_{i,k} = d$ and $y_k = \sum_i y_{i,k} > d$. This implies that there exists a firm $i$ such that $y_{i,k} > x_{i,k}$. By assumption, this implies that there exists a bundle $z_i$ in $D_i(p)$ such that $x_{i,k} < z_{i,k} \leq x_{i,k} + g_{i,k}(p) \leq x_{i,k} + \gamma_k(p)$. The bundle $z = \sum_{j \neq i} x_j + z_i$ is also in $D(p)$, and satisfies $0 < z_k - x_k \leq \gamma_k(p)$. ■

Now let $\gamma_k = \max_p\{\gamma_k(p)\}$. This the maximum gap in demand for good $k$, over all bidders and prices.

THEOREM 27 *For any pseudo-equilibrium price $p$ and good $k$, there exists a bundle $y \in D(p)$ such that*

$$|y_k - \bar{x}_k| \leq \gamma_k.$$

*Proof.* By definition, a pseudo-equilibrium price $p$ is such that $\bar{x}$ is in the convex hull of $D(p)$. In particular, there exist bundles $x, y$ in $D(p)$ such that $x_k \leq \bar{x}_k \leq y_k$. The result then follows from Lemma 4. ■

Thus we have a bound on excess demand or supply for good $k$. The next result uses the following definition. For any positive, $K$-dimensional vector $g$ and any bundle $x$, let $R(x,g) = \{z : \forall k \ |z_k - x_k| \leq g_k\}$ denote the hyperrectangle centered at $x$ and with span $2g_k$ along the $k^{th}$ coordinate.

Theorem 28 *If $K = 2$ and bidder $i$ has a concave valuation, then*

$$Co(D_i(p)) \subset \bigcup_{x \in D_i(p)} R(x, g_i(p))$$

*Proof.* Let $m \in Q = Co(D_i(p))$. If $m$ has integer coordinates, valuation concavity implies that it belongs to $D_i(p)$. Suppose otherwise. There exists an integral vector $(a, b) \in \mathbb{N}^2$ such that $m \in S = [a, a+1] \times [b, b+1]$. If any of the 4 integral vectors defining $S$ is in $D_i(p)$, the claim is true. Otherwise, $Q$ must cross $S$ on two sides. This, along with convexity of $Q$, implies that $Q \cap L$ is nonempty and contained in a slab of thickness strictly less than 1, where $L$ is one of the four hyperspaces tangent to $S$. Suppose without loss of generality that $L = \{x : x_2 \leq b\}$. Consider the set $\Lambda = D_i(p) \cap L$. If $m \notin \bigcup_{x \in \Lambda} R(x, g_i(p))$, this implies that projection of $D_i(p)$ on one of the two goods (depending on the orientation of the slab) does not contain any element between $m_k$ and $m_k + \varepsilon g_{i,k}(p)$ where $|\varepsilon| = 1$, which contradicts the definition of $g_{i,k}(p)$. ∎

Conjecture: the result may be true for all $K$ if bidders have class substitute valuations. Our goal is to show that, with enough control on individual demand functions, one can bound the excess demand (or supply), over all goods at the same time, at any pseudo-equilibrium. Thus, consider a positive vector $g \in \mathbb{R}^K$ such that

$$Co(D_i(p)) \subset \bigcup_{x \in D_i(p)} R(x, g)$$

for all $i$ and $p$. $g$ represents some notion of coarseness of individual demands. The case in which $g$ is a vector of ones implies the consecutive-integer property, for which there is no "hole" in the demand sets, and pseudo-equilibrium prices clear the market.[16] The general case is covered by the following theorem.

Theorem 29 *If $p$ is a pseudo-equilibrium price, there exists a bundle $y \in D(p)$ such that*

$$|y_k - \bar{x}_k| \leq \min\{K, n\} g_k$$

*for all $k$.*

*Proof.* First suppose that $n \leq K$. By definition of a pseudo-equilibrium, $\bar{x} \in Co(D(p)) = Co(\sum_i D_i(p)) = \sum_i Co(D_i(p))$. There exist $z_i \in Co(D_i(p))$ such that $\bar{x} = \sum z_i$. By assumption, there exists $y_i \in D_i(p)$ such that $|y_{i,k} - z_{i,k}| \leq g_k$ for all $k$. Letting $y = \sum y_i$, we have $y \in D(p)$ and $|\bar{x}_k - y_k| \leq n g_k$. Now suppose that $K < n$. Since $\bar{x} \in \sum_i Co(D_i(p))$,

[16] In that case, the result is actually sharper than the statement of Theorem 29.

the Shapley-Folkman theorem implies that there exist bundles $z_i \in Co(D_i(p))$ such that $\bar{x} = \sum z_i$ and $z_i \in D_i(p)$ except for $i$ in some index subset $J$ of cardinal weakly less than $K$. For $i \in J$, there exists, by assumption, a bundle $y_i \in D_i(p)$ such that $|y_{i,k} - z_{i,k}| \leq g_k$ for all $k$. Letting $y_i = z_i$ for $i \notin J$, and letting $y = \sum_i y_i$, we have $y \in D(p)$ and $|y_k - \bar{x}_k| \leq K g_k$ for all $k$. ■

We conclude this section with two additional properties of pseudo-equilibria.

PROPOSITION 5 *Either the economy has a Walrasian equilibrium, or the set of pseudo-equilibrium price vectors has measure zero.*

*Proof.* If there does not exist any Walrasian equilibrium, the subdifferential of $f : p \to \pi(p) - p\bar{x}$ must be multi-valued at any pseudo-equilibrium price vector. Since $f$ is convex, the set of singular points has measure zero, which proves the result. ■

PROPOSITION 6 *Suppose that bidders have concave valuations. If $p$ is a pseudo-equilibrium price vector at which at most one bidder has a multi-valued demand, then $p$ is an equilibrium.*

*Proof.* The aggregate excess demand set at $p$ is a translation of the singled-out bidder's optimal demand at $p$. By concavity of that bidder's valuation, the set must contain all integer bundles in its convex hull, including zero. Thus, the supply vector is in the aggregated demand set. ■

# 8 Divisible Goods

For the case of divisible goods, the notions of unitary and class substitutes need to be replaced. We instead use the concepts of *linear* and *nonlinear* substitutes. As the name suggests, linear substitutes only considers linear price vectors and thus constitutes the natural extension of class substitutes to economies with divisible goods. Concave nonlinear-substitute valuations possess many properties than one would expect from the extension of the unitary substitutes concept, as shown in this section.

DEFINITION 17 *$v$ is a linear-substitute valuation if whenever $p_j \leq p'_j$, $p_k = p'_k$ for all $k \neq j$, and $x \in D(p)$, there exists $x' \in D(p')$ such that $x'_k \geq x_k$ for all $k \neq j$.*

In the discrete case with individual item pricing, a rational consumer who buys $k$ units of some type of good always buys the cheapest $k$ units. Therefore, one way to describe individual item pricing is to specify that the cost of acquiring goods is a convex function of the number of goods acquired from each class and is additive across classes of goods. Higher prices mean that the marginal cost of acquiring additional units is higher. This characterization of the cost of acquiring goods and the corresponding representation of higher prices can be applied directly to the continuous case. That is the approach we adopt in this section.

Let $\mathcal{C}_1$ denote the space of continuously differentiable, convex functions from $\mathbb{R}_+$ to $\mathbb{R}$ which vanish at 0, and $\mathcal{C} = \mathcal{C}_1^K$. We endow $\mathcal{C}$ with the following partial order: $C \preceq \hat{C}$ if for all $k$, $c_k \leq \hat{c}_k$, where $c_k$ and $\hat{c}_k$ are the derivatives of $C_k$ and $\hat{C}_k$, respectively. With this order, $\mathcal{C}$ is a lattice, where for any $C$ and $\hat{C}$, the meet and the join satisfy, for all $k$ and $x_k \geq 0$, $(C \vee \hat{C})'_k(x_k) = \max\{c_k(x_k), \hat{c}_k(x_k\}$ and $(C \wedge \hat{C})'_k(x_k) = \min\{c_k(x_k), \hat{c}_k(x_k)\}$, respectively.[17] We extend the domain of any dual profit function $\pi$ from linear prices to $\mathcal{C}$ and denote $\bar{\pi}$ its extension:

$$\bar{\pi}(C) = \max_x \{v(x) - C(x)\},$$

where $C(x) = \sum_k C_k(x_k)$.

Definition 18 *$v$ is nonlinear-substitute valuation if whenever $C_j \leq C'_j$, $C_k = \hat{C}_k$ for all $k \neq j$, and $x \in D(C)$, there exists $x' \in D(\hat{C})$ such that $x'_k \geq x_k$ for all $k \neq j$.*

For the discrete case, we have seen that there are several properties distinguishing class substitutes and unitary substitutes, so there is scope for judgment in creating the analogue of unitary substitutes in the continuous case. For example, one could impose that the extended concept satisfy the law of aggregate demand. That would require that a dominant diagonal property hold for the matrix $[\partial x_i/\partial p_j]$ of partial derivatives of demand. The concept that we study below does not satisfy the law of aggregate demand.

In place of unitary substitutes, we study the concept of *concave, nonlinear-substitute* valuations. This definition preserves properties distinguishing unitary substitutes from class substitutes in the discrete setting, including robustness of the substitutes property with respect to nonlinear price changes and existence of Walrasian equilibria. Moreover, we find below that these valuations are characterized by dual submodularity on the domain of

[17] As can be easily checked, the marginal costs of $(C \wedge \hat{C})_k$ and $(C \vee \hat{C})_k$ are continuous and nondecreasing for all $k$, and constructed cost functions both vanish at 0, so that $C \wedge \hat{C}$ and $C \vee \hat{C}$ belong to $\mathcal{C}$.

nonlinear prices, which was also the characterization of unitary substitutes in the discrete case. We find further that, given concavity, the linear-substitute and nonlinear-substitute properties are equivalent. Therefore, our divisible-good extensions of the two concepts coincide in the case of concave valuations. The next two theorems develop all of these relationships.

Theorem 30 (Dual Submodularity) *If $v$ is a concave linear-substitute valuation, then $\bar{\pi}$ is submodular on $\mathcal{C}$.*

*Proof.* We replicate the proof of Theorem 14. We use a modified version of Proposition 1, whose proof is in the Appendix.

Proposition 7 *Suppose that $v$ is a concave linear-substitute valuation and let $C \in \mathcal{C}$. Then,*

$$\max_x \min_p \{\pi(p) + px - C(x)\} = \min_p \max_x \{\pi(p) + px - C(x)\}.$$

Given this result, we have[18]

$$\bar{\pi}(C) = \min_p \left\{ \pi(p) + \max_x \{px - C(x)\} \right\}.$$

The inner maximum equals

$$\sum_k \int_0^\infty (p_k - c_k(z_k))_+ dz_k.$$

Thus

$$\bar{\pi}(C) = \min_p \left\{ \pi(p) + \sum_k \int_0^\infty (p_k - c(z_k))_+ dz_k. \right\}$$

We now show that the function $h : (p, C) \to h(p, C) = \int_0^\infty (p - c(z))_+ dz$ is submodular on $\mathbb{R}_+ \times \mathcal{C}_1$. For $q < r$, $h(r, C) - h(q, C)$ is the area of the region $\{(z, p) : p \in [q, r] \text{ and } C(z) \leq p\}$, which is also equal to $\int_q^r z(p, C)dp$, where $z(p, C) = \sup\{z : c(z) \leq p\}$. Since $z(p, C)$ is nonincreasing in $C$ for all $p$, so is $h(r, C) - h(q, C)$, which proves submodularity of $h$. Linear substitutes implies that $\pi$ is submodular in $p$. Therefore, $\bar{\pi}$ is the minimum over $p \in \mathcal{P} = \mathbb{R}_+^K$ of an objective function that is submodular on $\mathcal{P} \times \mathcal{C}$. A Topkis theorem then implies that $\pi$ is submodular on $\mathcal{C}$. ■

Theorem 30 allows us to prove the equivalence of three candidate definitions for the divisible-good extension of unitary substitutes.

[18] See the proof of Theorem 14 for intermediary steps.

Theorem 31 *Suppose that $v$ is concave. Then the three following statements are equivalent.*

*(i) $v$ is a linear-substitute valuation.*

*(ii) $v$ is a nonlinear-substitute valuation.*

*(iii) $\pi$ is submodular on $\mathcal{C}$.*

*Proof.* Clearly, (ii) implies (i). From Theorem 30, (i) implies (iii). To conclude the proof, we show that (iii) implies (ii). We adapt the proof of Ausubel and Milgrom (2002), Theorem 10. We fix a direction of price increase for some good, and show that along this direction, the demand for any other good is nondecreasing. Fix goods $j \neq k$ and a direction of increase $\delta$ (i.e. $\delta$ is nondecreasing convex and vanishes at 0) for good $j$. Consider the function

$$\pi|_2(\lambda, \mu) = \max_x \{v(x) - C(x) - \lambda x_k - \mu\delta(x_j)\}$$

defined on $\mathbb{R}^2_+$. Since $\pi$ is submodular, so is $\pi|_2$. $\pi|_2$ is convex as the pointwise maximum of a family of functions that are affine in $(\lambda, \mu)$. In particular, $\partial\pi|_2/\partial\lambda$ exists almost everywhere. By an envelope theorem[19] $\partial\pi|_2/\partial\lambda$ exists everywhere that $x_k(\lambda, \mu)$ is a singleton and at those prices, $\partial\pi|_2/\partial\lambda = -x_k(\lambda, \mu)$. Submodularity of $\pi|_2$ implies that $\partial\pi|_2/\partial\lambda(\lambda, \mu)$ is nonincreasing in $\mu$ or, equivalently, that $x_k$ is nondecreasing in $\mu$. ■

Theorems 30 and 31 have an important consequence: concave nonlinear-substitute valuations are stable under perturbation by any concave modular function. Thus comparative statics results are robust with respect to such perturbations, as stated in the following theorem.

Theorem 32 *If $v$ is a concave nonlinear-substitute valuation, then $v + f$ is a concave nonlinear-substitute valuation for all $f$ modular and concave.*

*Proof.* Suppose that $v$ is a concave nonlinear-substitute valuation. Then, $v+f$ is concave whenever $f$ is concave. By Theorem 31, it remains to show that $v+f$ is a linear-substitute valuation. Let

$$x^f(p) = \arg\max_x \{v(x) + f(x) - px\}.$$

Without loss of generality, we can assume that $f_i(0) = 0$ for all terms of $f$. Let $C(x, p) = px - f(x)$. Since $f$ is modular and concave, $C$ is modular and for each $i$, $C_i$ is convex and

[19] Milgrom and Segal (2002), Corollary 4.

vanishes at 0. Therefore, $C$ belongs to $\mathcal{C}$. Moreover, increasing $p_k$ implies increasing $C_k$. Since $v$ is a nonlinear-substitute valuation and

$$x^f(p) = \arg\max_x \{v(x) - C(x,p)\},$$

$x_j^f(p)$ is nondecreasing in $C_k$, thus in $p_k$. ■

We now turn to the consequences of the substitutes properties in settings with multiple firms.

Repeating our previous analysis, we can show that the dual profit function of a valuation $v$ is submodular over linear prices if and only if $v$ is a linear-substitute valuation, and that linear-substitute valuations are closed under aggregation. With divisible goods, concavity is also closed under aggregation: the maximization

$$v(x) = \max_x \sum_i v_i(x_i)$$

subject to $\sum x_i \leq x$ has a concave objective function and a convex constraint function, so $v$ is concave[20] in the constraint bound $x$. This shows the following result, which extends Theorem 22.

Theorem 33 *Concave nonlinear-substitute valuations are closed under aggregation.*

*Proof.* The above discussion shows that concave linear-substitute valuations are closed under aggregation. This, along with Theorem 31, implies that the same is true of concave nonlinear-substitute valuations. ■

With divisible goods, concavity is a sufficient condition for the existence of a Walrasian equilibrium. If, in addition, firms have nonlinear-substitute valuations, then the Vickrey outcome is in the core.

Theorem 34 *If all bidders have concave nonlinear-substitute valuations, the Vickrey outcome is in the core.*

*Proof.* From Theorem 7 of Ausubel and Milgrom (2002), it is enough to show that the coalitional value function is bidder submodular. Therefore, we need to show that $w(S \cup \{l\}) - w(S)$ is nonincreasing in $S$. Let $x$ denote the quantity of goods available. We have

$$w(S \cup \{l\}) - w(S) = \max_{y \leq x} \{v_l(x-y) + v_S(y) - v_S(x)\},$$

[20]See for example Luenberger (1969, p.216).

where $v_T(z)$ denote the optimal value of bundle $z$ for coalition $T$. Therefore, it is enough to show that $v_S(y) - v_S(x)$ is non increasing in $S$ or simply that $v_S(z)$ is submodular in $(-z, S)$. Concavity of $v_S$[21] implies that

$$v_S(z) = \min_p \{\pi_S(p) + pz\}, \tag{4}$$

where $\pi_S$ is the dual profit function of $v_S$ and is equal to $\sum_{i \in S} \pi_i(p)$. Since $\pi$ is submodular in $p$, the objective in (4) is submodular in $(-z, S, p)$. Hence, $v_S(z)$ is submodular in $(-z, S)$, as required. ■

We have focused so far on monotone comparative statics of the demand function. In the discrete case, we saw that unitary substitutes not only implied that $x_j$ is nondecreasing in the (possibly nonlinear) price of other goods, but also that $\sum_j x_j(\tilde{p})$ was nonincreasing in $\tilde{p}$, which is the discrete law of aggregate demand. This property is no longer required for several of the theorems pertaining to divisible goods, as Theorems 30 and 34 illustrate.

Might there be some analogue of the law of aggregate demand for divisible substitute goods? One problem is to determine the units in which such a law might be expressed. For example, if one unit of good $i$ represents a 10-ride train pass between two cities, while one unit of good $j$ is a one-way bus ticket between the same cities. One expects that, starting from prices where a consumer chooses the train pass, a large price increase in the train pass results in the consumer buying several bus tickets to replace the train pass, implying that the sum $x_i + x_j$ *increases* as $p_i$ increases, which violates the law of aggregate demand. One way to pose the problem without units is to ask whether there exist constants $a_i$ such that $\sum_i a_i x_i$ be nondecreasing in prices? In the previous example, a natural choice would be $a_i = 1$ and $a_j = 10$, given the relative similarity of a train trip and a bus trip. More generally, we say that a valuation $v$ satisfies the *generalized law of aggregate demand* (GLoAD) if there exist increasing functions $f_i$ such that

$$\sum_i f_i(x_i(C))$$

is nonincreasing in $C$. It satisfies the *law of aggregate demand* if one can take $f_i(x_i) = x_i$ for all $i$. The GLoAD seems so much more flexible than the law of aggregate demand that one is led to wonder whether it is satisfied by linear-substitute valuations, or at least concave nonlinear-substitute valuations. However, the following theorem and its corollary show that the GLoAD is equivalent to the law of aggregate demand up to a mere convex

[21] See discussion preceding Theorem 33.

re-scaling of goods. For the remaining of this section we assume that the cost functions are nondecreasing.[22] To simplify the exposition, let $\circ$ denote the component-wise composition of any function with any modular function: $(f \circ g)(x) = f(g_1(x_1), \ldots, g_K(x_K))$, for any function $f$ and modular function $g$. Clearly, $f \circ g$ is modular if $f$ is also modular. Restricted to the class of increasing modular functions, denote $f^{-1}$ the component-wise inverse for the composition: $f^{-1}(x) = \sum_k f_k^{-1}(x_k)$. For functions of one variable, these definitions coincide with the usual ones.

THEOREM 35 *Let $v$ be a concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some function $f$, and $g$ be an increasing, concave, modular function. Then $\tilde{v} = v \circ g$ is a concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for the modular function $\tilde{f} = f \circ g$.*

*Proof.* Since $v$ and $g$ are nondecreasing concave, so is $\tilde{v}$. Let $\tilde{C}$ be a convex price schedule, and $y(\tilde{C}) = \arg\max \tilde{v}(y) - \tilde{C}(y)$. We wish to show that $y_j$ is nondecreasing in $\tilde{C}_k$ for $j \neq k$, and that there exists an increasing modular function $\tilde{f}$ such that $\tilde{f}(y(\tilde{C}))$ is nonincreasing in $\tilde{C}$. The function $\gamma = g^{-1}$ is increasing, convex, and modular. By assumption, there exists a modular function $f$ such that $f(x(C))$ is nondecreasing in $C$, where $x(C)$ is the demand of $v$ at the convex price schedule $C$. Let $C = \tilde{C} \circ \gamma$. Since all components of $\gamma$ and $\tilde{C}$ are nondecreasing convex, so are the components of $C$. Increasing $\tilde{C}_k$ to $\tilde{C}'_k$ is equivalent to increasing $C_k$ to $C'_k = \tilde{C}'_k \circ \gamma_k$. Therefore, if $j \neq k$, $y_j(\tilde{C}) = \gamma_j(x_j(C))$ is nondecreasing when $\tilde{C}_k$ increases. Moreover, letting $\tilde{f} = f \circ g$, we have $\tilde{f}(y(\tilde{C})) = f(x(\tilde{C} \circ \gamma))$, which is nonincreasing in $\tilde{C}$. ■

COROLLARY 4 *Suppose that $v$ is a concave nonlinear-substitute valuation satisfying the generalized law of aggregate demand for some convex function $f$. Then, $\tilde{v} = v \circ f^{-1}$ satisfies the law of aggregate demand.*

Thus, the generalized law of aggregate demand corresponds to a quantitative rather than a qualitative relaxation of the law of aggregate demand. In fact, it is possible to construct a concave nonlinear-substitute valuation that does not satisfy *any* generalized law of aggregate demand. We provide a sketch of this counter-example below, which establishes a fundamental difference between the cases of discrete and divisible goods.

[22]This assumption if used in the proof of Theorem 35. We did not make this assumption earlier in order to prove Theorem 32, where we consider $C(x) = px - f(x)$ and $f$ may be increasing.

Counter-Example 2 *There exist concave nonlinear-substitute valuations that do not satisfy the generalized law of aggregate demand.*

*Proof.* [Sketch] We consider the case of two goods. Let $x < x'$ and $y < y'$ be positive numbers, and consider the bundles $A = (x, y)$, $B = (x', y)$, $C = (x, y')$, and $D = (x', y')$. If GLoAD held, there would exist some increasing functions $f$ and $g$ such that $f(x(p, q)) + g(y(p, q))$ is nonincreasing in $(p, q)$, where $(p, q)$ is the price vector of the two goods, and $(x(p, q), y(p, q))$ is the demand in the goods at that price. Suppose that at $B$ and $C$, a small increase in price $p$ reduces $x(p, q)$ by a very small amount and increases $y(p, q)$ by a very large amount (as in the ticket/pass example above). For GLoAD to hold, this means that we must have $f'(x')$ much larger than $g'(y)$ (looking at B), and $f'(x)$ much larger than $g'(y')$ (looking at C). Now suppose that $A$ and $D$, a small increase in price $q$ reduces $y(q)$ by a very small amount, and increases $x(q)$ by a very large amount. For GLoAD to hold, this means that we must have[23] $g'(y)$ much larger than $f'(x)$ (looking at A) and $g'(y')$ much larger than $f'(x')$ (looking at D). These two sets of conditions are clearly incompatible, which shows that GLoAD cannot hold. To conclude our counter-example, it remains to show that there exist concave nonlinear-substitute valuations satisfying the demand behavior described at points A,B,C,D. Demand variations is determined by the Hessian of the valuation at these points. It is easy to choose Hessian matrices that are negative definite, with negative cross derivatives and that satisfy the requirements. We show that it is possible to extend these Hessian matrices over the whole bundle space while keeping negative definiteness and negative cross derivatives, by superposition of several concave submodular functions. This defines a valuation (up to an affine term) that is submodular and concave. In two dimensions, submodularity implies the linear-substitute property. By Theorem 31, the constructed valuation is therefore a concave nonlinear-substitute valuation. ■

# 9 Conclusion

The substitutes concepts play a critical role in equilibrium theory, particularly for discrete economies. For discrete economies, the unitary substitutes conditions gives a necessary and sufficient conditions for the robust existence of equilibrium and class substitutes drive the monotonicity that is exploited by current auction algorithms. Unitary (respectively, nonlinear) substitutes is also the condition that determines whether the Vickrey out-

[23]The proof is easily adapted if $f$ and $g$ are not differentiable at these points.

come is in the core for economies with discrete (respectively, divisible) goods. A related concept–the law of aggregate demand–has been the informal justification for the wide adoption of activity rules in practical auctions. Among our findings is that the law of aggregate demand is precisely the additional property that converts a concave class substitute valuation to a unitary-substitute valuation when goods are discrete, but that this difference vanishes when goods are divisible.

Our findings are also related to the literature on comparative statics in optimization. Prior to our paper, "robust comparative statics" analysis has focused mainly on models with complementarities, understood roughly as supermodularity of the objective. We have shown that there are also robust comparative statics results for the case of nonlinear substitutes, because that property is preserved by the addition of concave, modular functions.

# 10 Appendix: Proofs

## 10.1 Section 4

PROOF OF LEMMA 1. Consider a bundle $x$ such that $x_k \leq N_k - 1$ and $x_j \leq N_j - 2$. Take any binary representant $\tilde{x}$ of $x$, and call $l$ and $m$ two units of good $j$ not in $\tilde{x}$, and $s$ a unit of good $k$ not in $\tilde{x}$. Since $\tilde{v}$ satisfies the gross-substitute property, the set

$$\{\tilde{v}(\tilde{x}+e_l+e_m)-\tilde{v}(x+e_l)-\tilde{v}(\tilde{x}+e_m), \tilde{v}(\tilde{x}+e_l+e_s)-\tilde{v}(\tilde{x}+e_l)-\tilde{v}(\tilde{x}+e_s), \tilde{v}(\tilde{x}+e_m+e_s)-\tilde{v}(\tilde{x}+e_m)-\tilde{v}(\tilde{x}+e_s)\}$$

has two or more maximizers. Symmetry of $\tilde{v}$ implies that the last two arguments of that quantity are equal, and therefore greater than or equal to the first one. That is, written in multi-unit form,

$$v(x + e_k + e_j) - v(x + e_k) - v(x + e_j) \geq v(x + 2e_j) - 2v(x + ej),$$

which, after simplification, concludes the proof of Lemma 1. ■

PROOF OF THEOREM 12. Suppose by contradiction that the law of aggregate demand is violated: there exist $k$, $p$ and $x$ such that for all $\varepsilon$ small enough, we have (i) $x \in D(p-\varepsilon e_k)$, and (ii) for all $y \in D(p + \varepsilon e_k)$, $\|y\|_1 > \|x\|_1$. Clearly, for any such $y$, we have $y_k < x_k$. Let $D_k = D_k(p)$, $\underline{d} = \min D_k$ and $\bar{d} = x_k = \max D_k$. By continuity, we have (i) $x \in D(p)$, (ii) there exists some $y \in D(p)$ such that $y_k < x_k$, and (iii) for all $y \in D$ such that $y_k = \underline{d}$,

$\|y\|_1 > \|x\|_1$.

For each $d \in D_k$, define $g(d) = \min\{\|y_{-k}\|_1 : y_k = d \text{ and } y \in D(p)\}$. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}$ denote the largest convex function such that $\gamma(d) \leq g(d)$ for all $d \in D_k$. As can be easily verified, $\gamma$ is well defined, and is piecewise affine: there exists a partition $\Delta = \{\delta_l\}_{l \in \Lambda}$ of $\mathbb{R}_+$ such that $\gamma$ is affine on $[\delta_l, \delta_{l+1}]$. Moreover, $\bar{d}$ and $\underline{d}$ are elements of $\Delta$: there exist $\underline{l}$ and $\bar{l}$ such that $\underline{d} = \delta_{\underline{l}}$ and $\bar{d} = \delta_{\bar{l}}$. For $l \in \{\underline{l}+1, \bar{l}\}$, denote $H(l)$ the hyperplane containing the two $(K-2)$-dimensional affine varieties

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_l) \text{ and } z_k = \delta_l\}$$

and

$$\{z \in \mathbb{R}^K : \|z_{-k}\|_1 = \gamma(\delta_{l-1}) \text{ and } z_k = \delta_{l-1}\}.$$

As the reader can verify, there exists a unique hyperplane containing these two affine varieties, so $H(l)$ is well defined. Moreover, $H(l)$ lies below $D(p)$ and contains at least two elements $z$ and $y$ of $D(p)$ such that $z_k = \delta_l$ and $y_k = \delta_{l-1}$.

We claim that there exists $l \in \{\underline{l}+1, \bar{l}\}$ such that $\gamma(\delta_{l-1}) - \gamma(\delta_l) > \delta_l - \delta_{l-1}$. Suppose that the contrary holds. Then, $\gamma(\underline{d}) - \gamma(\bar{d}) \leq \bar{d} - \underline{d} = x_k - \underline{d}$. But then, there exists $y$ in $D(p)$ such that $y_k = \underline{d}$ and $\|y_{-k}\|_1 = \gamma(\underline{d})$, implying that $\|x\|_1 = x_k + \gamma(\bar{d}) \geq \underline{d} + \gamma(\underline{d}) = \|y\|_1$, which contradicts the hypothesized violation of the law of aggregate demand.

Consider an index $l$ as in the previous paragraph, and modify $p$ slightly so that the demand set becomes $D(p) \cap H(l)$. As can be easily verified, the price vector can be further modified so that the remaining bundles in the demand set are aligned on a unique straight line and, for the new price $\bar{p}$, there still exist $z$ and $y$ in $D(\bar{p})$ such that $z_k > y_k$ and $\|z\|_1 < \|y\|_1$. There are two cases: either there are two indices $i$ and $j$ such that $y_i > z_i$ and $y_j > z_j$, or there exists an index $i$ such that $y_i - x_i > x_k - y_k$. Since optimal bundles are aligned, the same properties hold for the extremities bundles of the segment containing $D(\bar{p})$, so we assume without loss of generality that $z$ and $y$ are these extreme bundles. In the first case, increasing $p_i$ slightly violates the class-substitute property, as the optimal quantity of good $j$ also decreases. In the second case, the convex-demand property is violated: the set $D_i(\bar{p})$ contains a hole between $z_i$ and $y_i$. ∎

Proof of Proposition 1 Trivially,

$$\max_x \min_p \{\pi(p) + px - (\tilde{p}, x)\} \leq \min_p \max_x \{\pi(p) + px - (\tilde{p}, x)\}. \tag{5}$$

We need to prove that the reverse inequality also holds. We fix $\tilde{p}$ throughout the proof. Consider a price $p$ solving $\min_p \max_x\{\pi(p)+px-(\tilde{p},x)\}$. Let $N(p) = \arg\max_x\{px-(\tilde{p},x)\}$. By inspection, $N(p)$ is a hyper-rectangle: there exist two bundles $r$ and $R$ with $r \leq R$ such that $N(p) = \{z \in \mathbb{Z}^K : r \leq z \leq R\}$.

Suppose that there exists a bundle $x$ in $N(p) \cap D(p)$. Then, the right-hand side of (5) equals

$$\pi(p) + px - (\tilde{p}, x) = v(x) - (\tilde{p}, x),$$

where the last equality comes from the fact that $x$ belongs to $D(p)$. Now consider any linear price vector $q$. We have $\pi(q)+qx-(\tilde{p},x) \geq v(x)-(\tilde{p},x)$, by definition of $\pi(q)$. This last inequality implies that the left-hand side of (5) is actually greater than or equal to its right-hand side. Therefore, we will have concluded the proof if we show that $N(p) \cap D(p)$ is nonempty, which we now turn to.

Let $Co(D(p))$ and $Co(N(p))$ denote the convex hulls of $D(p)$ and $N(p)$. We first show that $Co(D(p)) \cap Co(N(p))$ has a nonempty intersection. Suppose by contradiction that $Co(D(p)) \cap Co(N(p)) = \emptyset$. Then, since these two sets are closed and convex, the separating-hyperplane theorem implies that there exists a direction $\delta$ and a number $a$ such that $y\delta < a$ for $y \in N(p)$ and $x\delta > a$ for $x \in D(p)$. Now modify $p$ by an infinitesimal amount along the direction $\delta$, yielding a new level $q = p + \varepsilon\delta$. The objective function $\pi(p) + \max_z\{pz - (\tilde{p}, z)\}$ is affected by this change in two ways. First, through the sensitivity of $\pi$ with respect to $p$. Taking any $x \in D(q) \subset D(p)$, we have $\pi(p) = v(x) - px$ and $\pi(q) = v(x) - qx$. Therefore, the change of $\pi$ is $-\varepsilon x\delta$. Second, through the sensitivity of $\max_z\{pz-(\tilde{p},z)\}$ with respect to $p$. There exists $y \in N(p)$ such that $\max_z\{pz-(\tilde{p},z)\} = py-(\tilde{p},y)$ throughout the price change. Therefore, the effect on this term equals $\varepsilon y\delta$. The overall change of the objective function is then $\varepsilon(y-x)\delta < 0$, implying that $q$ leads to a strictly lower objective function than $p$, which contradicts optimality of $p$.

We have proved that the sets $Co(D(p))$ and $Co(N(p))$ have a non empty intersection. We now prove that this intersection contains a point with integer coordinates. Consider any polytope of $\mathbb{R}^K$. We say that an edge (i.e. a segment joining two vertices of the polytope) is *simply oriented* if either (i) it is parallel to one coordinate axis $\{\lambda e_i : \lambda \in \mathbb{R}\}$ of the space or (ii) there exist two coordinates $i$ and $j$ such that the edge is parallel to $e_i - e_j$. We say that a polytope is simply oriented if all its edges are simply oriented. Last, we recall that a polytope all of whose vertices have integer coordinates is called a *lattice polytope.*

LEMMA 5 *If a lattice polytope $P$ is simply oriented, and $H$ is the half space $\{x : x_k \geq q\}$, where $k \in \{1, \ldots, K\}$ and $q$ is an integer, then $P \cap H$ is either the empty set, or a simply oriented, lattice polytope.*

*Proof.* Suppose that $Q = P \cap H$ is nonempty. Then, it is a polytope. Its vertices are either vertices of $P$, in which case they are integral, or new vertices belonging to $H$. We prove that any such vertex also has integer coordinates. Any new vertex $x$ is the intersection of $H$ with an edge $E$ of $P$ that is not parallel to $H$. In particular, there exists an integral vertex $y$ of $P$ such that $x - y$ is parallel to $E$. Moreover, $y_k \neq q$, since the edge is not parallel to $H$. The edge is either parallel to $e_k$ or to $e_k - e_i$ for some $i \neq k$. In the first case, we have $x_j = y_j \in \mathbb{Z}$ for all $j \neq k$ and $x_k = q \in \mathbb{Z}$, so $x$ has integer coordinates. In the second case, $x_j = y_j \in \mathbb{Z}$ for all $j \notin \{i, k\}$, $x_k = q \in \mathbb{Z}$, and $x_i = y_i + (y_k - x_k) \in \mathbb{Z}$, so $x$ also has integer coordinates. We now prove that the edges of $Q$ are simply oriented. Thus consider an edge $E$ of $Q$, joining vertices $x$ and $y$. If either $x$ or $y$ are vertices of $P$, then $E$ is either an edge of $P$, or the result of such an edge being cut by $H$. In either case, it is simply oriented because $P$ is simply oriented. If both $x$ and $y$ are new vertices, $E$ is the intersection of a two-dimensional face $F$ of $P$ with $H$, where $F$ is not parallel to $H$. $F$ is defined by two linearly independent edges $E'$ and $E''$ of $P$ which are simply oriented, and at least one of which contains $e_k$. Suppose first that either $E'$ or $E''$, say $E'$, is orthogonal to $e_k$. Then it is easy to show that $E$ is parallel to $E''$ and therefore simply oriented. Now suppose that both $E'$ and $E''$ have a nonzero $k^{th}$ component. Because they are linearly independent, there exist $i$ and $j$ such that $F$ is generated by $e_k - e_i$ and $e_k - e_j$ (where the signs come from the fact that $P$ is simply oriented). In that case, as can be easily verified, $E$ is parallel to $e_i - e_j$, and therefore simply oriented.■

We observe that Lemma 5 still holds if the inequality sign is reversed in the definition of $H$.

$Co(D(p))$ is a lattice polytope since $D(p)$ consists of integral vectors. We now prove that $Co(D(p))$ is simply oriented. Thus consider any edge $E$ of $Co(D(p))$. There exists a vector $\delta$ of $\mathbb{R}^K$ such that $E$ is included in some straight line $\Delta = \{x_0 + \lambda\delta\}_{\lambda \in \mathbb{R}}$. We first show that $\delta$ has at most two nonzero components. Suppose on the contrary that $\delta$ has at least three components, say $i$, $j$, and $k$. Without loss of generality assume that $\delta_i$ and $\delta_j$ are positive. Since $E$ is a face of $Co(D(p))$, there exists an infinitesimal modification of the price vector $p$, such that $D(p) = E$. Moreover, $E$ contains two vectors $x$ and $y$ such that $x - y = \lambda\delta$ for some $\lambda > 0$. If we slightly increase $p_i$, $x$ becomes suboptimal, so the optimal quantity of good $j$ decreases, which violates the class-substitute property. Thus, $\delta$ has at most two nonzero components. We now prove that $E$ is simply oriented. If $\delta$ has only one nonzero component, the claim is trivial. Suppose that $\delta$ has two positive components, say $i$ and $j$. We show that $\delta_i = -\delta_j$. Since $E$ has integer vertices, we can assume that $\delta_i$ and $\delta_j$ are integers.[24] If $\delta_i\delta_j > 0$, slightly increasing $p_i$ reduces the optimal quantity of good $j$ which violates the class-substitute property. Thus, $\delta_i$ and $\delta_j$ have opposite signs. Now suppose that $|\delta_i| < |\delta_j|$. This implies that for all integral vectors $x$ and $y$ in $E$, we have $|x_j - y_j| \geq 2$, which violates the consecutive-integer property. Thus, $\delta_i = -\delta_j$, which concludes the proof.

We have shown that $Co(D(p))$ is a simply oriented lattice polytope. Since $Co(N(p))$ is a hyperrectangle of the form $\{x \in \mathbb{R}^K : a \leq x \leq b\}$ for some integral vectors $a$ and $b$, we have, denoting $H(k,q)_+ = \{x : x_k \geq q\}$ and $H(k,q)_- = \{x : x_k \leq q\}$,

$$Co(D(p)) \cap Co(N(p)) = Co(D(p)) \bigcap_{1 \leq k \leq K} (H_+(k, a_k) \cap H_-(k, b_k))\,.$$

Iterating Lemma 5 $2K$ times implies that $Co(D(p)) \cap Co(N(p))$ is either the empty set or a lattice polytope. Since we have already shown that this intersection is nonempty, it must contain an integral point, which concludes the proof of Proposition 1. ■

## 10.2 Section 5

Proof of Theorem 18. We extend part of the proof of Theorem 2 in Gul and Stacchetti (1999) to a multi-unit context. By assumption, there exist a price vector $\bar{p}$, a good $k$, and bundles $x$ and $x'$ such that (i) $\{x, x'\} \in D(\bar{p})$, (ii) $x'_k - x_k \geq 2$, and (iii) for all $z$ in $D(\bar{p})$, $z_k \notin (x_k, x'_k)$. This implies that at the price $p = \bar{p} - \eta e_k$, $x$ is only dominated by bundles $z$ such that $z_k \geq x_k + 2$. In particular, the single-improvement property is violated by $x$ at price $p$. Therefore, any bundle $y$ such that

$$y \in \arg\min_z \sum_k |x_k - z_k|$$

subject to

$$u_1(z,p) > u_1(x,p)$$

satisfies $y_k \geq x_k + 2$.

[24] See for example Korte and Vygen (2000).

Let $\rho = \sum_j (y_j - x_j)_+$. By hypothesis, $\rho \geq 2$. Let $\varepsilon = \frac{u_1(y,p) - u_1(x,p)}{2\rho}$. Let $I_+ = \{j : x_j < y_j\}$, $I_- = \{j : x_j > y_j\}$, and $I_0 = \{j : x_j = y_j\}$. If $j \in I_+$, introduce $N_j - y_j$ firms, call them "$C_j$", with unit-demand valuation $v_1(\mathcal{X}) + 2$ for a single unit of good $j$. If $j \in I_+ \setminus \{k\}$, introduce $y_j - x_j$ firms, call them "$c_j$", with unit-demand valuation $p_j + \varepsilon$ for a single unit of good $j$. If $j = k$, introduce $y_k - x_k - 1$ firms ("$c_k$") with unit-demand valuation $p_k + \varepsilon$ for a single unit of good $k$. If $j \in I_-$, introduce $N_j - x_j$ firms ($C_j$) with unit-demand valuation $v_1(\mathcal{X}) + 1$ for a single unit of good $j$, and $x_j - y_j$ firms ($c_j$) with unit-demand valuation $p_j$ for a single unit of good $j$. If $j \in I_0$, introduce $N_j - x_j$ firms with unit-demand $v_1(\mathcal{X}) + 1$. Last, introduce a special firm, "firm 2", with unit-demand $p_k + v_1(\mathcal{X}) + 1$ for a single unit of good $k$.

Now suppose that there exists a Walrasian equilibrium with price vector $t$, and let $X_i$ denote the bundle of the equilibrium received by firm $i$. Necessarily, $(X_1)_j \geq \min\{x_j, y_j\}$ for all $j$, since even if all unit-demand firms get one unit, there remain $\min\{x_j, y_j\}$ units of good $j$. Define a new price vector as follows: $q_j = t_j$ for $j \notin I_-$ and $q_j = p_j$ for $j \in I_-$. For $j \in I_-$, $N_j - x_j$ units go to firms $C_j$. The remaining $x_j$ units are shared between firm 1 and firms $c_j$, with at least $y_j$ units for firm 1. Now, if firm 1 has none of the remaining $x_j - y_j$ units, it means that $t_j \leq p_j$, and this share remains optimal when $t_j$ is increased to $p_j$. If firm 1 has all of the remaining units, it means that $t_j \geq p_j$, and this share remains optimal when $t_j$ is decreased $p_j$. If firm 1 has only a part of these remaining units, it means that $t_j$ is already equal to $p_j$. Thus $(X, q)$ is also a Walrasian equilibrium, such that $X_1 \geq x \wedge y$. Moreover, all $C_j$ get their units, so that $X_1 \leq x \vee y$. Therefore

$$x \wedge y \leq X_1 \leq x \vee y. \tag{6}$$

Firm 2 necessarily gets a unit of good $k \in I_+$. Therefore, $X_{1k} < y_k$. This, together with (6), implies that $\sum_k |x_k - X_{1k}| < \sum_k |x_k - y_k|$, and thus

$$u(X_1, p) \leq u(x, p). \tag{7}$$

Suppose that there exist some goods $j$ in $I_+$ such that $X_{1j} > x_j$. This implies that $q_j \geq p_j + \varepsilon$, since firms $c_j$ would otherwise want to get all the units. Combining these price inequalities with (7) yields $u_1(X_1, q) < u_1(x, q)$, which contradicts optimality of $X_1$ for firm 1.

Suppose instead that $X_{1j} \leq x_j$ for all $j$. Then, all units between $x_j$ and $y_j$ for $j \in I_+$ are consumed by firms $c_j$ and by firm 2. For $j \neq k$, this implies that $c_j$ have a positive value for the good: $q_j \leq p_j + \varepsilon$. For $j = k$, even though firm 2 takes one units of the $y_k - x_k$

available units of $k$, the fact that $y_k \geq x_k + 2$ implies that there is also a firm $c_k$ taking one unit of good $k$, which implies that $q_k \leq p_k + \varepsilon$. Since $X_1 = x$ on $I_+$ and $p_j = q_j$ for $j \notin I_+$, (7) implies

$$u_1(X_1, q) \leq u_1(x, q).$$

Since $q_j \leq p_j + \varepsilon$ for all $j \in I_+$, the value initially chosen for $\varepsilon$ implies that $u_1(x, q) < u_1(y, q)$, and thus $u_1(X_1, q) < u_1(y, q)$, which contradicts optimality of the bundle $X_1$ for firm 1. ■

Proof of Theorem 23 From A&M Theorem 7 (which allows for multiple units of goods), we know that the vector of Vickrey payoff vector is in the core if and only if the coalitional value function is bidder-submodular. We show that under the assumptions of Theorem 23, there always exist bidder valuations such that the coalitional value function is not bidder-submodular. Suppose that bidder 1's valuation violates the consecutive-integer property. There exist $\hat{p}$ and $k$ such that $D_k(\hat{p})$ does not consist of consecutive integers. Let $p = \hat{p} + \varepsilon e_k$ for $\varepsilon$ small enough. Then there exists $x$ and $z$ such that $x_k \geq z_k + 2$, and

$$v(z) - pz > v(x) - px > v(y) - py \tag{8}$$

for all $y$ such that $y_k \in (z_k, x_k)$. Introduce a second bidder with linear valuation $v_2(x) = p_{-k}x_{-k}$, and $x_k - z_k$ unit-demand bidders who only value good $k$. The total number of bidders is $x_k - z_k + 2 \leq N_k + 2 \leq \max_k N_k + 2$. From (8), we have

$$v(x) + p_{-k}(\bar{x} - x)_{-k} \geq v(y) + p_{-k}(\bar{x} - y)_{-k} + p_k(x_k - y_k)$$

whenever $x_k - y_k \leq x_k - z_k - 1$, and

$$v(z) + p_{-k}(\bar{x} - z)_{-k}p_k(x_k - z_k) > v(x) + p_{-k}(\bar{x} - x)_{-k}.$$

Therefore, denoting $S$ the set consisting of bidders 1, 2 and the $x_k - z_k - 2$ unit-demand firms, and $s$ and $t$ the last two unit-demand bidders, we have

$$w(S \cup \{s\}) = w(S)$$

and

$$w(S \cup \{s, t\}) > w(S \cup \{t\}),$$

which implies that $w$ is not bidder-submodular. ■

### 10.3 Section 6

PROOF OF THEOREM 24. The proof is based on three lemmas, proving respectively well-definedness, monotonicity, and confinement in $\mathcal{L}$.

LEMMA 6 (WELL-DEFINEDNESS) *The continuous SDA algorithm is well defined.*

*Proof.* On any region of the price space where excess demand is constant, the algorithm defines a straight trajectory of direction $z$, and is thus well-defined.[25] The only possible problem, thus, is to rule out the possibility that there are infinitely many region changes in an arbitrarily small amount of time. With the steepest-descent algorithm, the norm of $z$ is nondecreasing in time. Since $z$ is constant over any region where aggregate demand is constant, and the norm of $z$ strictly decreases each time it changes, any region that is left is never visited again. ■

LEMMA 7 (MONOTONICITY) *When bidders have class-substitute valuations and $z(0) \geq 0$, $p(\cdot)$ is nondecreasing.*

*Proof.* Suppose by contradiction that $z(t)$ fails to be nonnegative at some time $t$, and take the smallest such time. Since $z(0) \geq 0$, $t > 0$. By construction, $z(s) \geq 0$ on a left neighborhood of $t$. Let $m = z(t)$, $x = z(t_-)$, and $P$ be the opposite of the subdifferential of $f$ at $p(t)$. $P$ is a convex polytope, whose vertices are elements of the excess demand at $p(t)$, and $m$ is the element of $P$ with smallest norm. By assumption, $x$ is nonnegative. By continuity of demand, $x$ must also belong to $P$. Let $J = \{k : m_k < 0\}$. By assumption, $J \neq \emptyset$. Let $H$ be the affine hyperplane going through (the point) $m$ and orthogonal to (the vector) $m$. By assumption, $P$ is on one side of $H$ and touches $H$ at $m$. Let $F$ be the largest face of $P$ contained in $H$, $y$ be any vertex of $F$, and $C_y = \{z : \sum_J m_k z_k \geq \|m\|^2 - \sum_{J^c} m_s y_s\}$. Since $y - m$ is orthogonal to $m$, $C_y$ is a cone with vertex $y$. We will show that $C_y$ contains $P$ but not $x$, a contradiction.

Since $y-m$ is orthogonal to $m$, we have $\|m\|^2 - \sum_{J^c} m_s y_s = \sum_J m_k y_k = m_J y_J$, where the components of $m_J$ are equal to those of $m$ on $J$ and vanish on $J^c$, and a similar definition for $y_J$. By convexity of $F$, $m = y + \sum_l \alpha_l E_l$, where $\{E_l\}$ is the family of direction vectors of the edges of $F$ emanating from $y$. Taking the scalar product of the previous equality with $m_J$ yields $m m_J = y_J m_J + \sum_l \alpha_l E_l m_J$. We now prove that $E_l m_J = 0$ for all $l$. By

[25] The scalar function $\alpha$ is immaterial, as long as it is bounded away from 0 and $+\infty$.

construction of $F$,

$$m.E_l = 0. \tag{9}$$

Moreover the class substitute property implies that $E_l$ has at most two nonzero components, and any two nonzero components are of opposite sign (see the proof of Proposition 1). If $E_l$ has one nonzero component, it must be in $J^c$, otherwise it would violate (9). If it has two nonzero components, then either they are both in $J$ or both in $J^c$, for otherwise (9) would be violated. In any case, this implies that $E_l.m_J = 0$. Thus, $m_J v_J = m_J^2 > 0$. In particular $C_y = \{z : \sum_J m_k z_k \geq m_J^2\}$. Since the components of $x$ are nonnegative by construction, $x$ cannot belong to $C_y$.

To conclude the proof, we show that $C_y$ contains $P$. By convexity of $P$, it is enough to show that all edges of $P$ emanating from $y$ are going in the cone $C_y$. This will be the case if we show that for any such edge with direction $\delta$ (away from $y$), we have

$$\delta m_J \geq 0. \tag{10}$$

By definition of $F$, we have $\delta m \geq 0$ (i.e. any edge from $y$ must point outwards from $H$). Since bidders have class-substitute valuations, $\delta$ has at most two nonzero components. Suppose first that it has exactly two components, $\delta_i$ and $\delta_j$. If $i, j$ are in $J$, then (10) trivially holds. If $i, j$ are in $J^c$, then (10) is an equality. If $i \in J$ and $j \in J^c$, then $\delta m \geq 0$ and the fact that $\delta_i \delta_j < 0$ (by class-substitutes) implies that $\delta_i < 0$, and thus that (10) holds. If there is only one nonzero component, (10) holds trivially. ■

Lemma 8 (Confinement) *If bidders have class-substitute valuations, $p(0) \leq \underline{p}$ and $z(0) \geq 0$, then $p(t) \leq \underline{p}$ for all $t \geq 0$.*

*Proof.* Suppose not: there exists a time $t$ such that $p(t)$ crosses the hyperrectangle $R = \{z : z \leq \underline{p}\}$ from inside out. In particular, the index subset $I = \{j : p_j(t) = \underline{p}_j\}$ is nonempty, and we have $p_j(t) < \underline{p}_j$ for $j \notin I$. Moreover, $p(s) \not\leq \underline{p}$ for $s$ in a right neighborhood of $t$: there exists a nonempty subset $J \subset I$ such that $p_{s,j} > \underline{p}_j$ for $j \in J$ and $s \in (t, t + \varepsilon)$. By construction of the algorithm, this means that the vector $n$ of smallest norm in the opposite of the subdifferential of $p(t)$ satisfies $n_j > 0$ for $j \in J$. We will contradict this statement by showing that the vector $m$ defined by $m_j = n_j$ for $j \notin J$ and $m_j = 0$ for $j \in J$ is in the opposite of the subdifferential. $m$'s norm is strictly smaller than $n$'s, contradicting the assumption that $n$ is of smallest norm in the opposite of the subdifferential. By definition of the subdifferential, we need to show that, letting

$p = p(t)$,

$$m(q-p) \geq f(p) - f(q) \tag{11}$$

for all $q$. We first show this inequality in a neighborhood of $p$. By construction of $n$,

$$n(q-p) \geq f(p) - f(q)$$

for all $q$. Therefore, (11) is automatically satisfied for $q$ such that $q_j \leq p_j$ for $j \in J$. Now consider the case where $q_j > p_j$ for a subset $J(q)$ of $J$. Consider the vector $q'$ such that $q'_j = q_j$ for $j \notin J(q)$ and $q'_j = p_j$ for $j \in J(q)$. Since we are in a neighborhood of $p$, $q_j \leq \underline{p}_j$ for all $j \notin J(q)$. This implies that $q' \leq \underline{p}$ and, therefore, that $q' = q \wedge \underline{p}$. By submodularity of $f$, we have

$$f(\underline{p} \wedge q) + f(\underline{p} \vee q) \leq f(\underline{p}) + f(q).$$

The inequality, combined with the fact that $\underline{p}$ is a minimum of $f$, implies that $f(q') \leq f(q)$. By construction of $q'$, we have

$$m(q-p) = m(q'-p) \geq n(q'-p) \geq f(p) - f(q') \geq f(p) - f(q),$$

which concludes the proof on a neighborhood of $p$. To prove the result globally, consider any vector $q$ and let $q_\lambda = \lambda q + (1-\lambda)p$ where $\lambda \in (0,1)$. From the previous analysis we have, for $\lambda$ small enough,

$$m(q_\lambda - p) \geq f(p) - f(q_\lambda).$$

By convexity of $f$,

$$f(q_\lambda) \leq \lambda f(q) + (1-\lambda) f(p).$$

Combining the previous two inequalities and dividing by $\lambda$ yields the result. ■

We now conclude the proof of the theorem. Since $p(t)$ is nondecreasing and bounded, it must converge to some limit in $\mathcal{L}$. Since $\alpha$ is bounded away from zero, the rate of change of $p$ is bounded away from zero on any closed subset of the price space that does not contain any pseudo-equilibrium price. Since the only pseudo-equilibrium price contained in $\mathcal{L}$ is $\underline{p}$, this has to be the limit.

Proof of Lemma 2. By assumption, the excess demand set is an integer polytope of $\mathbb{R}^K$, bounded by the rectangle $[-\bar{x}, N\bar{x}]$. Therefore, $z$ can only take finitely many values. Since any such $z$ is the vector of minimum norm of an integral polytope, it has rational coordinates. Therefore, its direction can always be achieved on regular lattice.

That is, there exists a positive number $\alpha(z)$ such that $\alpha(z)z$ is a feasible direction of the lattice. Moreover, $\alpha(z)$ gets arbitrarily small as the grid gets arbitrarily thin. Since there are finitely many values of $z$, $\max_z\{\alpha(z)\}$ goes to zero as the grid thinness $\eta$ goes to zero.■

PROOF OF LEMMA 3. Without loss of generality, we can restrict attention to price vectors less than $M$. Since the number of bidders is finite, the function $f : p \to \pi(p) + \bar{x}p$ is piecewise affine, with finitely many regions. Moreover, directions of the hyperplanes supporting $f$ are determined by excess demand vectors, which take finitely many values (cf. proof of Lemma 2). Since $z$ is in the opposite of the differential of $f$, we have

$$f(q) - f(p) \geq z(p)(p - q)$$

for all $q$, with strict inequality if $p$ and $q$ are in distinct regions. The fact that $p$ is bounded by $M$ and that there are finitely many possible slopes for $f$ implies the existence of a constant $\rho > 0$ such that

$$f(q) - f(p) \geq \rho + z(p)(p - q) \tag{12}$$

whenever $p$ and $q$ are not in the same region. We now consider paths of the discrete steepest-descent algorithm starting from respective initial price vectors $p_0$ and $q_0$, with $\|p_0 - q_0\| < \varepsilon$. Trajectories are parallel until the two prices reach different regions, and thus leave the vector $p_t - q_t$ unchange until that time. Let $s \geq 0$ denote the first time that the two paths hit distinct regions. From (12), we have,

$$f(q_s) - f(p_s) \geq \rho + z(p_s)(p_s - q_s)$$

and

$$f(p_s) - f(q_s) \geq \rho + z(q_s)(q_s) - p_s).$$

Summing these inequalities yields[26]

$$(z(p_s) - z(q_s))(p_s - q_s) \leq -2\rho$$

Let $\alpha$ be the step size[27] of the steepest-descent algorithm: $p_{s+1} = p_s + \alpha z(p_s)$, and $q_{s+1} = q_s + \alpha z(q_s)$

$$\|p_{s+1} - q_{s+1}\|^2 = \|p_s - q_s\|^2 + \|\alpha(z(p_s) - z(q_s))\|^2 + 2\alpha(z(p_s) - z(q_s)) \cdot (p_s - q_s).$$

[26]This proof strategy introduces a strict version of the theory of maximally monotone mapping. See Rockafellar (1970).

[27]The result holds if $\alpha$ depends on $t$ and $p$, as long as it is continuous in $p$.

Therefore,

$$\|p_{s+1} - q_{s+1}\|^2 - \|p_s - q_s\|^2 \leq -4\rho\alpha + O(\alpha^2),$$

which is negative for $\alpha$ small enough, which we impose by appropriately setting $\bar{\alpha}$. Thus, we have proved that $\|p_t - q_t\|$ remains constant when prices are in the same region, and decreases otherwise. ■

## 10.4 Section 8

Proof of Proposition 7.

We adapt the proof of Proposition 1. Let $p$ denote any solution of $\min_p \max_x\{\pi(p) + px - C(x)\}$. With divisible goods and concave demand, $Co(D(p)) = D(p)$ (closedness of $D(p)$ is guaranteed by continuity of the objective function $v(x) - px$) and $Co(N(p)) = N(p)$ where $N(p) = \arg\max_x\{px - C(x)\}$ (closedness of $N(p)$ is guaranteed by continuity of the cost function). Therefore, it is enough to show that $Co(D(p)) \bigcap Co(N(p)) \neq \emptyset$. Suppose otherwise. Since these two sets are closed and convex, the separating-hyperplane theorem then implies the existence of a direction $\delta$ and a number $a$ such that $y\delta < a$ for $y \in N(p)$ and $x\delta > a$ for $x \in D(p)$. Now modify $p$ by an infinitesimal amount along the direction $\delta$, yielding a new level $q = p + \varepsilon\delta$. For $p_\lambda = (1-\lambda)p + \lambda q$, let $g(p_\lambda) = \pi(p_\lambda) + \max_z\{p_\lambda z - (\tilde{p}, z)\}$. By the integral form of the envelope theorem (see Milgrom and Segal (2002)),

$$g(q) - g(p) = \int_0^1 -\varepsilon\delta x(\lambda) + \int_0^1 \varepsilon\delta y(\lambda),$$

where $x(\lambda)$ is any element of $D(p\lambda)$ and $y(\lambda)$ is any element of $N(p_\lambda)$. Since objective functions are continuous, Berge's theorem implies that for $\varepsilon$ small enough, $x(\lambda)\delta$ is uniformly strictly above $a$ and $y(\lambda)\delta$ is uniformly strictly below $a$. This implies that $g(q) < g(p)$, which contradicts optimality of $p$. Therefore, $D(p) \cap N(p)$ has a nonempty intersection. The rest of the proof is identical to the first part of the proof of Proposition 1. ■

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