Generalised empirical likelihood-based kernel density estimation

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Abstract

If additional information about the distribution of a random variable is available in the form of moment conditions, a weighted kernel density estimate reflecting the extra information can be constructed by replacing the uniform weights with the generalised empirical likelihood probabilities. It is shown that the resultant density estimator provides an improved approximation to the moment constraints. Moreover, a reduction in variance is achieved due to the systematic use of the extra moment information.
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1 Introduction

Nonparametric density estimation is an important tool in applied econometrics, finance, and many other areas, where it is often used for exploratory data analysis or as a part of another estimator; see e.g. Pagan and Ullah (1999), Wand and Jones (1995), Silverman (1986) and Li and Racine (2007).

Let $X = (X_1, \ldots, X_d)^\intercal$ be a random vector which has a continuous probability density function f. Given a random sample $X_i = (X_{1,i}, \ldots, X_{d,i})^\intercal$, $i = 1, \ldots, n$, the kernel density estimator (KDE) of f at an arbitrary point $x \in \mathbb{R}^d$ is given by

$$\hat{f}(x; H_n) = n^{-1} \sum_{i=1}^{n} K_{H_n}(x - X_i), \tag{1}$$

where $K_{H_n}(z) = \det(H_n)^{-1/2} K\left(H_n^{-1/2} z\right)$, $K(\cdot)$ is a d-variate kernel function, and H_n is a non-stochastic sequence of symmetric positive definite bandwidth matrices. This estimator was proposed by Rosenblatt (1956) and Parzen (1962) and can be motivated as a smoothed version of a histogram. We will write $\hat{f}(x)$ for $\hat{f}(x; H_n)$ and H for H_n; the dependence of $\hat{f}$ on bandwidth and of H on sample size being implicit.

In some applications it may be necessary to construct an estimator of a probability density function (pdf) which obeys certain constraints. For instance, the mean of X may be known or there may be a known relationship between the moments, perhaps implied by estimating equations. Extra distributional information may be due to a certain physical law as in the example considered in Chen (1997) where according to the line transect theory the distribution of the perpendicular sighting distances in an aerial line transect survey should have mean zero.

Assumptions about the relationship between the mean and variance of the observations underlies the standard quasi-likelihood estimation; see Wedderburn (1974) and Godambe and Thompson (1989). If the variance of X is a known function of the (unknown) mean, μ, the information about f can be expressed in the form of two moment conditions, viz. $\mathrm{E}\, X = \mu$ and $\mathrm{E}\,(X - \mu)^2 = \psi(\mu)$, where $\psi(\cdot)$ is a known function. The method presented in this paper allows such information to be incorporated into an estimate of f.

Incorporating auxiliary population information is also of interested when using survey data; see e.g. Qin and Lawless (1994, p. 301) and Chen and Qin (1993). For example, one may be interested in estimating the density of household income based on a survey data. If the average income is known from, say, census data, it can be treated as a known population mean and incorporated into the estimate.

This paper considers the case when the extra information can be formulated in the form of moment conditions on X. This case has been examined by Chen (1997) who proposes re-weighting the Rosenblatt-Parzen KDE (RPKDE)

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using *empirical likelihood* weights instead of equal probability weights, n^{-1}, placed at every data point. A similar approach has been applied by Hall and Presnell (1999).

Specifically, suppose that additional information about f is available in the form

$$\mathrm{E}\left[g\left(X;\beta_0\right)\right] = 0, \tag{2}$$

where $g\left(x;\beta\right) = \left[g_1(x;\beta), \ldots, g_q(x;\beta)\right]^{\mathsf{T}}$ is a *known* real vector-valued function representing q moment conditions, $\beta \in \mathscr{B} \subseteq \mathbb{R}^p$ is a $p \times 1$ vector of unknown parameters, $p \leq q$, and expectation is taken with respect to the distribution of X.

In this paper, we seek an estimator of f, $\tilde{f}(\cdot)$, which satisfies constraints (2) in the sense that $\int g_l(u;\beta_0)\tilde{f}(u)du = 0$, $l = 1, \ldots, q$. As shown in section 2, RPKDE will not in general possess this property. The reweighted KDE defined in section 4 will better satisfy conditions (2).

This work extends the previous analysis by allowing parameters in the moment conditions to be *estimated* using *generalised* empirical likelihood (GEL) estimation, described in section 3.

Prior to computing an estimate with the constraints imposed, one should test whether the constraints are consistent with the data. For example, in a simple case when the mean is hypothesized to be known a standard t-test can be employed. GEL-based tests can be used as described in section 3. As GEL estimation is part of the proposed procedure, such test statistics can be computed at no extra cost.

Properties of the GEL-based estimator are presented in section 4. In particular, it is shown that, provided moment conditions contain some overidentifying information, a reweighted estimate will have smaller variance than the standard kernel estimate. We show that the reduction in the variance occurs in the second order term and is the *same* for all members of the GEL family.

Section 5 analyses the performance of the proposed density estimator in small and medium samples via a Monte-Carlo study. The final section concludes.

Throughout the paper, derivatives of a scalar-valued function of a scalar argument are denoted as $h^{(j)}(z_0) = \partial^j h(z)/\partial z^j\big|_{z=z_0}$. For a vector-valued function $h(z) : \mathbb{R}^p \mapsto \mathbb{R}^q$, $\mathscr{D}_z h(z_0) = \left[\partial h_i(z_0)/\partial z_j\right]_{i,j=1}^{q,p}$ is the $q \times p$ Jacobian matrix, and $\mathscr{H}_z h(z_0)$ is the $qp \times p$ matrix of second derivatives obtained by vertically stacking the $p \times p$ matrices $\mathscr{H}_z h_i(z_0) = \left[\partial^2 h_i(z_0)/\partial z_j \partial z_k\right]_{j,k=1}^{p,p}$. It will also be convenient to define $\mathscr{H}_z^{\circ} h(z_0) = \mathrm{reshape}\left(K_{p,qp} \,\mathrm{vec}\left(\mathscr{H}_z h(z_0)\right), q, p^2\right)$, where $K_{p,qp}$ is the $qp^2 \times qp^2$ commutation matrix, and $\mathrm{vec}\left(\cdot\right)$ is the vectorisation operator. $\otimes$ denotes the Kronecker product; $\mathrm{tr}\left(A\right)$ and $\det(A)$ are the trace and determinant of a square matrix A, respectively; I_q denotes the $q \times q$ identity matrix, ι_q is the $q \times 1$ vector of ones. $\int$ denotes a definite (multivariate) integral, usually over $\mathbb{R}$ ($\mathbb{R}^d$).

2 Rosenblatt–Parzen kernel density estimator

The Rozenblatt-Parzen KDE has been studied extensively and its properties are well-documented. Asymptotic approximations to the mean integrated squared error (MISE) of $\hat{f}$ can be obtained under additional assumptions on kernel, bandwidth, and density of X.

Assumption 1 (a) $K(\cdot)$ is a d-variate kernel satisfying $\int K(z)dz = 1$, $\int zK(z)dz = 0$ and $\int zz^{\mathsf{T}}K(z)dz = \mu_2(K)I_d$, where $\mu_2(K) = \int z_i^2 K(z)dz < \infty$, independent of i; (b) all entries of the sequence of bandwidth matrices $H = H_n$ approach zero as $n \to \infty$ and $\lim_{n\to\infty} n^{-1}\det(H)^{-1/2} = 0$; and (c) all entries of $\mathscr{H}_x f(x)$ are bounded, continuous and square integrable.

If Assumption 1 holds, the asymptotic mean and variance of $\hat{f}(x)$ are

$$\mathrm{E}\,\hat{f}(x) = f(x) + \frac{1}{2}\mu_2(K)\,\mathrm{tr}\left(H\mathscr{H}_x f(x)\right) + o\left(\mathrm{tr}\left(H\right)\right),$$
$$\mathrm{Var}\,\hat{f}(x) = n^{-1}\det(H)^{-1/2}R(K)f(x) + o\left(n^{-1}\det(H)^{-1/2}\right),$$

where $R(\psi) = \int \psi^2(x)dx$ for any square-integrable function ψ; see e.g. Wand (1992). Thus the mean integrated squared error (MISE) of $\hat{f}$ is

$$\mathrm{MISE}\,\hat{f}(\cdot) = n^{-1}\det(H)^{-1/2}R(K) + \frac{1}{4}\mu_2^2(K)\int \mathrm{tr}\left(H\mathscr{H}_x f(x)\right)^2 dx + o\left(n^{-1}\det(H)^{-1/2} + \mathrm{tr}\left(H\right)^2\right). \tag{3}$$

The first two terms in (3) give the asymptotic MISE (AMISE) of $\hat{f}$, which provides a useful large-sample approximation to MISE.

For general H, minimisation of AMISE can be performed numerically (Wand, 1992). A considerable simplification occurs when the choice of bandwidth is restricted to $H = h^2 I_d$. In this case it is easy to see that the bias term is of order h^4, whereas the variance term is of order $n^{-1}h^{-d}$. Hence the bandwidth is to be chosen to balance the bias-variance trade-off: smaller values of h reduce bias but increase variance. The asymptotically optimal (AMISE-minimising) bandwidth, is proportional to $n^{-1/(d+4)}$, setting both terms in AMISE to be of the same order, $n^{-4/(d+4)}$. In practice, the choice of bandwidth is very important; see Sheather (2004) for a recent review and the references given above.

In general, the RPKDE will not satisfy conditions (2).

Example 1 In the univariate case, $d = 1$, $\mathrm{E}_{\hat{f}} X^j = n^{-1} \sum_{i=1}^n \int (x_i + zh)^j K(z) dz$, and since $K(\cdot)$ is a symmetric density function, $\mathrm{E}_{\hat{f}} X = n^{-1} \sum_{i=1}^n x_i$, the sample average. Hence the constraint that the mean is μ, say, will not generally be satisfied in finite samples. The same is true for higher moments, e.g. $\mathrm{E}_{\hat{f}} X^2 = n^{-1} \sum_{i=1}^n x_i^2 + h^2 \mu_2(K)$, $\mathrm{E}_{\hat{f}} X^3 = n^{-1} \sum_{i=1}^n x_i^3 + 3h^2 \mu_2(K) n^{-1} \sum_{i=1}^n x_i$, etc. $\square$

Suppose that $g(\cdot)$ satisfies the following conditions:

Assumption 2 For all $\beta \in \mathrm{B}_r(\beta_0)$, an open ball around β_0, all entries of $\mathscr{H}_x g(x;\beta)$ are bounded, continuous and square integrable.

If Assumptions 1 and 2 hold, then for general $g(x;\beta)$ and $\beta \in \mathrm{B}_r(\beta_0)$, for $l = 1, \ldots, q$,

$$\int g_l(x;\beta) \hat{f}(x) dx = \widehat{g}_l(\beta) + \frac{1}{2} \mu_2(K) \operatorname{tr}(H J_l(\beta)) + o_p(\operatorname{tr}(H)). \tag{4}$$

where $\widehat{g}_l(\beta) = n^{-1} \sum_{i=1}^n g_l(x_i;\beta)$ and $J_l(\beta) = \mathrm{E}\, \mathscr{H}_x g_l(x_i;\beta)$. The results follows by the change of coordinates; detailed proofs are given in the Supplement.

As shown in section 4, the reweighted estimator provides an improved approximation to the moment conditions; in particular, the first term in (4) is zero.

3 Generalised empirical likelihood

Implied probabilities, obtained as a by-product of the GEL estimation, can be used to reweight the RPKDE so that the resultant density estimator better approximates conditions (2).

GEL is an estimation method for models based on moment conditions of the form (2); see inter alia Smith (1997), Imbens (2002) and Newey and Smith (2004), NS. To give a brief overview of GEL, introduce the *carrier* function $\rho(\cdot) : \mathscr{V} \to \mathbb{R}$, a *concave* real-valued scalar function defined on an open interval $\mathscr{V} \subseteq \mathbb{R}$ containing zero. Let $\rho^{(k)}(v) = \partial^k \rho(v)/\partial v^k$ denote the k-th derivative of $\rho(\cdot)$, $k = 0, 1, 2, \ldots$. It will be convenient to impose the innocuous normalisation $\rho^{(1)}(0) = \rho^{(2)}(0) = -1$.

Special cases of GEL include empirical likelihood (EL), exponentially tilting (ET) and continuously updating estimators (CUE). These correspond to $\rho(v) = \ln(1-v)$ for $v < 1$, $\rho(v) = -\exp(v)$ and $\rho(v) = -v^2/2 - v$ respectively, all of which are members of the Cressie and Read (1984) family, $\rho(v) = \frac{-1}{\gamma+1}(1+\gamma v)^{\frac{\gamma+1}{\gamma}}$; see also NS.

Assume further that

Assumption 3 (a) $\beta_0 \in \mathscr{B}$ is the unique solution to $\mathrm{E}[g(X_i;\beta)] = 0$, $\mathscr{B}$ is compact and β_0 is in the interior of $\mathscr{B}$; (b) $g(x,\beta)$ is continuous at each $\beta \in \mathscr{B}$ with probability one and $\mathrm{E}\left[\sup_{\beta \in \mathscr{B}} \|g(X_i,\beta)\|^2\right] < \infty$; (c) $\Omega = \mathrm{E}\left[g(X_i;\beta_0) g(X_i;\beta_0)^{\mathsf{T}}\right]$ is nonsingular; (d) in an open ball $\mathrm{B}_r(\beta_0)$ around β_0, $\mathrm{E}\left[\sup_{\beta \in \mathrm{B}_r(\beta_0)} \|g(X_i,\beta)\|^6\right] < \infty$, $\mathscr{D}_\beta g(X_i;\beta)$ is continuous, $\mathrm{E}\left[\sup_{\beta \in \mathrm{B}_r(\beta_0)} \|\mathscr{D}_\beta g(X_i,\beta)\|^2\right] < \infty$, , and there exists $d(x)$ with $\mathrm{E}[d(X)] < \infty$ such that for each $\beta \in \mathrm{B}_r(\beta_0)$, $\|\mathscr{H}_\beta g(x,\beta) - \mathscr{H}_\beta g(x,\beta_0)\| \leq d(x)\|\beta - \beta_0\|$, and $\mathrm{E}\left[\sup_{\beta \in \mathrm{B}_r(\beta_0)} \|\mathscr{H}_\beta g(X_i,\beta)\|^2\right] < \infty$; (e) $G = \mathrm{E}[\mathscr{D}_\beta g(X_i;\beta_0)]$ has rank p; (f) $\rho(v)$ is four times continuously differentiable with Lipschitz fourth derivative in a neighbourhood of zero.

The class of GEL criteria considered here is defined as

$$P_n(\lambda,\beta) = n^{-1} \sum_{i=1}^n \rho\left(\lambda^{\mathsf{T}} g_i(\beta)\right) - \rho(0), \tag{5}$$

where $g_i(\beta) = g(x_i;\beta)$. The estimator of β, $\widehat{\beta}$, solves the saddle point problem

$$\widehat{\beta} = \underset{\beta \in \mathscr{B}}{\operatorname{argmin}} \sup_{\lambda \in \Lambda_n(\beta)} P_n(\lambda,\beta), \tag{6}$$

where $\Lambda_n(\beta) = \left\{\lambda : \lambda^{\mathsf{T}} g_i(\beta) \in \mathscr{V},\ i = 1, \ldots, n\right\}$. For given β, the vector of auxiliary parameters (Lagrange multipliers), $\widehat{\lambda} = \widehat{\lambda}(\beta)$, solves the first-order conditions

$$Q_{\lambda,n}(\widehat{\lambda}(\beta)) = n^{-1} \sum_{i=1}^n \rho^{(1)}\left(\widehat{\lambda}^{\mathsf{T}} g_i(\beta)\right) g_i(\beta) = 0. \tag{7}$$

The *implied probabilities* are then defined as

$$\hat{\pi}_i = \rho^{(1)}\left(\widehat{\lambda}^{\mathsf{T}} g_i\left(\widehat{\beta}\right)\right) \Big/ \sum_{j=1}^n \rho^{(1)}\left(\widehat{\lambda}^{\mathsf{T}} g_i\left(\widehat{\beta}\right)\right). \tag{8}$$

By construction, the $\hat{\pi}_i$'s sum to unity over $i = 1, \ldots, n$. Furthermore, the first order conditions imply that $\sum_{i=1}^n \hat{\pi}_i g_i\left(\widehat{\beta}\right) = 0$. It is this latter property that eliminates the first term in (4) when the expectation is taken over the reweighted density estimator.

As shown in NS, $\widehat{\beta}$ is a consistent and asymptotically normal estimator of β_0, the solution to the inner optimisation in (6) when $\beta = \widehat{\beta}$ exists with probability approaching one, and $\widehat{\lambda} = O_p\left(n^{-1/2}\right)$. The latter result of course holds when β_0 is known.

If β_0 is known, only the inner optimisation in (6) has to be carried out, and the implied probabilities are defined by (8) with β_0 replacing $\widehat{\beta}$.

GEL allows the construction of the tests for overidentifying restrictions that are similar to the classical likelihood ratio, Wald, and Lagrange multiplier tests. For example, the normalised GEL criterion evaluated at the estimated parameters $\widehat{\beta}$ and $\widehat{\lambda}$, $2n\widehat{P}_n\left(\widehat{\lambda}, \widehat{\beta}\right)$, possesses a chi-square limiting distribution with $q-p$ degrees of freedom, χ^2_{q-p}. Other test statistics can be constructed as described in, inter alia, Smith (1997), Kitamura and Stutzer (1997), and Ramalho and Smith (2005).

3.1 Computational aspects

It should be noted that the solution to (7) does not always exist. In particular, there is no solution when zero is not in $\mathrm{CH}\left(\{g_i(\beta)\}_{i=1}^n\right)$, the convex hull of $\{g_1(\beta), \ldots, g_n(\beta)\}$; see e.g. Kitamura (2006, sec. 8.1). When β_0 is known, it is only required that $\mathbf{0} \in \mathrm{CH}\left(\{g_i(\beta_0)\}_{i=1}^n\right)$, but when β is estimated, zero must be in the convex hull of $\{g_i(\beta)\}_{i=1}^n$ for all β at which the GEL criterion is evaluated.

Example 2 Let X_i, $i = 1, \ldots, n$, be an i.i.d. sample from a standard normal distribution and $g(x_i; \beta_0) = x_i$; i.e. it is known that the mean is zero. Then with probability 2^{-n+1} in a sample of size n, the x_i's will be either all positive or all negative, and there will be no solution to (7); see also Qin and Lawless (1994, example 2).

It is interesting to note that when all the sample values are positive, $P_n(\lambda)$ is a decreasing function of λ for both EL and ET, and the maximum is achieved at $\widehat{\lambda} = -\infty$. (A similar argument applies to the case when all sample values are negative). The EL probabilities then become $\lim_{\lambda \to -\infty\ \lambda \in \Lambda_n} \hat{\pi}_i = 1/\left[x_i \sum_{j=1}^n (1/x_j)\right]$, and $\sum_{i=1}^n \pi_i x_i = n \Big/ \sum_{j=1}^n (1/x_j) = H_x$, the harmonic average of x_i's.

ET in this case assigns weight one to the smallest observation (assuming no ties in the data) and zero to all other data points. CUE avoids this problem, but at a cost that some of the implied probabilities are *negative*. □

One possibility then is to use *adjusted GEL*, whereby an artificial observation, $g_{n+1}(\beta)$, is added to the data such that zero is in the convex hull of $\{g_1(\beta), \ldots, g_n(\beta)\} \cup g_{n+1}(\beta)$. In particular, adding $g_{n+1}(\beta) = -a_n \widehat{g}(\beta)$, where $\widehat{g}(\beta) = n^{-1}\sum_{i=1}^n g_i(\beta)$ and $a_n > 0$ ensures that $0_q \in \mathrm{CH}\left(\{g_1(\beta), \ldots, g_n(\beta), g_{n+1}(\beta)\}\right)$; see Chen, Variyath, and Abraham (2008) and Liu and Chen (2010, sec. 3). Their suggestion is to set $a_n = \max(1, \ln(n)/2)$ and to use a trimmed mean of the $g_i(\beta)$'s in place of $\widehat{g}(\beta)$ if desired.

In computations performed in section 5 the following approach was employed:

1. β_0 known.

   IF $0_q \in \mathrm{CH}\left(\{g_i(\beta_0)\}_{i=1}^n\right)$ use unadjusted GEL;

   ELSE use adjusted GEL.

2. β_0 unknown.

   ◦ Obtain a preliminary estimate of β, $\widehat{\beta}_{init}$, by GMM or another appropriate method;

   IF $0_q \notin \mathrm{CH}\left(\{g_i(\beta_{init})\}_{i=1}^n\right)$ use adjusted GEL.

   ELSE try estimation using unadjusted GEL;

   IF unadjusted GEL fails, use adjusted GEL.

Finally, the outer optimisation can also be challenging as several local minima may exist; see e.g. Guggenberger (2008). Whilst in low dimensions, a grid search over β may be feasible, as the dimension of β becomes large, stochastic optimisation methods such as simulated annealing can be used, perhaps combined with a direct search near the final value.

4 GEL-based KDE

The GEL-based KDE (GELKDE) is obtained by replacing the empirical probabilities n^{-1} by the implied probabilities (8), i.e.

$$\tilde{f}_\rho(x) = \sum_{i=1}^n \hat{\pi}_i K_H(x - x_i). \tag{9}$$

Because the GEL weights, $\hat{\pi}_i$, are not always non-negative, $\tilde{f}_\rho(x)$ may also take negative values (typically, in the tails of the distribution). In this case, one can 'shrink'[1] the implied probabilities, for example, by transforming to

$$\begin{aligned}\hat{\pi}_i^\star &= \frac{1}{1+\epsilon_n}\hat{\pi}_i + \frac{\epsilon_n}{1+\epsilon_n}\cdot\frac{1}{n},\\ &= \frac{\hat{\pi}_i + \epsilon_n/n}{\sum_{i=1}^n\left(\hat{\pi}_i + \epsilon_n/n\right)}, \qquad \text{where } \epsilon_n = -n\min\left[\min_{1\le i\le n}\hat{\pi}_i\,, 0\right];\end{aligned}$$

see Smith (2011) and Antoine, Bonnal, and Renault (2007). Consequently, $\hat{\pi}_i^\star \geq 0$ and sum to one by construction, thus ensuring that $\tilde{f}_\rho(\cdot)$ is a proper density.

Because $\sum_{i=1}^n \hat{\pi}_i g_i\left(\hat{\beta}\right) = 0$, GELKDE approximates the constraints (2) better than RPKDE. For example, in the univariate case, since $\mathrm{E}_{\tilde{f}_\rho} g_l\left(X;\beta\right) = \sum_{i=1}^n \hat{\pi}_i \int g\left(x_i + zh;\beta\right) K(z)dz$, when the mean is known to be μ, $g\left(x_i;\mu\right) = x_i - \mu$, $\mathrm{E}_{\tilde{f}_\rho}\left(X\right) = \sum_{i=1}^n \hat{\pi}_i x_i = \mu$, i.e. the constraint is satisfied exactly (provided the solution to (7) exists). Note also that $\mathrm{E}_{\tilde{f}_\rho}\left(X^2\right) = \sum_{i=1}^n \hat{\pi}_i x_i^2 + h^2\mu_2(K)$. Hence if the constraint is $\mathrm{E}\left(X^2\right) = m^2$, say, although it will not be met exactly, $\mathrm{E}_{\tilde{f}_\rho}\left(X^2\right) = m^2 + h^2\mu_2(K)$, the GELKDE approximates this constraint better than RPKDE; cf. Example 1.

For general $g\left(x_i;\beta\right)$, $l = 1,\ldots,q$, $\beta \in \mathrm{B}_r\left(\beta_0\right)$, one obtains

$$\int g_l\left(x;\beta\right)\tilde{f}_\rho(x)dx = \frac{1}{2}\mu_2(K)\,\mathrm{tr}\left(HJ_l\left(\beta\right)\right) + o_p\left(\mathrm{tr}\left(H\right)\right). \tag{10}$$

Note that the first term in (10) is the same as the second term in (4), whereas the first term in (4) vanishes. Hence in general GELKDE provides a better approximation to moment conditions than RPKDE.

4.1 Bias and variance

Since the GEL estimator is defined implicitly, the exact expectation of GELKDE cannot be obtained. Hence, an asymptotic analysis is required.

Let $\Sigma = \left(G^\mathsf{T}\Omega^{-1}G\right)^{-1}$, $Q = \Sigma G^\mathsf{T}\Omega^{-1}$, $P = \Omega^{-1} - \Omega^{-1}GQ$, $\Psi = \mathrm{E}\left[\left(g_i\left(\beta_0\right)g_i\left(\beta_0\right)^\mathsf{T}\right)\otimes g_i\left(\beta_0\right)^\mathsf{T}\right]$, $\Gamma = \mathrm{E}\left[\mathscr{H}_\beta^\circ g_i\left(\beta_0\right)\right]$, $T = \mathrm{E}\left[g_i\left(\beta_0\right)^\mathsf{T}\otimes\mathscr{D}_\beta g_i\left(\beta_0\right)\right]$, and $k_\rho = 1 + \rho_0^{(3)}/2$. Remark that $k_\rho = 0$ for EL and any other carrier function with $\rho_0^{(3)} = -2$, $1/2$ for ET, and 1 for CUE.

Theorem 1 *If Assumptions 1, 2, and 3 hold, then*

$$\mathrm{E}\,\tilde{f}_\rho(x) = \mathrm{E}\,\hat{f}(x) + n^{-1}\left(k_\rho B_1(x) + B_2(x)\right)f(x) + O\left(n^{-1}\,\mathrm{tr}\left(H\right)\right), \tag{11}$$

where

$$\begin{aligned}B_1(x) &= -g^\mathsf{T}(x;\beta_0)Pg(x;\beta_0) + \mathrm{vec}\left(P\right)^\mathsf{T}\Psi^\mathsf{T}Pg\left(x;\beta_0\right) + q - p;\\ B_2(x) &= \mathrm{vec}\left(Q\right)^\mathsf{T}T^\mathsf{T}Pg\left(x;\beta_0\right) - \frac{1}{2}\mathrm{vec}\left(\Sigma\right)^\mathsf{T}\Gamma^\mathsf{T}Pg\left(x;\beta_0\right).\end{aligned}$$

$$\mathrm{Var}\,\tilde{f}_\rho(x) = \mathrm{Var}\,\hat{f}(x) - n^{-1}g^\mathsf{T}\left(x;\beta_0\right)Pg\left(x;\beta_0\right)f^2(x) + O\left(n^{-1}\,\mathrm{tr}\left(H\right)\right). \tag{12}$$

$$\mathrm{MISE}\,\tilde{f}_\rho(\cdot) = \mathrm{MISE}\,\hat{f}(\cdot) - n^{-1}C_{ivar} + O\left(n^{-1}\,\mathrm{tr}\left(H\right)\right), \tag{13}$$

where $C_{ivar} = \int g^\mathsf{T}\left(x;\beta_0\right)Pg\left(x;\beta_0\right)f^2(x)dx$.

If β_0 *is known, equations* (11), (12), *and* (13) *are valid with* $p = 0$, $P = \Omega^{-1}$,

$$B_1(x) = -g^\mathsf{T}(x;\beta_0)\Omega^{-1}g(x;\beta_0) + \mathrm{vec}\left(\Omega^{-1}\right)^\mathsf{T}\Psi^\mathsf{T}\Omega^{-1}g\left(x;\beta_0\right) + q,$$

and $B_2(x) = 0$. □

The proof is given in the Supplement. For the univariate case, Oryshchenko (2011, Prop. 3&4) derives approximations up to an order $o\left(n^{-1}h^2\right)$ listing the terms of order $n^{-1}h^2$ explicitly.

An immediate consequence of Theorem 1 is that the asymptotically optimal bandwidth remains unchanged since the leading terms in MISE $\tilde{f}_\rho(\cdot)$ are the same as in MISE $\hat{f}(\cdot)$.

If β_0 is known, then whenever $k_\rho = 0$ (e.g. for EL), the n^{-1} bias term in (11) vanishes. The derivations indicate that under sufficient smoothness EL-based KDE will have the same expectation as $\hat{f}$, asymptotically, to a higher order than $O\left(n^{-1}h^2\right)$.

[1] Alternatively, one can simply take a positive part of $\tilde{f}_\rho(x)$, $\tilde{f}_\rho^+(x) = \max\left(\tilde{f}_\rho(x),\,0\right)$, to be the final estimate. In this case $\tilde{f}_\rho^+(\cdot)$ should be renormalized to ensure it integrates to one. However, as the latter is computationally difficult, one may prefer to shrink the implied probabilities if any are negative.

As C_{ivar} is non-negative, there is always an $1/n$ *reduction* in variance, which does *not* depend on the GEL carrier function. Asymptotically the effect entering via variance dominates and GELKDE enjoys a $1/n$ reduction in mean integrated squared error.

The relative difference between $\operatorname{ISB} \tilde{f}_\rho(\cdot)$ and $\operatorname{ISB} \hat{f}(\cdot)$, $\Delta\% \operatorname{ISB}_{\tilde{f}_\rho,\hat{f}} = \left[\operatorname{ISB} \tilde{f}_\rho(\cdot) - \operatorname{ISB} \hat{f}(\cdot)\right] \Big/ \operatorname{ISB} \hat{f}(\cdot) \sim n^{-1} \operatorname{tr}(H)^{-1}$, whereas $\Delta\% \operatorname{IVar}_{\tilde{f}_\rho,\hat{f}} \sim \det(H)^{1/2}$. With asymptotically optimal bandwidth, $n^{-1}\det(H)^{-1/2} \sim \operatorname{tr}(H)^2$, $\Delta\% \operatorname{ISB}_{\tilde{f}_\rho,\hat{f}}$ vanishes at a faster rate than $\Delta\% \operatorname{IVar}_{\tilde{f}_\rho,\hat{f}}$.

Provided $q > p$, these results hold in the case when β is estimated. However, unlike the known β_0 case, in general it is no longer true that the n^{-1} bias term may be set to zero for a particular choice of carrier function as in general $B_2(x) \neq 0$.

4.2 Univariate density

Let $K(\cdot)$ be a symmetric second-order univariate kernel with $\mu_6(K) < \infty$ and $\mu_4(K^2) < \infty$, where $\mu_j(\psi) = \int x^j \psi(x)dx$, and let X be a scalar random variable with density $f(x)$. Then under sufficient smoothness of $f(x)$, it can be shown that

$$\operatorname{ISB} \hat{f}(\cdot) = \frac{1}{4}h^4\mu_2^2(K)R\left(f^{(2)}(x)\right) + \frac{1}{24}h^6\mu_2(K)\mu_4(K)\int f^{(2)}(x)f^{(4)}(x)dx + O\left(h^8\right),$$

$$\operatorname{IVar} \hat{f}(\cdot) = (nh)^{-1}R(K) - n^{-1}R(f) + \frac{1}{2}n^{-1}h\mu_2\left(K^2\right)\int f^{(2)}(x)dx - n^{-1}h^2\mu_2(K)\int f(x)f^{(2)}(x)dx + O\left(n^{-1}h^3\right),$$

and the asymptotically optimal bandwidth is $h_{AMISE} = cn^{-1/5}$, where $c = \left[R(K)\big/\left(\mu_2^2(K)R\left(f^{(2)}\right)\right)\right]^{1/5}$. Furthermore,

$$\operatorname{ISB} \tilde{f}_\rho(\cdot) = \operatorname{ISB} \hat{f}(\cdot) + n^{-1}h^2\mu_2(K)C_{isb} + O\left(n^{-3/2}\right),$$

$$\operatorname{IVar} \tilde{f}_\rho(\cdot) = \operatorname{IVar} \hat{f}(\cdot) - n^{-1}C_{ivar} + n^{-1}h^2\mu_2(K)C^\star_{ivar} + O\left(n^{-3/2}\right),$$

where $C_{isb} = \int \left(k_\rho B_1(x) + B_2(x)\right) f(x)f^{(2)}(x)dx$ and the terms entering $C^\star_{ivar}$ are given explicitly in Oryshchenko (2011, Eq.(3.16d)). Hence,

$$\Delta\% \operatorname{ISB}_{\tilde{f}_\rho,\hat{f}} = \frac{4C_{isb}}{\mu_2(K)R\left(f^{(2)}\right)}n^{-1}h^{-2} + O\left(n^{-3/2}h^{-4}\right). \tag{14a}$$

$$\Delta\% \operatorname{IVar}_{\tilde{f}_\rho,\hat{f}} = -\frac{C_{ivar}}{R(K)}h - \frac{C_{ivar}R(f)}{R(K)^2}h^2 + O\left(h^3\right). \tag{14b}$$

With $h \sim n^{-1/5}$, $\Delta\% \operatorname{ISB}_{\tilde{f}_\rho,\hat{f}}$ vanishes at rate $n^{-3/5}$, whereas $\Delta\% \operatorname{IVar}_{\tilde{f}_\rho,\hat{f}}$, and hence $\Delta\% \operatorname{MISE}_{\tilde{f}_\rho,\hat{f}}$, approach zero at a much slower rate, $n^{-1/5}$. Specifically, when $h = h_{AMISE}$,

$$\Delta\% \operatorname{MISE}_{\tilde{f}_\rho,\hat{f}} = -\frac{4}{5}\frac{C_{ivar}}{R(K)}cn^{-1/5} - \frac{16}{25}\frac{C_{ivar}R(f)}{R(K)^2}c^2n^{-2/5} + O\left(n^{-3/5}\right), \tag{14c}$$

where the $O\left(n^{-3/5}\right)$ term depends, inter alia, on C_{isb}, entering with a positive sign.

Example 3 Suppose that X_i, $i = 1, \ldots, n$, is an i.i.d. sample from the Gaussian distribution, and that a Gaussian kernel is used, i.e. $f(x) = \phi(x)$ and $K(x) = \phi(x)$, where $\phi(\cdot)$ denotes the standard normal density. In this case $R(\phi) = \frac{1}{2\sqrt{\pi}}$, $R(\phi^{(2)}) = \frac{3}{8\sqrt{\pi}}$, and $\mu_2(\phi) = 1$, and the constant in the asymptotically optimal bandwidth is $c = (4/3)^{1/5} \approx 1.0592$. Then,

$$\Delta\% \operatorname{ISB}_{\tilde{f}_\rho,\hat{f}} = \frac{32\sqrt{\pi}C_{isb}}{3}n^{-1}h^{-2} + O\left(n^{-3/2}h^{-4}\right);$$

$$\Delta\% \operatorname{IVar}_{\tilde{f}_\rho,\hat{f}} = -2\sqrt{\pi}C_{ivar}h - 2\sqrt{\pi}C_{ivar}h^2 + O\left(h^3\right).$$

Table 1 presents the constants C_{isb} and C_{ivar} for five examples of moment constraints. Remark that in scenarios

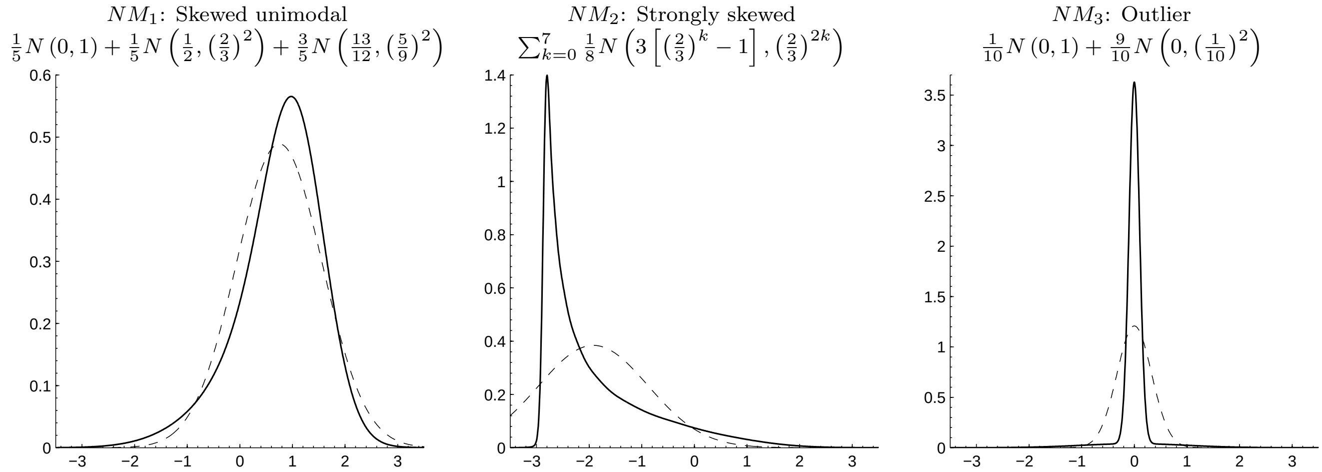
Table 1: Leading constants for GEL-KDE with Gaussian data

Moment constraints	Leading constants	
	C_{isb}	C_{ivar}
1. Known mean	$-3k_\rho/(8\sqrt{\pi})$	$1/(4\sqrt{\pi})$
2. Known mean and variance	$15k_\rho/(32\sqrt{\pi})$	$7/(16\sqrt{\pi})$
3. Known mean and third moment	$-41k_\rho/(64\sqrt{\pi})$	$13/(32\sqrt{\pi})$
4. Unknown mean and known variance	$15k_\rho/(32\sqrt{\pi}) + 3/(16\sqrt{\pi})$	$3/(16\sqrt{\pi})$
5. Unknown mean and known third central moment	$-17k_\rho/(64\sqrt{\pi})$	$5/(32\sqrt{\pi})$

2 and 4 reweighting leads to an increase in the integrated squared bias. In particular, in scenario 2, with h_{AMISE}, $\Delta\%\,\mathrm{ISB}_{\tilde{f}_\rho,\hat{f}} \approx 4.5k_\rho n^{-3/5}$ and $\Delta\%\,\mathrm{IVar}_{\tilde{f}_\rho,\hat{f}} \approx -0.93n^{-1/5} - 0.98n^{-2/5}$. Thus, for example, with $n = 100$, ISB will increase by about 28% for CUE and by about 14% for ET-based KDE, whereas IVar will decrease by about 53% for all GEL-based estimators. □

The following example examines some departures from normality.

Example 4 Mixtures of normal densities can approximate many interesting densities thus providing a powerful tool to study the performance of kernel estimators. The three densities selected here are the skewed unimodal, the strongly skewed, and the outlier densities shown in Figure 1 alongside the normal density with the same mean and variance plotted for reference. These are, respectively, densities #2, #3, and #5 of Marron and Wand (1992). Densities NM_1 and NM_2 are constructed to resemble the extreme value and the lognormal densities respectively. The outlier density is similar to the normal but with 10% of observations being strong outliers.



Legend: solid line—normal mixture density, dashed line—normal density with the same mean and variance as NM_j.

Figure 1: Selected normal mixture densities

For each mixture density Table 2 lists the mean, μ, variance, σ^2, third central moment, m_2, as well as $R(f)$, $R\left(f^{(2)}\right)$, and the constant in the asymptotically optimal bandwidth, c. The leading constants, C_{isb} and C_{ivar}, for the case when the mean in known are given in the last two columns.

Table 2: Leading constants for GEL-KDE of selected normal mixture densities

Density	Moments			$R(f)$	$R\left(f^{(2)}\right)$	c	Leading constants	
	μ	σ^2	m_3				C_{isb}	C_{ivar}
NM_1	0.75	0.6657	-0.3968	0.3768	1.9630	0.6784	-0.4562	0.1600
NM_2	-1.9189	1.0778	1.6352	0.5569	7508.5	0.1303	13.8243	0.3135
NM_3	0	0.109	0	2.3592	17137.5	0.1105	-120.2040	0.1242

Remark that qualitative predictions for the moderately skewed density NM_1 are similar to those for the normal density (scenario 1 in Example 3). For the strongly skewed density, however, the ISB increases, and the relative change in variance is small. Thus, $\Delta\%\,\mathrm{MISE}_{\tilde{f}_\rho,\hat{f}} = -0.1159n^{-1/5} - 0.0238n^{-2/5} + O\left(n^{-3/5}\right)$, which predicts only a 5% decrease in MISE with $n = 100$ and 3.1% decrease with $n = 1000$. A very moderate decrease in MISE is predicted for the outlier density, too. In this case, $\Delta\%\,\mathrm{MISE}_{\tilde{f}_\rho,\hat{f}} = -0.0389n^{-1/5} - 0.0288n^{-2/5} + O\left(n^{-3/5}\right)$, which predicts a 2% decrease in MISE with $n = 100$ and 1.2% with $n = 1000$. □

4.3 Bias correction

Although the contribution from the $1/n$ bias term in (11) to the MISE of GELKDE is of a lower order than the contribution from the variance, in small and moderate samples the bias effect can be substantial, offsetting the reduction in variance. Simulation evidence presented in the next section suggests that for moderate and large sample sizes the reduction in variance dominates, but in very small samples MISE may increase.

As the direction of the bias is not known *a priori*, unless the true density is known, it may be advisable to bias-correct GELKDE by estimating and subtracting the $1/n$ bias term. To be specific, the bias-corrected GELKDE is defined as

$$\tilde{f}^{bc}_\rho(x) = \tilde{f}_\rho(x) - n^{-1}\left(k_\rho \widetilde{B}_1(x) + \widetilde{B}_2(x)\right)\tilde{f}_\rho(x), \tag{15}$$

where $\widetilde{B}_1(x)$ and $\widetilde{B}_2(x)$ are suitable estimates of $B_1(x)$ and $B_2(x)$. We suggest using implied probabilities to obtain weighted estimators of quantities entering $B_1(x)$ and $B_2(x)$; see e.g. Smith (2011, sec. 3). To ensure that the bias-corrected estimate is a density, any negative values can be set to zero and renormalised as necessary.

4.4 Higher-order kernels

A common bias-reduction technique in kernel density estimation consists in employing higher-order kernels. Consider a univariate density f and a symmetric kernel $K(\cdot)$ such that $\mu_0(K) = 1$, $\mu_j(K) = 0$ for $j = 1, \dots, k-1$, and finite $\mu_k(K) \neq 0$, for some even $k \geq 2$. Then $K(\cdot)$ is said to be a k-th order kernel. Then it is easy to see that if f possesses a k-th derivative which is continuous and square integrable, then the bias of RPKDE is of order h^k, $\operatorname{E}\hat{f}(x) = f(x) + \frac{1}{k!}\mu_k(K)h^k f^{(k)}(x) + o\left(h^k\right)$, whereas the leading terms in the variance remain the same, viz. $\operatorname{Var}\hat{f}(x) = (nh)^{-1}R(K) - n^{-1}f^2(x) + o\left(n^{-1}\right)$. The asymptotically optimal bandwidth is proportional to $n^{-1/(2k+1)}$, and $\operatorname{MISE}\hat{f} \sim n^{-1/[1+1/(2k)]}$, which can be set arbitrarily close to n^{-1} as $k \to \infty$. It should be noted that kernels of order higher than two necessarily take negative values, and thus the resulting density estimate can take negative values.

From the proof of Theorem 1 it is evident that $O\left(n^{-1}\right)$ terms in (11) and (12) remain when higher-order kernel is employed, but the reminder terms become of a smaller order. Hence, the main results remain valid.

5 Monte-Carlo study

The small sample performance of GELKDE in situations presented in Examples 3 and 4 is investigated here via simulation. The exact MISE of the unweighted KDE is evaluated analytically (Fryer, 1976; Marron and Wand, 1992), and the exact MISE-minimising bandwidth, h_{MISE}, is used throughout. The setup is thus the most favourable for PRKDE. All results are based on $500,000$ replications. Multiple-segment trapezoidal rule numerical integration over a grid of $1,000$ points on the $[-4, 4]$ interval is used to obtain the ISB, IVar and MISE of GELKDE; the simulated values for the unweighted KDE are checked against the exact values. In all cases, implied probabilities are shrunk where necessary to ensure $\tilde{f}$ is nonnegative. More detailed simulation results are given in the Supplement.

For each of the cases presented in Examples 3 and 4, the performance of the unweighted KDE is compared to the GEL-based estimators (9) using the three special cases of GEL: EL, ET, and CUE. In the figures these are marked by a triangle, $\triangledown$, square, $\square$, and circle, $\bigcirc$, respectively; the shaded markers correspond to the bias-corrected GEL-based estimators, (15). Remark than when the parameters are known, the $1/n$ bias term of the EL-based KDE is zero, and hence, no bias-correction is performed.

Figures 2 and 3 show simulation results for scenarios 1–3 and 4–5 of Example 3, and Figure 4—for scenarios in Example 4, respectively. In these figures horizontal axes show sample size on a common logarithm scale, and vertical axes show percentage change in ISB, IVar, and MISE on a linear scale.

The results generally confirm the conclusions of the previous section. In the examples considered, the asymptotic approximation is good for samples of more than about 100 observations. In very small samples an increase in ISB can dominate the reduction in variance and lead to an increase in MISE. This is especially evident when the amount of overidentifying information is small and, additionally, estimation error contributes to the variance term (Figure 3). For instance, ISB of GELKDE is bigger than that of RPKDE in cases 2 and 4 in Example 3, but while in case 2 the reduction in variance offsets the effect of ISB in samples as small as 25 observations, in case 4 there is less extra information available (only variance is known, mean is estimated), and hence the bias effect dominates for $n < 100$.

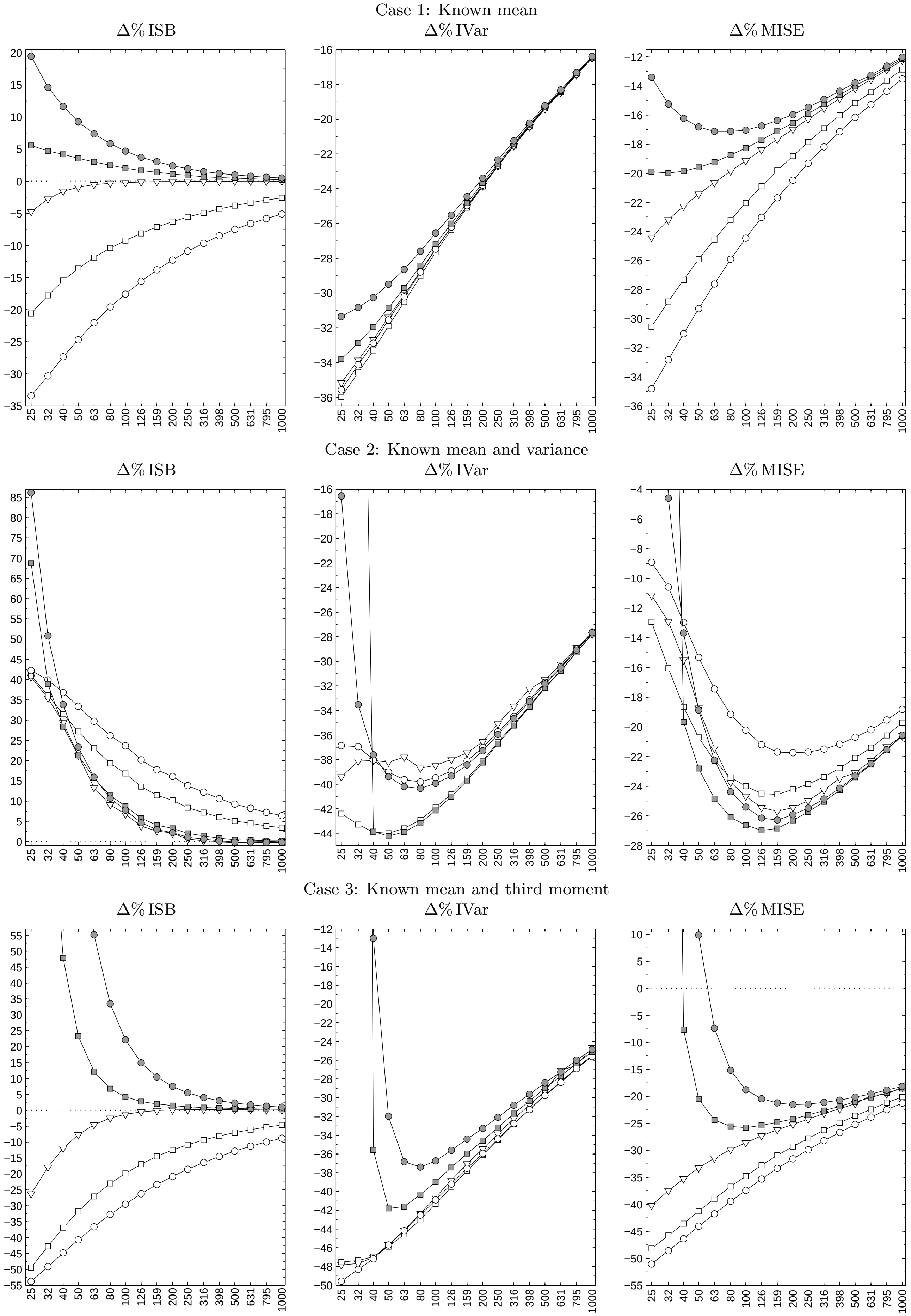
In general, neither of the considered GEL-based estimators dominate in small samples, although they are equivalent asymptotically. When $C_{isb} > 0$, EL should perform better asymptotically as the $1/n$ bias term is zero for EL. However, this is not necessarily true in small samples: in case 4, for instance, ET-based KDE has a lower MISE for sample sizes less than about 80 observations.

Bias-corrected GELKDE behave as EL-based KDE asymptotically. As evidenced in the simulation results, however, it may be ill-advised to bias-correct GELKDE when the sample size is very small, say, less than 100, and moment conditions involve higher moments. Remark that in scenario 5, Example 3, the $1/n$ bias term happens to be zero for EL, and both the bias-corrected and uncorrected EL-based KDEs have an almost identical ISB, but the bias-corrected KDE has a slightly higher variance.

The only additional information used by GEL-based estimators of the normal mixture densities[2] listed in Example 4 is that the mean is known. For the mildly skewed density, NM_1, the results are qualitatively similar to case 1 in Figure 2, but the reduction in MISE is smaller. For the strongly skewed density, NM_2, ISB increases and the reduction in MISE is quite small. With 25 observations MISE of the CUE-based KDE is 1.65% larger than that of the unweighted KDE; otherwise, the results are consistent with those suggested by the asymptotic analysis.

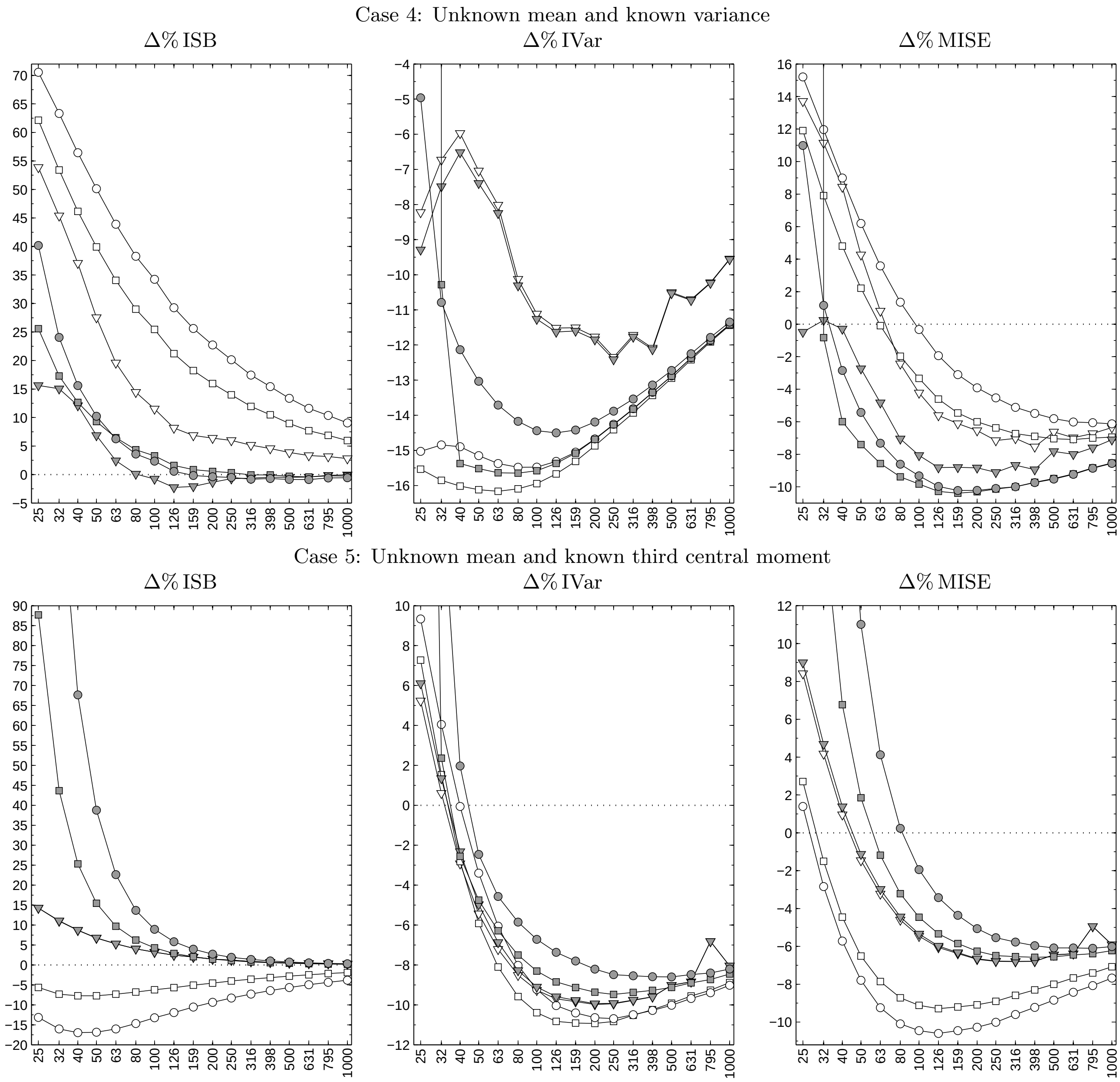
Perhaps the most interesting results are those for the outlier density. Relative to RPKDE, the EL-based KDE has a significantly larger variance and MISE for small and moderate sample sizes ($n < 200$). CUE and ET estimators,

[2]Oryshchenko (2011, Sec.3.5) also presents simulation results for the Student's t-distribution with degrees of freedom $\nu = 16, 8, 4$, and 2. Qualitatively, the performance of GELKDE there is similar to the case when the data is Gaussian. However, as the tails become heavier, the reduction in variance is smaller.



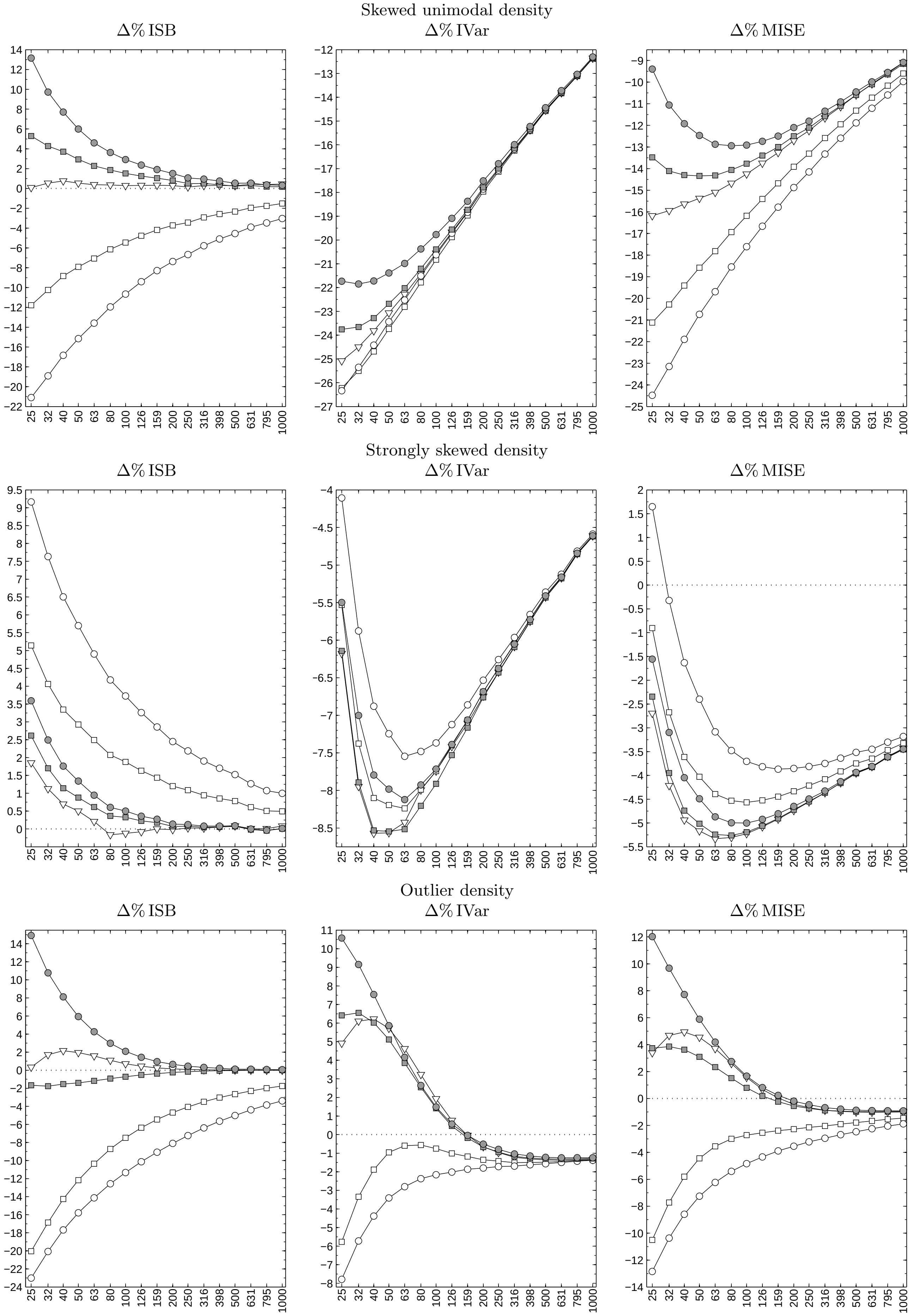
Legend: $\triangledown$—$\tilde{f}_{EL}$, $\square$—$\tilde{f}_{ET}$, $\bigcirc$—$\tilde{f}_{CUE}$; $\blacksquare$—$\tilde{f}^{bc}_{ET}$, $\bullet$—$\tilde{f}^{bc}_{CUE}$.
Horizontal axes—samples size, n, $\log_{10}$-scale; vertical axes—%, linear scale.

Figure 2: Performance of GELKDE with Gaussian data and known parameters



Legend: ▽—$\tilde{f}_{EL}$, □—$\tilde{f}_{ET}$, ○—$\tilde{f}_{CUE}$; ▼—$\tilde{f}^{bc}_{EL}$, ■—$\tilde{f}^{bc}_{ET}$, ●—$\tilde{f}^{bc}_{CUE}$.
Horizontal axes—samples size, n, $\log_{10}$-scale; vertical axes—%, linear scale.

Figure 3: Performance of GELKDE with Gaussian data and estimated parameters



Legend: $\triangledown$—$\tilde{f}_{EL}$, $\square$—$\tilde{f}_{ET}$, $\bigcirc$—$\tilde{f}_{CUE}$; $\blacksquare$—$\tilde{f}^{bc}_{ET}$, $\bullet$—$\tilde{f}^{bc}_{CUE}$.
Horizontal axes—samples size, n, $\log_{10}$-scale; vertical axes—%, linear scale.

Figure 4: Performance of GELKDE with non-Gaussian data and known mean

however, perform remarkably well, even though the achievable reduction in MISE is quite small in this case. This is consistent with the empirical evidence suggesting sensitivity of EL to outliers.

6 Conclusions

Additional information concerning the distribution of a random variable formulated in terms of moment conditions depending on a finite-dimensional parameter vector, which may or may not be known, can be incorporated by reweighting a kernel density estimate using implied GEL probabilities.

The resultant density estimator better approximates the moment conditions than the unweighted, Rosenblatt-Parzen, estimator. Furthermore, a reduction in variance is achieved due to the use of the extra moment information, provided that, if the parameter vector is unknown, it is overidentified. The effect on variance does not depend on the GEL carrier function and dominates the bias effect asymptotically. Simulation evidence suggests that the above conclusions hold in moderate and large samples, whereas in small samples bias can increase and dominate the reduction in variance. The bias of GELKDE depends on the carrier function; however, bias-corrected estimators may be formulated to eliminate the bias.

An extension of these methods for dependent processes may be of interest in economics and finance. Preliminary simulation evidence (Oryshchenko, 2011, App. 3.C) suggests that incorporating information about the dependence structure gives a reduction in variance and mean integrated squared error as compared with RPKDE. GEL methods need to be modified appropriately to deal with dependent data. One possibility is to use a version of GEL defined via smoothed moment indicators, developed in Smith (2011), which extends this class of estimators to weakly dependent data. Extensions to long-range dependence may be possible using frequency domain empirical likelihood; see e.g. Nordman and Lahiri (2006). Furthermore, GEL methods can be coupled with penalisation methods thus combining model selection and estimation steps; see inter alia Otsu (2007) and Shahidi (2009). This may be of particular relevance for dependent data when the dependence structure is unknown.

Other possible extensions include the estimation of *conditional* densities and nonparametric regression with extra moment conditions. De Gooijer and Zerom (2003) propose an ad hoc reweighting of a Nadaraya-Watson estimator of a conditional density which is an improvement over the unweighted case and enjoys superior bias properties of the local linear smoother. In particular, EL is used to make the Nadaraya-Watson weights more resemble local linear weights. The encouraging results of this paper suggest that further extensions may be developed for the estimation of conditional densities.

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Supplement to 'Generalised empirical likelihood-based kernel density estimation'

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Contents

1 Proof of Theorem 1

1.1 Consistency and asymptotic normality of GEL

Assumption 1. *(a)* $\beta_0 \in \mathscr{B}$ *is the unique solution to* $\mathrm{E}\left[g(X_i,\beta)\right]=0$*; (b)* $\mathscr{B}$ *is compact; (c)* $g(x,\beta)$ *is continuous at each* $\beta \in \mathscr{B}$ *with probability one; (d)* $\mathrm{E}\left[\sup_{\beta\in\mathscr{B}}\|g(X_i,\beta)\|^2\right]<\infty$*; (e)* $\Omega=\mathrm{E}\left[g(X_i,\beta_0)g(X_i,\beta_0)^{\mathsf{T}}\right]$ *is nonsingular; (f)* $\rho(v)$ *is twice continuously differentiable in a neighbourhood of zero.*

Except for (d), Assumption 1 is the same as Assumption 1 in NS. The requirement in NS Assumption 1(d) that $\mathrm{E}\left[\sup_{\beta\in\mathscr{B}}\|g(X_i,\beta)\|^{\alpha}\right]<\infty$ for some $\alpha>2$ can be relaxed to $\alpha=2$ using Lemma 3 of Owen (1990), which is reproduced below; see also Guggenberger and Smith (2005, p. 673).

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Lemma 1. *(Owen, 1990, Lemma 3.)*
Let $Y_i \geq 0$ be i.i.d. random variables. If $\mathrm{E}[Y_1^2] < \infty$, then

$$\max_{1\leq i\leq n} Y_i = o_p\left(n^{1/2}\right)$$

with probability one as $n \to \infty$.

Let $b_i = \sup_{\beta\in\mathscr{B}}\|g_i(\beta)\|$ and $\Lambda_n = \{\lambda :\ \|\lambda\| \leq n^{-1/2}\}$. Then by Assumption 1(d) and Lemma 1, $\max_{1\leq i\leq n} b_i = o_p\left(n^{1/2}\right)$. Hence, by Cauchy-Schwarz,

$$\sup_{\beta\in\mathscr{B},\ \lambda\in\Lambda_n,\ 1\leq i\leq n} |\lambda^\mathsf{T} g_i(\beta)| \leq n^{-1/2} \max_{1\leq i\leq n} b_i = o_p\left(1\right).$$

Lemma 2. *(Newey and Smith, 2004, Lemma A1)*
If Assumption 1 is satisfied, then for $\Lambda_n = \{\lambda :\ \|\lambda\| \leq n^{-1/2}\}$, $\sup_{\beta\in\mathscr{B},\ \lambda\in\Lambda_n,\ 1\leq i\leq n}|\lambda^\mathsf{T} g_i(\beta)| \xrightarrow{p} 0$, and with probability approaching one, $\Lambda_n \subseteq \widehat{\Lambda}_n(\beta)$ for all $\beta \in \mathscr{B}$.

Theorem 1. *(Newey and Smith, 2004, Theorem 3.1)*
If Assumption 1 is satisfied, then $\hat{\beta} \xrightarrow{p} \beta_0$, $\hat{g}(\hat{\beta}) = O_p\left(n^{-1/2}\right)$, $\hat{\lambda} = \operatorname{argmax}_{\lambda\in\widehat{\Lambda}_n(\hat{\beta})} P_n(\lambda, \hat{\beta})$ exists with probability approaching one, and $\hat{\lambda} = O_p\left(n^{-1/2}\right)$.

Assumption 2. *(a) $\beta_0 \in \operatorname{int}(\mathscr{B})$, the interior of $\mathscr{B}$; (b) $g(x, \beta)$ is continuously differentiable in a neigbourhood $\mathrm{B}_r\left(\beta_0\right)$ of β_0, and $\mathrm{E}\left[\sup_{\beta\in\mathrm{B}_r(\beta_0)}\|\mathscr{D}_\beta g(X_i, \beta)\|\right] < \infty$; (c) $G = \mathrm{E}\left[\mathscr{D}_\beta g(X_i, \beta_0)\right]$ has rank p.*

Theorem 2. *(Newey and Smith, 2004, Theorem 3.2)*
If Assumptions 1 and 2 are satisfied, then

$$n^{1/2}\begin{bmatrix}\widehat{\beta}-\beta_0\\ \widehat{\lambda}\end{bmatrix} \xrightarrow{d} N\left(0_{p+q},\ \begin{bmatrix}\Sigma & 0_{p\times q}\\ 0_{q\times p} & P\end{bmatrix}\right),$$

where $\Sigma = \left(G^\mathsf{T}\Omega^{-1}G\right)^{-1}$ and $P = \Omega^{-1} - \Omega^{-1}G\Sigma G^\mathsf{T}\Omega^{-1}$.

1.2 GEL expansions

Strengthen Assumptions 1(f) and 2(b) to

Assumption 3. *(a)-(i) $\mathrm{E}\left[\sup_{\beta\in\mathrm{B}_r(\beta_0)}\|g(X_i, \beta)\|^6\right] < \infty$, (ii) $\mathrm{E}\left[\sup_{\beta\in\mathrm{B}_r(\beta_0)}\|\mathscr{D}_\beta g(X_i, \beta)\|^2\right] < \infty$, and (iii) there exists $d(x)$ with $\mathrm{E}[d(X)] < \infty$ such that for each $\beta \in \mathrm{B}_r\left(\beta_0\right)$, $\|\mathscr{H}_\beta g(x, \beta) - \mathscr{H}_\beta g(x, \beta_0)\| \leq d(x)\|\beta - \beta_0\|$, and $\mathrm{E}\left[\sup_{\beta\in\mathrm{B}_r(\beta_0)}\|\mathscr{H}_\beta g(X_i, \beta)\|^2\right] < \infty$; (b) $\rho(v)$ is four times continuously differentiable with Lipschitz fourth derivative in a neighbourhood of zero.*

Recall that $Q = \Sigma G^\mathsf{T}\Omega^{-1}$, $\Psi = \mathrm{E}\left[\left(g_i(\beta_0)g_i(\beta_0)^\mathsf{T}\right) \otimes g_i(\beta_0)^\mathsf{T}\right]$, $\Gamma = \mathrm{E}\left[\mathscr{H}_\beta^\circ g_i(\beta_0)\right]$, $T = \mathrm{E}\left[g_i(\beta_0)^\mathsf{T} \otimes \mathscr{D}_\beta g_i(\beta_0)\right]$, and $T^\circ = \mathrm{E}\left[g_i(\beta_0)^\mathsf{T} \otimes \mathscr{D}_\beta^\mathsf{T} g_i(\beta_0)\right]$. Note that Ω, Σ, and P are symmetric matrices. Furthermore, note that $P\Omega P = P$, $Q\Omega Q^\mathsf{T} = \Sigma$, and $P\Omega Q^\mathsf{T} = 0_{q\times p}$.

For notational convenience, let $\widehat{g}_i = g_i(\widehat{\beta})$, $\hat{v}_i = \widehat{\lambda}^\mathsf{T}\widehat{g}_i$, $\widehat{g}_0 = \widehat{g}(\beta_0)$, $\widehat{\Omega} = n^{-1}\sum_{i=1}^n g_i(\beta_0)g_i(\beta_0)^\mathsf{T} - \Omega$, $G_i(\beta) = \mathscr{D}_\beta g_i(\beta)$, $\widehat{G} = n^{-1}\sum_{i=1}^n G_i(\beta_0) - G$, and similarly, $\Gamma_i(\beta) = \mathscr{H}_\beta^\circ g_i(\beta)$, and $\widehat{\Gamma} = n^{-1}\sum_{i=1}^n \Gamma_i(\beta_0) - \Gamma$. Also let $\widehat{T} = n^{-1}\sum_{i=1}^n g_i(\beta_0)^\mathsf{T} \otimes \mathscr{D}_\beta g_i(\beta_0) - T$, and $\widehat{\Psi} = n^{-1}\sum_{i=1}^n \left(g_i(\beta_0)g_i(\beta_0)^\mathsf{T}\right) \otimes g_i(\beta_0)^\mathsf{T} - \Psi$.

Lemma 3. *Let $X_1, X_2, \ldots,$ be independent, identically distributed random variables, $\mathrm{E}\,X_1 = 0$, and $0 < \mathrm{E}|X_1|^2 < \infty$. Put $S_n = \sum_{i=1}^n X_i$. Then $n^{-1/2}S_n = O_p\left(1\right)$.*

Proof. By independence, $\mathrm{E}\,X_iX_j = 0$ for $i \neq j$; then $\mathrm{E}|n^{-1/2}S_n|^2 = \mathrm{E}|X_1|^2$ for all n. Given $\varepsilon > 0$, for M large enough, $\mathrm{E}|X_1|^2/M^2 < \varepsilon$. Then by the Bienaymé-Chebyshev inequality, $\mathbb{P}\left(|n^{-1/2}S_n| > M\right) < \varepsilon$, i.e. the law of $n^{-1/2}S_n$ is uniformly tight. ■

Then by Lemma 3, invoking Assumption 3(a)-(i), $\widehat{\Omega}$ and $\widehat{\Psi}$ are $O_p\left(n^{-1/2}\right)$; and invoking Assumptions 3(a)-(ii) and (iii), $\widehat{G}$, $\widehat{\Gamma}$, and $\widehat{T}$ are $O_p\left(n^{-1/2}\right)$.

1.2.1 First order conditions

Consider the GEL first order conditions

$$Q_{\lambda,n} = n^{-1}\sum_{i=1}^n \rho^{(1)}(\hat{v}_i)\widehat{g}_i, \tag{1a}$$

$$Q_{\beta,n} = n^{-1}\sum_{i=1}^n \rho^{(1)}(\hat{v}_i)G_i\left(\widehat{\beta}\right)^\mathsf{T}\widehat{\lambda}. \tag{1b}$$

Expanding $\rho^{(1)}(\hat{v}_i)$ around $\hat{\lambda}=0$ zero gives $\rho^{(1)}(\hat{v}_i)=-1-\hat{v}_i+\frac{1}{2}\rho_0^{(3)}\hat{v}_i^2+\frac{1}{6}\rho^{(4)}\left(\bar{\lambda}^{\mathsf{T}}\widehat{g}_i\right)\hat{v}_i^3$, where $\rho_0^{(j)}=\rho^{(j)}(0)$, $\rho_0^{(1)}=\rho_0^{(2)}=-1$ by normalisation, and $\bar{\lambda}=t\widehat{\lambda}$, $t\in[0,1]$, is a point on a line segment joining $\hat{\lambda}$ and zero. By Assumption 3(b) and Lemma 2, for some constant $L>0$, $\max_{1\le i\le n}|\rho^{(4)}\left(\bar{\lambda}^{\mathsf{T}}\widehat{g}_i\right)-\rho_0^{(4)}|\le L\cdot\max_{1\le i\le n}|\bar{\lambda}^{\mathsf{T}}\widehat{g}_i|\xrightarrow{p}0$. Substituting into the first order conditions for λ gives

$$\begin{aligned}
Q_{\lambda,n}&=n^{-1}\sum_{i=1}^{n}\left[-1-\hat{v}_i+\frac{1}{2}\rho_0^{(3)}\hat{v}_i^2+\frac{1}{6}\rho_0^{(4)}\hat{v}_i^3\left(1+o_p\left(1\right)\right)\right]\widehat{g}_i\\
&=-\widehat{g}\left(\widehat{\beta}\right)-n^{-1}\sum_{i=1}^{n}\hat{v}_i\widehat{g}_i+\frac{1}{2}\rho_0^{(3)}n^{-1}\sum_{i=1}^{n}\hat{v}_i^2\widehat{g}_i+\frac{1}{6}\rho_0^{(4)}n^{-1}\sum_{i=1}^{n}\hat{v}_i^3\widehat{g}_i\left(1+o_p\left(1\right)\right)\\
&=-\widehat{g}\left(\widehat{\beta}\right)-n^{-1}\sum_{i=1}^{n}\widehat{g}_i\widehat{g}_i^{\mathsf{T}}\widehat{\lambda}+\frac{1}{2}\rho_0^{(3)}n^{-1}\sum_{i=1}^{n}\left[\left(\widehat{g}_i\widehat{g}_i^{\mathsf{T}}\right)\otimes\widehat{g}_i^{\mathsf{T}}\right]\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)+O_p\left(n^{-3/2}\right),
\end{aligned}$$

because $n^{-1}\sum_{i=1}^{n}\hat{v}_i^3\widehat{g}_i=O_p\left(n^{-3/2}\right)$ by Markov inequality, invoking Assumption 3(a)-(i) and Theorem 1.

Let $\widetilde{\beta}=\widehat{\beta}-\beta_0$ and $\bar{\beta}=\beta_0+t\left(\widehat{\beta}-\beta_0\right)$, $t\in[0,1]$, be a point on the line joining β_0 and $\widehat{\beta}$. As $\widehat{\beta}\xrightarrow{p}\beta_0$, and $\widetilde{\beta}=O_p\left(n^{-1/2}\right)$, an expansion around β_0 yields

$$\widehat{g}\left(\widehat{\beta}\right)=\widehat{g}_0+G\widetilde{\beta}+\widehat{G}\widetilde{\beta}+\frac{1}{2}\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right)+\frac{1}{2}\left[\widehat{\Gamma}+n^{-1}\sum_{i=1}^{n}\left(\Gamma_i(\bar{\beta})-\Gamma_i(\beta_0)\right)\right]\left(\widetilde{\beta}\otimes\widetilde{\beta}\right). \tag{2a}$$

By Assumption 3(a)-(iii), $n^{-1}\sum_{i=1}^{n}\|\Gamma_i(\bar{\beta})-\Gamma_i(\beta_0)\|\le n^{-1}\sum_{i=1}^{n}d(z_i)\|\bar{\beta}-\beta_0\|=O_p\left(n^{-1/2}\right)$. Hence, the last term in (2a) is $O_p\left(n^{-3/2}\right)$.

By the Mean Value Theorem, $\widehat{g}_i=g_i(\beta_0)+G_i(\bar{\beta})\widetilde{\beta}$. Writing $G_i(\beta)\widetilde{\beta}g_i(\beta)^{\mathsf{T}}=\left(g_i(\beta)^{\mathsf{T}}\otimes G_i(\beta)\right)\left(I_q\otimes\widetilde{\beta}\right)$, one obtains

$$\begin{aligned}
n^{-1}\sum_{i=1}^{n}\widehat{g}_i\widehat{g}_i^{\mathsf{T}}\widehat{\lambda}=\Omega\widehat{\lambda}+\widehat{\Omega}\widehat{\lambda}&+\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\widehat{\lambda}+T\left(I_q\otimes\widetilde{\beta}\right)\widehat{\lambda}\\
&+\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)\left\{\widehat{T}+n^{-1}\sum_{i=1}^{n}\left[g_i(\beta_0)^{\mathsf{T}}\otimes\left(G_i(\bar{\beta})-G_i(\beta_0)\right)\right]\right\}^{\mathsf{T}}\widehat{\lambda}\\
&+\left\{\widehat{T}+n^{-1}\sum_{i=1}^{n}\left[g_i(\beta_0)^{\mathsf{T}}\otimes\left(G_i(\bar{\beta})-G_i(\beta_0)\right)\right]\right\}\left(I_q\otimes\widetilde{\beta}\right)\widehat{\lambda}\\
&+n^{-1}\sum_{i=1}^{n}G_i(\bar{\beta})\widetilde{\beta}\widetilde{\beta}^{\mathsf{T}}G_i(\bar{\beta})^{\mathsf{T}}\widehat{\lambda}.
\end{aligned}\tag{2b}$$

The last three terms are $O_p\left(n^{-3/2}\right)$ by a similar argument. Finally,

$$n^{-1}\sum_{i=1}^{n}\left[\left(\widehat{g}_i\widehat{g}_i^{\mathsf{T}}\right)\otimes\widehat{g}_i^{\mathsf{T}}\right]\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)=\Psi\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)+O_p\left(n^{-3/2}\right). \tag{2c}$$

Combining terms gives (3a).

Note that one can write $\hat{v}_iG_i(\beta)^{\mathsf{T}}\widehat{\lambda}=\left(\widehat{g}_i^{\mathsf{T}}\otimes G_i(\beta)^{\mathsf{T}}\right)\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)$. Then by an analogous argument,

$$\begin{aligned}
Q_{\beta,n}&=n^{-1}\sum_{i=1}^{n}\left[-1-\hat{v}_i+\frac{1}{2}\rho_0^{(3)}\hat{v}_i^2\left(1+o_p\left(1\right)\right)\right]G_i\left(\widehat{\beta}\right)^{\mathsf{T}}\widehat{\lambda}\\
&=-n^{-1}\sum_{i=1}^{n}G_i\left(\widehat{\beta}\right)^{\mathsf{T}}\widehat{\lambda}-n^{-1}\sum_{i=1}^{n}\hat{v}_iG_i\left(\widehat{\beta}\right)^{\mathsf{T}}\widehat{\lambda}+\frac{1}{2}\rho_0^{(3)}n^{-1}\sum_{i=1}^{n}\hat{v}_i^2G_i\left(\widehat{\beta}\right)^{\mathsf{T}}\widehat{\lambda}\left(1+o_p\left(1\right)\right)\\
&=-G^{\mathsf{T}}\widehat{\lambda}-\widehat{G}^{\mathsf{T}}\widehat{\lambda}-\left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\widehat{\lambda}-T^{\circ}\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)\\
&\quad-\left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\widehat{\Gamma}^{\mathsf{T}}\widehat{\lambda}-\left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)n^{-1}\sum_{i=1}^{n}\left[\Gamma_i(\bar{\beta})-\Gamma\right]^{\mathsf{T}}\widehat{\lambda}-\widehat{T}^{\circ}\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)\\
&\quad-n^{-1}\sum_{i=1}^{n}\left[\widehat{g}_i^{\mathsf{T}}\otimes G_i\left(\widehat{\beta}\right)^{\mathsf{T}}-g_i(\beta_0)^{\mathsf{T}}\otimes G_i(\beta_0)^{\mathsf{T}}\right]\left(\widehat{\lambda}\otimes\widehat{\lambda}\right)\\
&\quad+\frac{1}{2}\rho_0^{(3)}n^{-1}\sum_{i=1}^{n}\hat{v}_i^2G_i\left(\widehat{\beta}\right)^{\mathsf{T}}\widehat{\lambda}\left(1+o_p\left(1\right)\right),
\end{aligned}$$

which gives (3b). Hence, up to an order $n^{-3/2}$, the first order conditions are

$$\begin{aligned} Q_{\lambda,n} &= -\Omega\widehat{\lambda} - \widehat{g}_0 - G\widetilde{\beta} - \widehat{G}\widetilde{\beta} - \widehat{\Omega}\widehat{\lambda} \\ &\quad - \frac{1}{2}\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right) - T\left(I_q\otimes\widetilde{\beta}\right)\widehat{\lambda} - \left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\widehat{\lambda} + \frac{1}{2}\rho_0^{(3)}\Psi\left(\widehat{\lambda}\otimes\widehat{\lambda}\right) + O_p\left(n^{-3/2}\right) \end{aligned} \tag{3a}$$

$$Q_{\beta,n} = -G^{\mathsf{T}}\widehat{\lambda} - \widehat{G}^{\mathsf{T}}\widehat{\lambda} - \left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\widehat{\lambda} - T^{\circ}\left(\widehat{\lambda}\otimes\widehat{\lambda}\right) + O_p\left(n^{-3/2}\right). \tag{3b}$$

If β_0 is known, expansion (3a) is valid with $\widetilde{\beta}=0$, i.e. $\widehat{\lambda}$ solves

$$Q_{\lambda,n} = -\Omega\widehat{\lambda} - \widehat{g}_0 - \widehat{\Omega}\widehat{\lambda} + \frac{1}{2}\rho_0^{(3)}\Psi\left(\widehat{\lambda}\otimes\widehat{\lambda}\right) + O_p\left(n^{-3/2}\right). \tag{3c}$$

1.2.2 Lagrange multipliers

From (3a), setting $Q_{\lambda,n}=0$ and premultiplying by Ω^{-1} gives

$$\begin{aligned} \widehat{\lambda} &= -\Omega^{-1}\widehat{g}_0 - \Omega^{-1}G\widetilde{\beta} - \Omega^{-1}\widehat{G}\widetilde{\beta} - \Omega^{-1}\widehat{\Omega}\widehat{\lambda} - \frac{1}{2}\Omega^{-1}\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right) + \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\left(\widehat{\lambda}\otimes\widehat{\lambda}\right) \\ &\quad - \Omega^{-1}T\left(I_q\otimes\widetilde{\beta}\right)\widehat{\lambda} - \Omega^{-1}\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\widehat{\lambda} + O_p\left(n^{-3/2}\right). \end{aligned}$$

Iteratively solving for $\widehat{\lambda}$ one obtains

$$\begin{aligned} -\Omega^{-1}\widehat{\Omega}\widehat{\lambda} &= \Omega^{-1}\widehat{\Omega}\Omega^{-1}\widehat{g}_0 + \Omega^{-1}\widehat{\Omega}\Omega^{-1}G\widetilde{\beta} + O_p\left(n^{-3/2}\right), \\ \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\left(\widehat{\lambda}\otimes\widehat{\lambda}\right) &= \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] + O_p\left(n^{-3/2}\right), \\ -\Omega^{-1}T\left(I_q\otimes\widetilde{\beta}\right)\widehat{\lambda} &= \Omega^{-1}T\left(I_q\otimes\widetilde{\beta}\right)\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) + O_p\left(n^{-3/2}\right), \\ -\Omega^{-1}\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\widehat{\lambda} &= \Omega^{-1}\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) + O_p\left(n^{-3/2}\right), \end{aligned}$$

and hence,

$$\begin{aligned} \widehat{\lambda} &= -\Omega^{-1}\widehat{g}_0 - \Omega^{-1}G\widetilde{\beta} - \Omega^{-1}\widehat{G}\widetilde{\beta} - \frac{1}{2}\Omega^{-1}\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right) + \Omega^{-1}\widehat{\Omega}\Omega^{-1}\widehat{g}_0 + \Omega^{-1}\widehat{\Omega}\Omega^{-1}G\widetilde{\beta} \\ &\quad + \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] + \Omega^{-1}T\left(I_q\otimes\widetilde{\beta}\right)\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) \\ &\quad + \Omega^{-1}\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) + O_p\left(n^{-3/2}\right). \end{aligned} \tag{4}$$

Substituting from (4) into (3b) gives

$$\begin{aligned} Q_{\beta,n} &= G^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 + G^{\mathsf{T}}\Omega^{-1}G\widetilde{\beta} + G^{\mathsf{T}}\Omega^{-1}\widehat{G}\widetilde{\beta} + \frac{1}{2}G^{\mathsf{T}}\Omega^{-1}\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right) - G^{\mathsf{T}}\Omega^{-1}\widehat{\Omega}\Omega^{-1}\widehat{g}_0 - G^{\mathsf{T}}\Omega^{-1}\widehat{\Omega}\Omega^{-1}G\widetilde{\beta} \\ &\quad - \frac{1}{2}\rho_0^{(3)}G^{\mathsf{T}}\Omega^{-1}\Psi\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] - G^{\mathsf{T}}\Omega^{-1}T\left(I_q\otimes\widetilde{\beta}\right)\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) \\ &\quad - G^{\mathsf{T}}\Omega^{-1}\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) + \widehat{G}^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 + \widehat{G}^{\mathsf{T}}\Omega^{-1}G\widetilde{\beta} + \left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 \\ &\quad + \left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}G\widetilde{\beta} - T^{\circ}\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] + O_p\left(n^{-3/2}\right) \end{aligned}$$

Setting $Q_{\beta,n}=0$ and premultiplying by $\Sigma = \left(G^{\mathsf{T}}\Omega^{-1}G\right)^{-1}$ gives

$$\begin{aligned} \widetilde{\beta} &= -Q\widehat{g}_0 - Q\widehat{G}\widetilde{\beta} - \frac{1}{2}Q\Gamma\left(\widetilde{\beta}\otimes\widetilde{\beta}\right) + Q\widehat{\Omega}\Omega^{-1}\widehat{g}_0 + Q\widehat{\Omega}\Omega^{-1}G\widetilde{\beta} + \frac{1}{2}\rho_0^{(3)}Q\Psi\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] \\ &\quad + QT\left(I_q\otimes\widetilde{\beta}\right)\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) + Q\left(I_q\otimes\widetilde{\beta}^{\mathsf{T}}\right)T^{\mathsf{T}}\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right) - \Sigma\widehat{G}^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 - \Sigma\widehat{G}^{\mathsf{T}}\Omega^{-1}G\widetilde{\beta} \\ &\quad - \Sigma\left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 - \Sigma\left(I_p\otimes\widetilde{\beta}^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}G\widetilde{\beta} + \Sigma T^{\circ}\left[\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\otimes\left(\Omega^{-1}\widehat{g}_0 + \Omega^{-1}G\widetilde{\beta}\right)\right] + O_p\left(n^{-3/2}\right). \end{aligned}$$

Since $\widetilde{\beta} = -Q\widehat{g}_0 + O_p\left(n^{-1}\right)$, iteratively solving for $\widetilde{\beta}$ gives

$$\begin{aligned} \widetilde{\beta} &= -Q\widehat{g}_0 + Q\widehat{G}Q\widehat{g}_0 - \frac{1}{2}Q\Gamma\left[(Q\widehat{g}_0)\otimes(Q\widehat{g}_0)\right] + Q\widehat{\Omega}\Omega^{-1}\widehat{g}_0 - Q\widehat{\Omega}\Omega^{-1}GQ\widehat{g}_0 - \Sigma\widehat{G}^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 + \Sigma\widehat{G}^{\mathsf{T}}\Omega^{-1}GQ\widehat{g}_0 \\ &\quad + \frac{1}{2}\rho_0^{(3)}Q\Psi\left[\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right)\otimes\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right)\right] - QT\left(I_q\otimes(Q\widehat{g}_0)\right)\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right) \\ &\quad - Q\left(I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right)T^{\mathsf{T}}\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right) + \Sigma\left(I_p\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 \\ &\quad - \Sigma\left(I_p\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right)\Gamma^{\mathsf{T}}\Omega^{-1}GQ\widehat{g}_0 + \Sigma T^{\circ}\left[\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right)\otimes\left(\Omega^{-1}\widehat{g}_0 - \Omega^{-1}GQ\widehat{g}_0\right)\right] + O_p\left(n^{-3/2}\right). \end{aligned}$$

Combining terms, one obtains

$$
\begin{aligned}
\widetilde{\beta} &= -Q\widehat{g}_0 + Q\widehat{G}Q\widehat{g}_0 - \frac{1}{2}Q\Gamma\left[(Q\widehat{g}_0)\otimes(Q\widehat{g}_0)\right] + Q\widehat{\Omega}P\widehat{g}_0 + \frac{1}{2}\rho_0^{(3)}Q\Psi\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right] \\
&\quad - \Sigma\widehat{G}^{\mathsf{T}}P\widehat{g}_0 + \Sigma T^{\circ}\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right] - QT\left[I_q\otimes(Q\widehat{g}_0)\right]P\widehat{g}_0 - Q\left[I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]T^{\mathsf{T}}P\widehat{g}_0 \\
&\quad + \Sigma\left[I_p\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]\Gamma^{\mathsf{T}}P\widehat{g}_0 + O_p\left(n^{-3/2}\right) \\
&= -Q\widehat{g}_0 + Q\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{G}\right)\operatorname{vec}(Q) - \frac{1}{2}Q\Gamma\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right) + Q\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{\Omega}\right)\operatorname{vec}(P) + \frac{1}{2}\rho_0^{(3)}Q\Psi\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) \\
&\quad - \Sigma\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{G}^{\mathsf{T}}\right)\operatorname{vec}(P) + \Sigma T^{\circ}\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) - QT\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) - \left[Q\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(T) \\
&\quad + \left[\Sigma\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(\Gamma) + O_p\left(n^{-3/2}\right)
\end{aligned} \tag{5}
$$

where the second equality follows because one can write $Q\widehat{G}Q\widehat{g}_0 = Q\operatorname{vec}\left(\widehat{G}Q\widehat{g}_0\right) = Q\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{G}\right)\operatorname{vec}(Q)$, and similarly for $Q\widehat{\Omega}P\widehat{g}_0$ and $\Sigma\widehat{G}^{\mathsf{T}}P\widehat{g}_0$; also, $Q\Psi\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right] = Q\Psi\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right)$, and similarly for $Q\Gamma\left[(Q\widehat{g}_0)\otimes(Q\widehat{g}_0)\right]$ and $\Sigma T^{\circ}\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right]$. Furthermore,

$$
\left[I_q\otimes(Q\widehat{g}_0)\right]P\widehat{g}_0 = \left[I_q\otimes(Q\widehat{g}_0)\right]\left[(P\widehat{g}_0)\otimes 1\right] = (P\widehat{g}_0)\otimes(Q\widehat{g}_0) = \operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right).
$$

Finally,

$$
\begin{aligned}
Q\left[I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]T^{\mathsf{T}}P\widehat{g}_0 &= \operatorname{vec}\left(Q\left[I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]T^{\mathsf{T}}P\widehat{g}_0\right) = \left\{\left(\widehat{g}_0^{\mathsf{T}}P\right)\otimes\left[Q\left(I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right)\right]\right\}\operatorname{vec}\left(T^{\mathsf{T}}\right) \\
&= \left\{\left(\widehat{g}_0^{\mathsf{T}}P\right)\otimes\left[Q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]\right\}\operatorname{vec}\left(T^{\mathsf{T}}\right) = \left\{(P\widehat{g}_0)\otimes\left[Q^{\mathsf{T}}\otimes(Q\widehat{g}_0)\right]\right\}^{\mathsf{T}}\operatorname{vec}\left(T^{\mathsf{T}}\right) \\
&= \left\{K_{q,qp}\left[Q^{\mathsf{T}}\otimes(Q\widehat{g}_0)\right]\otimes(P\widehat{g}_0)\right\}^{\mathsf{T}}\operatorname{vec}\left(T^{\mathsf{T}}\right) = \left\{Q\otimes\left[(Q\widehat{g}_0)\otimes(P\widehat{g}_0)\right]^{\mathsf{T}}\right\}K_{qp,q}\operatorname{vec}\left(T^{\mathsf{T}}\right) \\
&= \left[Q\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(T),
\end{aligned}
$$

and similarly, $\Sigma\left[I_p\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]\Gamma^{\mathsf{T}}P\widehat{g}_0 = \left[\Sigma\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(\Gamma)$.

Remark that

$$
\mathrm{E}\left[\widehat{\beta}\right] = \beta_0 + n^{-1}QT\operatorname{vec}(Q) - \frac{1}{2}n^{-1}Q\Gamma\operatorname{vec}(\Sigma) + n^{-1}\left(1+\rho_0^{(3)}/2\right)Q\Psi\operatorname{vec}(P) + O\left(n^{-3/2}\right),
$$

or $\operatorname{Bias}\left[\widehat{\beta}\right] = B_I + \left(1+\rho_0^{(3)}\right)B_\Omega$, where $B_I = -n^{-1}\frac{1}{2}Q\Gamma\operatorname{vec}(\Sigma) + n^{-1}QT\operatorname{vec}(Q)$ and $B_\Omega = n^{-1}Q\Psi\operatorname{vec}(P)$, cf. NS, Theorem 4.2.

Substituting from (5) into (4) gives

$$
\begin{aligned}
\widehat{\lambda} &= -P\widehat{g}_0 + P\widehat{\Omega}P\widehat{g}_0 + \frac{1}{2}\rho_0^{(3)}P\Psi\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right] + P\widehat{G}Q\widehat{g}_0 + Q^{\mathsf{T}}\widehat{G}^{\mathsf{T}}P\widehat{g}_0 - Q^{\mathsf{T}}T^{\circ}\left[(P\widehat{g}_0)\otimes(P\widehat{g}_0)\right] \\
&\quad - \frac{1}{2}P\Gamma\left[(Q\widehat{g}_0)\otimes(Q\widehat{g}_0)\right] - Q^{\mathsf{T}}\left[I_p\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]\Gamma^{\mathsf{T}}P\widehat{g}_0 \\
&\quad - PT\left[I_q\otimes(Q\widehat{g}_0)\right]P\widehat{g}_0 - P\left[I_q\otimes(Q\widehat{g}_0)^{\mathsf{T}}\right]T^{\mathsf{T}}P\widehat{g}_0 + O_p\left(n^{-3/2}\right) \\
&= -P\widehat{g}_0 + P\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{\Omega}\right)\operatorname{vec}(P) + \frac{1}{2}\rho_0^{(3)}P\Psi\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) + P\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{G}\right)\operatorname{vec}(Q) + Q^{\mathsf{T}}\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{G}^{\mathsf{T}}\right)\operatorname{vec}(P) \\
&\quad - Q^{\mathsf{T}}T^{\circ}\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) - \frac{1}{2}P\Gamma\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right) - \left[Q^{\mathsf{T}}\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(\Gamma) \\
&\quad - PT\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right) - \left[P\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\right]\operatorname{vec}(T) + O_p\left(n^{-3/2}\right).
\end{aligned} \tag{6a}
$$

If β_0 is known, expansion (6a) is valid with $Q=0$ and $P=\Omega^{-1}$, i.e.

$$
\begin{aligned}
\widehat{\lambda} &= -\Omega^{-1}\widehat{g}_0 + \Omega^{-1}\widehat{\Omega}\Omega^{-1}\widehat{g}_0 + \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\left[\left(\Omega^{-1}\widehat{g}_0\right)\otimes\left(\Omega^{-1}\widehat{g}_0\right)\right] + O_p\left(n^{-3/2}\right) \\
&= -\Omega^{-1}\widehat{g}_0 + \Omega^{-1}\left(\widehat{g}_0^{\mathsf{T}}\otimes\widehat{\Omega}\right)\operatorname{vec}\left(\Omega^{-1}\right) + \frac{1}{2}\rho_0^{(3)}\Omega^{-1}\Psi\operatorname{vec}\left(\Omega^{-1}\widehat{g}_0\widehat{g}_0^{\mathsf{T}}\Omega^{-1}\right) + O_p\left(n^{-3/2}\right).
\end{aligned} \tag{6b}
$$

1.2.3 Implied probabilities

The implied probabilities are defined as

$$
\hat{\pi}_i = \rho^{(1)}\left(\hat{v}_i\right)\Big/\sum_{j=1}^{n}\rho^{(1)}\left(\hat{v}_j\right).
$$

Using (2a) and (5), $\widehat{g}\left(\widehat{\beta}\right) = [I_q - GQ]\,\widehat{g}_0 + O_p\left(n^{-1}\right)$; and from (6a), $\widehat{\lambda} = -P\widehat{g}_0 + O_p\left(n^{-1}\right)$. Hence, expanding the denominator yields

$$\begin{aligned}\left[\sum_{i=1}^{n}\rho^{(1)}(\hat{v}_i)\right]^{-1} &= -\frac{1}{n} + \frac{1}{n}\widehat{\lambda}^{\mathsf{T}}\left[I_q - GQ\right]\widehat{g}_0 - \frac{1}{2}\rho_0^{(3)}\frac{1}{n}\widehat{\lambda}^{\mathsf{T}}\Omega\widehat{\lambda} + O_p\left(n^{-5/2}\right) \\ &= -n^{-1} - n^{-1}\left(1+\rho_0^{(3)}/2\right)\widehat{g}_0^{\mathsf{T}}P\widehat{g}_0 + O_p\left(n^{-5/2}\right).\end{aligned} \tag{7a}$$

As in subsection 1.2.1, the numerator is

$$\rho^{(1)}(\hat{v}_i) = -1 - \widehat{\lambda}^{\mathsf{T}}\widehat{g}_i + \frac{1}{2}\rho_0^{(3)}\widehat{g}_i^{\mathsf{T}}\widehat{\lambda}\widehat{\lambda}^{\mathsf{T}}\widehat{g}_i + \frac{1}{6}\rho_0^{(4)}\left(\widehat{\lambda}^{\mathsf{T}}\widehat{g}_i\right)^3\left(1 + o_p\left(1\right)\right).$$

Let g_i denote $g_i(\beta_0)$, and as before, $\bar{\beta} = \beta_0 + t(\widehat{\beta} - \beta_0)$, $t \in [0,1]$. Then $\widehat{g}_i^{\mathsf{T}}\widehat{\lambda}\widehat{\lambda}^{\mathsf{T}}\widehat{g}_i = g_i^{\mathsf{T}}\widehat{\lambda}\widehat{\lambda}^{\mathsf{T}}g_i + \left(g_i^{\mathsf{T}} \otimes G_i(\bar{\beta})\right)\cdot O_p\left(n^{-3/2}\right) + \left(g_i^{\mathsf{T}} \otimes G_i(\bar{\beta})\right)^{\mathsf{T}}\cdot O_p\left(n^{-3/2}\right)$, and $\widehat{\lambda}^{\mathsf{T}}\widehat{g}_i = \widehat{\lambda}^{\mathsf{T}}g_i + \widehat{g}_0^{\mathsf{T}}PG_i(\beta_0)Q\widehat{g}_0 + G_i(\beta_0)\cdot O_p\left(n^{-3/2}\right) + \Gamma_i(\bar{\beta})\cdot O_p\left(n^{-3/2}\right)$. Thus,

$$\rho^{(1)}(\hat{v}_i) = -1 - \widehat{\lambda}^{\mathsf{T}}g_i - \widehat{g}_0^{\mathsf{T}}PG_i(\beta_0)Q\widehat{g}_0 + \frac{1}{2}\rho_0^{(3)}g_i^{\mathsf{T}}P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Pg_i + R_n^*, \tag{7b}$$

where R_n^* contains $g_ig_i^{\mathsf{T}}$, $G_i(\beta_0)$, $\Gamma_i(\bar{\beta})$, $\left(g_i^{\mathsf{T}} \otimes G_i(\bar{\beta})\right)$, $\left(g_i^{\mathsf{T}} \otimes G_i(\bar{\beta})\right)$, and $(g_ig_i^{\mathsf{T}}) \otimes g_i^{\mathsf{T}}$ multiplying a $O_p\left(n^{-3/2}\right)$ term. Multiplying (7a) and (7b), and substituting for $\widehat{\lambda}$ from (6a) yields

$$\begin{aligned}\hat{\pi}_i = n^{-1} &- n^{-1}\widehat{g}_0^{\mathsf{T}}Pg_i - n^{-1}\frac{1}{2}\rho_0^{(3)}g_i^{\mathsf{T}}P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Pg_i + n^{-1}\operatorname{vec}(P)^{\mathsf{T}}\left(\widehat{g}_0 \otimes \widehat{\Omega}^{\mathsf{T}}\right)Pg_i + \frac{1}{2}\rho_0^{(3)}n^{-1}\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right)^{\mathsf{T}}\Psi^{\mathsf{T}}Pg_i \\ &+ n^{-1}\left(1+\rho_0^{(3)}/2\right)\widehat{g}_0^{\mathsf{T}}P\widehat{g}_0 + n^{-1}\operatorname{vec}(Q)^{\mathsf{T}}\left(\widehat{g}_0 \otimes \widehat{G}^{\mathsf{T}}\right)Pg_i + n^{-1}\operatorname{vec}(P)^{\mathsf{T}}\left(\widehat{g}_0 \otimes \widehat{G}\right)Qg_i \\ &- n^{-1}\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right)^{\mathsf{T}}(T^{\circ})^{\mathsf{T}}Qg_i - \frac{1}{2}n^{-1}\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)^{\mathsf{T}}\Gamma^{\mathsf{T}}Pg_i - n^{-1}\operatorname{vec}(\Gamma)^{\mathsf{T}}\left[Q \otimes \operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)\right]g_i \\ &- n^{-1}\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\mathsf{T}}P\right)^{\mathsf{T}}T^{\mathsf{T}}Pg_i - n^{-1}\operatorname{vec}(T)^{\mathsf{T}}\left[P \otimes \operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Q^{\mathsf{T}}\right)\right]g_i + n^{-1}\widehat{g}_0^{\mathsf{T}}PG_i(\beta_0)Q\widehat{g}_0 + R_n^{[\pi]},\end{aligned} \tag{8a}$$

where

$$\begin{aligned}R_n^{[\pi]} = \iota_q^{\mathsf{T}}\big[g_i + g_ig_i^{\mathsf{T}}\iota_q + \left((g_ig_i^{\mathsf{T}}) \otimes g_i^{\mathsf{T}}\right)\iota_{q^2}(1 + o_p(1)) + G_i(\beta_0)\iota_p + \Gamma_i(\beta_0)\iota_{p^2}(1 + o_p(1)) \\ + \left(g_i^{\mathsf{T}} \otimes G_i(\beta_0)\right)\iota_{pq}(1 + o_p(1))\big]\cdot O_p\left(n^{-5/2}\right),\end{aligned}$$

and ι_k is a $k \times 1$ vector of units.

If β_0 is known, (7a) is valid with $P = \Omega^{-1}$, and (7b) with $Q = 0$, and no derivatives in the remainder term. Hence, (8a) is valid with $P = \Omega^{-1}$, $Q = 0$, i.e.

$$\begin{aligned}\hat{\pi}_i = n^{-1} - n^{-1}\widehat{g}_0^{\mathsf{T}}\Omega^{-1}g_i + n^{-1}\operatorname{vec}\left(\Omega^{-1}\right)^{\mathsf{T}}\left(\widehat{g}_0 \otimes \widehat{\Omega}^{\mathsf{T}}\right)\Omega^{-1}g_i + \frac{1}{2}\rho_0^{(3)}n^{-1}\operatorname{vec}\left(\Omega^{-1}\widehat{g}_0\widehat{g}_0^{\mathsf{T}}\Omega^{-1}\right)^{\mathsf{T}}\Psi^{\mathsf{T}}\Omega^{-1}g_i \\ - n^{-1}\frac{1}{2}\rho_0^{(3)}g_i^{\mathsf{T}}\Omega^{-1}\widehat{g}_0\widehat{g}_0^{\mathsf{T}}\Omega^{-1}g_i + n^{-1}\left(1+\rho_0^{(3)}/2\right)\widehat{g}_0^{\mathsf{T}}\Omega^{-1}\widehat{g}_0 + R_n^{[\pi]},\end{aligned} \tag{8b}$$

where $R_n^{[\pi]} = \iota_q^{\mathsf{T}}\left[g_i + g_ig_i^{\mathsf{T}}\iota_q + \left((g_ig_i^{\mathsf{T}}) \otimes g_i^{\mathsf{T}}\right)\iota_{q^2}(1 + o_p(1))\right]\cdot O_p\left(n^{-5/2}\right)$.

1.3 GELKDE

1.3.1 Expectation

Write

$$\operatorname{E}\tilde{f}_{\rho}(x) = \operatorname{E}\hat{f}(x) + \operatorname{E}\left[(n\hat{\pi}_1 - 1)\,K_H\left(x - X_1\right)\right].$$

Considering each term in (8a) in turn, one obtains

$$\begin{aligned}\operatorname{E}\left[\widehat{g}_0^{\mathsf{T}}Pg_1(\beta_0)\,K_H(x - X_1)\right] &= n^{-1}\operatorname{E}\left[g_1^{\mathsf{T}}(\beta_0)\,Pg_1(\beta_0)\,K_H(x - X_1)\right] \\ &= n^{-1}\int g^{\mathsf{T}}(y;\beta_0)\,Pg(y;\beta_0)\,K_H(x - y)\,f(y)dy \\ &= n^{-1}\int g^{\mathsf{T}}\left(x - H^{-1/2}z;\beta_0\right)Pg\left(x - H^{-1/2}z;\beta_0\right)K(z)\,f\left(x - H^{-1/2}z\right)dz \\ &= n^{-1}g^{\mathsf{T}}(x;\beta_0)Pg(x;\beta_0)f(x) + O\left(n^{-1}\operatorname{tr}(H)\right).\end{aligned} \tag{9a}$$

Similarly,

$$\begin{aligned}\operatorname{E}\left[g_1^{\mathsf{T}}(\beta_0)\,P\widehat{g}_0\widehat{g}_0^{\mathsf{T}}Pg_1(\beta_0)\,K_H(x - X_1)\right] &= n^{-1}\operatorname{E}\left[g_1^{\mathsf{T}}(\beta_0)\,Pg_2(\beta_0)\,g_2^{\mathsf{T}}(\beta_0)\,Pg_1(\beta_0)\,K_H(x - X_1)\right] + O\left(n^{-2}\right) \\ = n^{-1}\operatorname{E}\left[g_1^{\mathsf{T}}(\beta_0)\,P\Omega Pg_1(\beta_0)\,K_H(x - X_1)\right] + O\left(n^{-2}\right) &= n^{-1}g^{\mathsf{T}}(x;\beta_0)Pg(x;\beta_0)f(x) + O\left(n^{-1}\operatorname{tr}(H)\right).\end{aligned} \tag{9b}$$

$$\mathrm{E}\left[\operatorname{vec}(P)^{\top}\left(\widehat{g}_0\otimes\widehat{\Omega}\right)Pg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right). \tag{9c}$$

$$\mathrm{E}\left[\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\top}P\right)^{\top}\Psi^{\top}Pg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right). \tag{9d}$$

$$\begin{aligned}\mathrm{E}\left[\widehat{g}_0^{\top}P\widehat{g}_0K_H(x-X_1)\right]&=n^{-1}\,\mathrm{E}\operatorname{tr}\left(g_2(\beta_0)g_2^{\top}(\beta_0)P\right)\mathrm{E}\,K_H(x-X_1)+O\left(n^{-2}\right)\\&=n^{-1}\operatorname{tr}(\Omega P)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right)=n^{-1}(q-p)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right),\end{aligned} \tag{9e}$$

because $\operatorname{tr}(\Omega P)=\operatorname{tr}(I_q-GQ)=q-\operatorname{tr}\left(\Sigma G^{\top}\Omega^{-1}G\right)=q-\operatorname{tr}(I_p)=q-p$.

$$\mathrm{E}\left[\operatorname{vec}(Q)^{\top}\left(\widehat{g}_0\otimes\widehat{G}^{\top}\right)Pg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(Q)^{\top}T^{\top}Pg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right). \tag{9f}$$

$$\mathrm{E}\left[\operatorname{vec}(P)^{\top}\left(\widehat{g}_0\otimes\widehat{G}\right)Qg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(P)^{\top}(T^{\circ})^{\top}Qg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right). \tag{9g}$$

$$\mathrm{E}\left[\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\top}P\right)^{\top}(T^{\circ})^{\top}Qg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(P)^{\top}(T^{\circ})^{\top}Qg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right). \tag{9h}$$

$$\mathrm{E}\left[\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\top}Q^{\top}\right)^{\top}\Gamma^{\top}Pg_1(\beta_0)K_H(x-X_1)\right]=n^{-1}\operatorname{vec}(\Sigma)^{\top}\Gamma^{\top}Pg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right), \tag{9i}$$

because $Q\Omega Q^{\top}=\Sigma G^{\top}\Omega^{-1}G\Sigma=\Sigma$. Finally, the last four terms give contribution of order n^{-2}, viz.

$$\begin{aligned}&\mathrm{E}\left[\operatorname{vec}(\Gamma)^{\top}\left[Q\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\top}Q^{\top}\right)\right]g_1(\beta_0)K_H(x-X_1)\right]\\&\qquad=\operatorname{vec}(\Gamma)^{\top}\left[Q\otimes\operatorname{vec}\left(P\Omega Q^{\top}\right)\right]\mathrm{E}\left[g_1(\beta_0)K_H(x-X_1)\right]+O\left(n^{-2}\right)=O\left(n^{-2}\right),\end{aligned} \tag{9j}$$

because $P\Omega Q^{\top}=PG\Sigma=\Omega^{-1}G\Sigma-\Omega^{-1}GQG\Sigma=0_{q\times p}$ since $QG=\Sigma G^{\top}\Omega^{-1}G=I_p$. Similarly,

$$\mathrm{E}\left[\operatorname{vec}\left(Q\widehat{g}_0\widehat{g}_0^{\top}P\right)^{\top}T^{\top}Pg_1(\beta_0)K_H(x-X_1)\right]=O\left(n^{-2}\right). \tag{9k}$$

$$\mathrm{E}\left[\operatorname{vec}(T)^{\top}\left[P\otimes\operatorname{vec}\left(P\widehat{g}_0\widehat{g}_0^{\top}Q^{\top}\right)\right]g_1(\beta_0)K_H(x-X_1)\right]=O\left(n^{-2}\right). \tag{9l}$$

$$\begin{aligned}\mathrm{E}\left[\widehat{g}_0^{\top}PG_1(\beta_0)Q\widehat{g}_0K_H(x-X_1)\right]&=n^{-1}\,\mathrm{E}\left[g_2^{\top}(\beta_0)PG_1(\beta_0)Qg_2(\beta_0)K_H(x-X_1)\right]+O\left(n^{-2}\right)\\&=n^{-1}\,\mathrm{E}_{X_1}\left\{\mathrm{E}_{X_2|X_1}\left[\operatorname{tr}\left(g_2^{\top}(\beta_0)PG_1(\beta_0)Qg_2(\beta_0)\right)\right]K_H(x-X_1)\right\}+O\left(n^{-2}\right)\\&=n^{-1}\,\mathrm{E}_{X_1}\left[\operatorname{tr}\left(Q\Omega PG_1(\beta_0)\right)K_H(x-X_1)\right]+O\left(n^{-2}\right)=O\left(n^{-2}\right).\end{aligned} \tag{9m}$$

Combining terms gives

$$\begin{aligned}\mathrm{E}\left[(n\hat{\pi}_1-1)K_H(x-X_1)\right]&=-n^{-1}g^{\top}(x;\beta_0)Pg(x;\beta_0)f(x)-\frac{1}{2}\rho_0^{(3)}n^{-1}g^{\top}(x;\beta_0)Pg(x;\beta_0)f(x)\\&\quad+n^{-1}\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)f(x)+\frac{1}{2}\rho_0^{(3)}n^{-1}\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)f(x)+\left(1+\frac{1}{2}\rho_0^{(3)}\right)n^{-1}(q-p)f(x)\\&\quad+n^{-1}\operatorname{vec}(Q)^{\top}T^{\top}Pg(x;\beta_0)f(x)+n^{-1}\operatorname{vec}(P)^{\top}(T^{\circ})^{\top}Qg(x;\beta_0)f(x)-n^{-1}\operatorname{vec}(P)^{\top}(T^{\circ})^{\top}Qg(x;\beta_0)f(x)\\&\quad-\frac{1}{2}n^{-1}\operatorname{vec}(\Sigma)^{\top}\Gamma^{\top}Pg(x;\beta_0)f(x)+O\left(n^{-1}\operatorname{tr}(H)\right)\\&=n^{-1}\left(1+\frac{1}{2}\rho_0^{(3)}\right)\left[-g^{\top}(x;\beta_0)Pg(x;\beta_0)+\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)+(q-p)\right]f(x)\\&\quad+n^{-1}\left[\operatorname{vec}(Q)^{\top}T^{\top}Pg(x;\beta_0)-\frac{1}{2}\operatorname{vec}(\Sigma)^{\top}\Gamma^{\top}Pg(x;\beta_0)\right]f(x)+O\left(n^{-1}\operatorname{tr}(H)\right).\end{aligned}$$

Hence,

$$\mathrm{E}\,\tilde{f}_\rho(x)=\mathrm{E}\,\hat{f}(x)+n^{-1}\left[k_\rho B_1(x)+B_2(x)\right]f(x)+O\left(n^{-1}\operatorname{tr}(H)\right), \tag{10}$$

where

$$\begin{aligned}B_1(x)&=-g^{\top}(x;\beta_0)Pg(x;\beta_0)+\operatorname{vec}(P)^{\top}\Psi^{\top}Pg(x;\beta_0)+q-p,\\B_2(x)&=\operatorname{vec}(Q)^{\top}T^{\top}Pg(x;\beta_0)-\frac{1}{2}\operatorname{vec}(\Sigma)^{\top}\Gamma^{\top}Pg(x;\beta_0),\end{aligned}$$

and $k_\rho=1+\rho_0^{(3)}/2$. If β_0 is known, (10) is valid with

$$\begin{aligned}&B_1(x)=-g^{\top}(x;\beta_0)\Omega^{-1}g(x;\beta_0)+\operatorname{vec}\left(\Omega^{-1}\right)^{\top}\Psi^{\top}\Omega^{-1}g(x;\beta_0)+q,\\&B_2(x)=0.\end{aligned}$$

It then also follows immediately that

$$\mathrm{ISB}\,\tilde{f}_\rho(\cdot)=\mathrm{ISB}\,\hat{f}(\cdot)+n^{-1}\mu_2(K)\int\operatorname{tr}\left(H\mathscr{H}_xf(x)\right)\left(k_\rho B_1(x)+B_2(x)\right)f(x)dx+o\left(n^{-1}\operatorname{tr}(H)\right)$$

1.3.2 Variance

Write $\tilde{f}_\rho(x) = \hat{f}(x) + \widehat{S}(x)$, where $\widehat{S}(x) = n^{-1}\sum_{i=1}^{n}(n\hat{\pi}_i - 1) K_H(x - X_i)$ and let

$$S(x) = \mathrm{E}\,\widehat{S}(x) = n^{-1}(k_\rho B_1(x) + B_2(x)) f(x) + O\left(n^{-1}\operatorname{tr}(H)\right).$$

Recall that $\mathrm{E}\,\tilde{f}_\rho(x) = \mathrm{E}\,\hat{f}(x) + S(x)$, where $\mathrm{E}\,\hat{f}(x) = f(x) + O(\operatorname{tr}(H))$. Then $\mathrm{E}^2\,\tilde{f}_\rho(x) = \mathrm{E}^2\,\hat{f}(x) + 2S(x)\,\mathrm{E}\,\hat{f}(x) + S^2(x)$ and $\mathrm{E}\,\tilde{f}_\rho^2(x) = \mathrm{E}\,\hat{f}^2(x) + 2\,\mathrm{E}\left[\hat{f}(x)\widehat{S}(x)\right] + \mathrm{E}\,\widehat{S}^2(x)$. Hence,

$$\operatorname{Var}\tilde{f}_\rho(x) = \operatorname{Var}\hat{f}(x) + 2\,\mathrm{E}\left[\hat{f}(x)\widehat{S}(x)\right] - 2S(x)\,\mathrm{E}\,\hat{f}(x) + \mathrm{E}\,\widehat{S}^2(x) - S^2(x).$$

Considering each term in (8a) one obtains that

$$\begin{aligned}\mathrm{E}\left[\hat{f}(x)\widehat{S}(x)\right] &= n^{-2}\,\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(n\hat{\pi}_i - 1) K_H(x - X_i) K_H(x - X_j)\right] \\ &= S(x)\,\mathrm{E}\,\hat{f}(x) - n^{-1}\,\mathrm{E}\left[g_2^\top(\beta_0) P g_1(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right),\end{aligned}$$

where the extra $O\left(n^{-1}\right)$ terms comes from the first term in (8a), i.e. $-\widehat{g}_0^\top P g_i(\beta_0)$. Indeed,

$$\begin{aligned}n^{-2}\,\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{g}_0^\top P g_i(\beta_0) K_H(x - X_i) K_H(x - X_j)\right] &= n^{-1}\,\mathrm{E}\left[g_1^\top(\beta_0) P g_1(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] \\ &\quad + n^{-1}\,\mathrm{E}\left[g_2^\top(\beta_0) P g_1(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right) \\ &= n^{-1}\,\mathrm{E}\left[g_1^\top(\beta_0) P g_1(\beta_0) K_H(x - X_1)\right]\mathrm{E}\,\hat{f}(x) + n^{-1} g^\top(x;\beta_0) P g(x;\beta_0) f^2(x) + O\left(n^{-1}\operatorname{tr}(H)\right).\end{aligned}$$

It is also easy to see that the only $O\left(n^{-1}\right)$ contribution to $\mathrm{E}\,\widehat{S}^2(x)$ comes from the first term in (8a), viz.

$$\begin{aligned}\mathrm{E}\,\widehat{S}^2(x) &= n^{-2}\,\mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(n\hat{\pi}_i - 1)(n\hat{\pi}_j - 1) K_H(x - X_i) K_H(x - X_j)\right] \\ &= n^{-1}\,\mathrm{E}\left[g_1^\top(\beta_0) P g_2(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right),\end{aligned}$$

because

$$\begin{aligned}n^{-2}\,\mathrm{E}&\left[\sum_{i=1}^{n}\sum_{j=1}^{n} g_i^\top(\beta_0) P\widehat{g}_0\widehat{g}_0^\top P g_j(\beta_0) K_H(x - X_i) K_H(x - X_j)\right] \\ &= n^{-1}\,\mathrm{E}\left[g_1^\top(\beta_0) P g_3(\beta_0) g_3^\top(\beta_0) P g_2(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right) \\ &= n^{-1}\,\mathrm{E}\left[g_1^\top(\beta_0) P g_2(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Var}\tilde{f}_\rho(x) &= \operatorname{Var}\hat{f}(x) - n^{-1}\,\mathrm{E}\left[g_2^\top(\beta_0) P g_1(\beta_0) K_H(x - X_1) K_H(x - X_2)\right] + O\left(n^{-2}\right) \\ &= \operatorname{Var}\hat{f}(x) - n^{-1} g^\top(x;\beta_0) P g(x;\beta_0) f^2(x) + O\left(n^{-1}\operatorname{tr}(H)\right).\end{aligned} \tag{11}$$

Then

$$\operatorname{IVar}\tilde{f}_\rho(x) = \operatorname{IVar}\hat{f}(x) - n^{-1}\int g^\top(x;\beta_0) P g(x;\beta_0) f^2(x)dx + O\left(n^{-1}\operatorname{tr}(H)\right). \tag{12}$$

If β_0 is known, (11) and (12) are valid with $P = \Omega^{-1}$. This completes the proof.

2 Moment conditions under GELKDE

Assuming $g(x;\beta)$ possesses continuous and square integrable second derivatives with respect to x, equation (3) follows since for for given $\beta \in \mathrm{B}_r(\beta_0)$, and $l = 1,\ldots,q$,

$$\begin{aligned}
\int g_l(x;\beta)\hat{f}(x)dx &= n^{-1}\sum_{i=1}^{n}\int g_l(x;\beta)K_H(x-x_i)\,dx = n^{-1}\sum_{i=1}^{n}\int g_l\left(x_i + H^{1/2}z;\beta\right)K(z)\,dz \\
&= n^{-1}\sum_{i=1}^{n}\int\left(g_l(x_i;\beta) + \mathscr{D}_x g_l(x_i;\beta)H^{1/2}z + \frac{1}{2}z^{\mathsf{T}}H^{1/2}\mathscr{H}_x g_l(x_i;\beta)H^{1/2}z + o_p(\mathrm{tr}(H))\right)K(z)\,dz \\
&= n^{-1}\sum_{i=1}^{n}g_l(x_i;\beta) + \frac{1}{2}n^{-1}\sum_{i=1}^{n}\int z^{\mathsf{T}}H^{1/2}\mathscr{H}_x g_l(x_i;\beta)H^{1/2}zK(z)\,dz + o_p(\mathrm{tr}(H)) \\
&= n^{-1}\sum_{i=1}^{n}g_l(x_i;\beta) + \frac{1}{2}n^{-1}\sum_{i=1}^{n}\mathrm{tr}\left(H^{1/2}\mathscr{H}_x g_l(x_i;\beta)H^{1/2}\int zz^{\mathsf{T}}K(z)\,dz\right) + o_p(\mathrm{tr}(H)) \\
&= n^{-1}\sum_{i=1}^{n}g_l(x_i;\beta) + \frac{1}{2}\mu_2(K)\,\mathrm{tr}\left(Hn^{-1}\sum_{i=1}^{n}\mathscr{H}_x g_l(x_i;\beta)\right) + o_p(\mathrm{tr}(H)) \\
&= \widehat{g}_l(\beta) + \frac{1}{2}\mu_2(K)\,\mathrm{tr}(HJ_l(\beta)) + o_p(\mathrm{tr}(H)), \qquad (13)
\end{aligned}$$

where $\widehat{g}_l(\beta) = n^{-1}\sum_{i=1}^{n}g_l(x_i;\beta)$, and the last equality obtains by writing $n^{-1}\sum_{i=1}^{n}\mathscr{H}_x g_l(x_i;\beta) = J_l(\beta) + o_p(1)$, where $J_l(\beta) = \mathrm{E}\,\mathscr{H}_x g_l(x_i;\beta)$. Equation (10) follows because

$$\begin{aligned}
\int g_l(x;\beta)\tilde{f}_\rho(x)dx &= \sum_{i=1}^{n}\hat{\pi}_i\int g_l(x;\beta)K_H(x-x_i)\,dx = \sum_{i=1}^{n}\hat{\pi}_i\int g_l\left(x_i + H^{1/2}z;\beta\right)K(z)\,dz \\
&= \sum_{i=1}^{n}\hat{\pi}_i g_l(x_i;\beta) + \frac{1}{2}\sum_{i=1}^{n}\hat{\pi}_i\int z^{\mathsf{T}}H^{1/2}\mathscr{H}_x g_l(x_i;\beta)H^{1/2}zK(z)\,dz + o_p(\mathrm{tr}(H)) \\
&= \frac{1}{2}\mu_2(K)\,\mathrm{tr}\left(Hn^{-1}\sum_{i=1}^{n}\mathscr{H}_x g_l(x_i;\beta)(1+o_p(1))\right) + o_p(\mathrm{tr}(H)) = \frac{1}{2}\mu_2(K)\,\mathrm{tr}(HJ_l(\beta)) + o_p(\mathrm{tr}(H)), \qquad (14)
\end{aligned}$$

where we used the fact that $\hat{\pi}_i = n^{-1}(1+o_p(1))$.

3 Higher order expansion of $\hat{f}(x)$

The mean and variance of $\hat{f}(x)$ are $\mathrm{E}\,\hat{f}(x) = (K_h * f)(x)$ and $\mathrm{Var}\,\hat{f}(x) = n^{-1}\left[\left(K_h^2 * f\right)(x) - (K_h * f)^2(x)\right]$, where $*$ denotes convolution, i.e. $(f*g)(x) = \int f(x-y)g(y)dy$. Assuming sufficient smoothness of $f(x)$, one obtains that

$$\begin{aligned}
\mathrm{E}\,\hat{f}(x) &= \int K_h(x-y)f(y)dy = \int f(x+hz)K(z)dz \\
&= \int\left[f(x) + hzf^{(1)}(x) + \frac{1}{2}h^2z^2f^{(2)}(x) + \frac{1}{6}h^3z^3f^{(3)}(x) + \frac{1}{24}h^4z^4f^{(4)}(x) + \frac{1}{120}h^5z^5f^{(5)}(x) + O\left(h^6\right)\right]K(z)dz \\
&= f(x) + \frac{1}{2}h^2\mu_2(K)f^{(2)}(x) + \frac{1}{24}h^4\mu_4(K)f^{(4)}(x) + O\left(h^6\right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{ISB}\,\hat{f}(x) &= \int\left[\frac{1}{2}h^2\mu_2(K)f^{(2)}(x) + \frac{1}{24}h^4\mu_4(K)f^{(4)}(x) + O\left(h^6\right)\right]^2dx \\
&= \frac{1}{4}h^4\mu_2^2(K)\int\left[f^{(2)}(x)\right]^2dx + \frac{1}{24}h^6\mu_2(K)\mu_4(K)\int f^{(2)}(x)f^{(4)}(x)dx + O\left(h^8\right).
\end{aligned}$$

Remark that since $K(-z) = K(z)$, $K^2(-z) = K^2(z)$, hence for odd j, $\int z^jK^2(z)dz = 0$.

$$\begin{aligned}
\left(K_h^2 * f\right)(x) &= \int K_h^2(x-y)f(y)dy = h^{-1}\int f(x+hz)K^2(z)dz \\
&= h^{-1}\int\Big[f(x) + hzf^{(1)}(x) + \frac{1}{2}h^2z^2f^{(2)}(x) + \frac{1}{6}h^3z^3f^{(3)}(x) + \frac{1}{24}h^4z^4f^{(4)}(x) + \frac{1}{120}h^5z^5f^{(5)}(x) \\
&\qquad + O\left(h^6\right)\Big]K^2(z)dz \\
&= h^{-1}R(K)f(x) + \frac{1}{2}h\mu_2\left(K^2\right)f^{(2)}(x) + \frac{1}{24}h^3\mu_4\left(K^2\right)f^{(4)}(x) + O\left(h^5\right).
\end{aligned}$$

$$(K_h * f)^2(x) = f^2(x) + h^2\mu_2(K)f(x)f^{(2)}(x) + h^4\left[\frac{1}{4}\mu_2^2(K)\left(f^{(2)}(x)\right)^2 + \frac{1}{12}\mu_4(K)f(x)f^{(4)}(x)\right] + O\left(h^6\right).$$

Hence

$$\operatorname{Var}\hat{f}(x) = (nh)^{-1}R(K) - n^{-1}f^2(x) + \frac{1}{2}n^{-1}h\mu_2\left(K^2\right)f^{(2)}(x) - n^{-1}h^2\mu_2(K)f(x)f^{(2)}(x) + O\left(n^{-1}h^3\right).$$

$$\operatorname{IVar}\hat{f}(x) = (nh)^{-1}R(K) - n^{-1}R(f) + \frac{1}{2}n^{-1}h\mu_2\left(K^2\right)\int f^{(2)}(x)dx - n^{-1}h^2\mu_2(K)\int f(x)f^{(2)}(x)dx + O\left(n^{-1}h^3\right).$$

3.1 Relative difference

Let $\Delta\%\mathrm{W}_{\tilde{f}_\rho,\hat{f}} = \left[\mathrm{W}\tilde{f}_\rho(\cdot) - \mathrm{W}\hat{f}(\cdot)\right]/\mathrm{W}\hat{f}(\cdot)$, where W can denote ISB, IVar, or MISE. By an expansion around $h=0$, $\left[h^{-4}\operatorname{ISB}\hat{f}(\cdot)\right]^{-1} = 4\mu_2^{-2}(K)R^{-1}\left(f^{(2)}\right) + O\left(h^2\right)$, and hence,

$$\begin{aligned}\Delta\%\operatorname{ISB}_{\tilde{f}_\rho,\hat{f}} = \frac{h^{-4}\left[\operatorname{ISB}\tilde{f}_\rho(\cdot) - \operatorname{ISB}\hat{f}(\cdot)\right]}{h^{-4}\operatorname{ISB}\hat{f}(\cdot)} &= \left[n^{-1}h^{-2}\mu_2(K)C_{isb} + O\left(n^{-3/2}h^{-4}\right)\right]\left[4\mu_2^{-2}(K)R^{-1}\left(f^{(2)}\right) + O\left(h^2\right)\right] \\ &= \frac{4C_{isb}}{\mu_2(K)R\left(f^{(2)}\right)}n^{-1}h^{-2} + O\left(n^{-3/2}h^{-4}\right).\end{aligned}$$

Similarly, as $\left[nh\operatorname{IVar}\hat{f}(\cdot)\right]^{-1} = R^{-1}(K) + R(f)R^{-2}(K)h + O\left(h^2\right)$,

$$\begin{aligned}\Delta\%\operatorname{IVar}_{\tilde{f}_\rho,\hat{f}} &= \left[-hC_{ivar} + h^3\mu_2(K)C_{ivar}^\star + O\left(n^{-1/2}h\right)\right]\left[R^{-1}(K) + R(f)R^{-2}(K)h + O\left(h^2\right)\right] \\ &= -\frac{C_{ivar}}{R(K)}h - \frac{C_{ivar}R(f)}{R(K)^2}h^2 + O\left(h^3\right).\end{aligned}$$

With asymptotically optimal bandwidth, $h = cn^{-1/5}$, where $c = \left[R(K)/\left(\mu_2^2(K)R\left(f^{(2)}\right)\right)\right]^{1/5}$,

$$\operatorname{MISE}\hat{f}(\cdot) = \frac{5}{4}c^{-1}R(K)n^{-4/5} - R(f)n^{-1} + A_{-6/5}n^{-6/5} + O\left(n^{-7/5}\right),$$

where $A_{-6/5} = \frac{1}{24}c^6\mu_2(K)\mu_4(K)\int f^{(2)}(x)f^{(4)}(x)dx + \frac{1}{2}c\mu_2\left(K^2\right)\int f^{(2)}(x)dx$. Also,

$$\operatorname{MISE}\tilde{f}_\rho(\cdot) = \operatorname{MISE}\hat{f}(\cdot) - C_{ivar}n^{-1} + \left(C_{isb} + C_{ivar}^\star\right)\mu_2(K)c^2n^{-7/5} + O\left(n^{-3/2}\right),$$

and hence

$$\Delta\%\operatorname{MISE}_{\tilde{f}_\rho,\hat{f}} = -\frac{4}{5}\frac{C_{ivar}}{R(K)}cn^{-1/5} - \frac{16}{25}\frac{C_{ivar}R(f)}{R(K)^2}c^2n^{-2/5} + A_{-3/5}c^3n^{-3/5} + O\left(n^{-7/10}\right),$$

where

$$A_{-3/5} = \frac{4}{5}\frac{\mu_2(K)}{R(K)}\left(C_{isb} + C_{ivar}^\star\right) - C_{ivar}\left(\frac{64}{125}\frac{R(f)^2}{R(K)^3} - \frac{8}{25}\frac{\mu_2\left(K^2\right)\int f^{(2)}(x)dx}{R(K)^2} - \frac{2}{75}\frac{\mu_4(K)\int f^{(2)}(x)f^{(4)}(x)dx}{R(K)\mu_2(K)R\left(f^{(2)}\right)}\right).$$

3.2 Gaussian density

Let $f(x) = \phi(x)$ and $K(x) = \phi(x)$, where $\phi(x)$ denotes the standard normal density. Let $\operatorname{OF}(2r) = (2r-1)(2r-3)\cdots 1 = (2r)!/(2^r r!)$ denote the 'Odd Factorial'. Remark that for the Gaussian density, for $r_1 + r_2$ even and $r_1 \geq r_2$,

$$\int x^{r_1}\phi_\sigma^{(r_2)}(x)dx = (-1)^{r_1}\frac{r_1!}{(r_1-r_2)!}\sigma^{r_1-r_2}\operatorname{OF}(r_1-r_2);$$

otherwise $\int x^{r_1}\phi_\sigma^{(r_2)}(x)dx = 0$ (Aldershof, Marron, Park, and Wand, 1995, Corollary 3.4). For $r_1 + r_2$ even

$$\int \phi_{\sigma_1}^{(r_1)}(x)\phi_{\sigma_2}^{(r_2)}(x)dx = (-1)^{(r_1+r_2)/2}(2\pi)^{-1/2}\operatorname{OF}(r_1+r_2)/\tilde{\sigma}^{r_1+r_2+1},$$

where $\tilde{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2}$, and for $r_1 + r_2$ odd $\int \phi_{\sigma_1}^{(r_1)}(x)\phi_{\sigma_2}^{(r_2)}(x)dx = 0$ (Aldershof et al., 1995, Corollary 5.3). Also,

$$R\left(\phi^{(r)}\right) = \int\left[\phi^{(r)}(x)\right]^2 dx = \frac{\operatorname{OF}(2r)}{2^{r+1}\sqrt{\pi}},$$

(Aldershof et al., 1995, Corollary 5.4). Thus, $R(\phi) = \frac{1}{2\sqrt{\pi}}$, $R\left(\phi^{(2)}\right) = \frac{3}{8\sqrt{\pi}}$, $\int \phi^{(2)}(x)dx = 0$, $\int \phi(x)\phi^{(2)}(x)dx = -\frac{1}{4\sqrt{\pi}}$, and $\int \phi^{(2)}(x)\phi^{(4)}(x)dx = -\frac{15}{16\sqrt{\pi}}$. Also, $\mu_2(\phi) = 1$, $\mu_4(\phi) = 3$, hence

$$\begin{aligned}\operatorname{ISB}\hat{f}(x) &= \frac{1}{4}h^4\mu_2^2(\phi)R\left(\phi^{(2)}\right) + \frac{1}{24}h^6\mu_2(\phi)\mu_4(\phi)\int \phi^{(2)}(x)\phi^{(4)}(x)dx + O\left(h^8\right)\\ &= \frac{3}{32\sqrt{\pi}}h^4 - \frac{15}{128\sqrt{\pi}}h^6 + O\left(h^8\right).\end{aligned}$$

$$\begin{aligned}\operatorname{IVar}\hat{f}(x) &= (nh)^{-1}R(\phi) - n^{-1}R(\phi) + \frac{1}{2}n^{-1}h\mu_2\left(\phi^2\right)\int \phi^{(2)}(x)dx - n^{-1}h^2\mu_2(\phi)\int \phi(x)\phi^{(2)}(x)dx + O\left(n^{-1}h^3\right)\\ &= \frac{1}{2\sqrt{\pi}}(nh)^{-1} - \frac{1}{2\sqrt{\pi}}n^{-1} + \frac{1}{4\sqrt{\pi}}n^{-1}h^2 + O\left(n^{-1}h^3\right).\end{aligned}$$

The asymptotically optimal bandwidth in this case is $h_{AMISE} = (4/3)^{1/5}n^{-1/5}$. Therefore,

$$\Delta\%\operatorname{ISB}_{\tilde{f}_\rho,\hat{f}} = \frac{32\sqrt{\pi}C_{isb}}{3}n^{-1}h^{-2} + O\left(n^{-3/2}h^{-4}\right) = \frac{16\cdot 2^{1/5}\sqrt{\pi}C_{isb}}{3^{3/5}}n^{-3/5} + O\left(n^{-7/10}\right).$$

$$\Delta\%\operatorname{IVar}_{\tilde{f}_\rho,\hat{f}} = -2\sqrt{\pi}C_{ivar}h - 2\sqrt{\pi}C_{ivar}h^2 + O\left(h^3\right) = -\frac{2\cdot 2^{2/5}\sqrt{\pi}C_{ivar}}{3^{1/5}}n^{-1/5} - \frac{2\cdot 2^{4/5}\sqrt{\pi}C_{ivar}}{3^{2/5}}n^{-2/5} + O\left(n^{-3/5}\right).$$

4 Example 3

In this example, X_i, $i = 1, \ldots, n$, are i.i.d. $N(0,1)$. Thus, $\mathscr{H}_x\phi(x) = \left(x^2 - 1\right)\phi(x)$, and

$$\begin{aligned}C_{isb} &= \int \left(k_\rho B_1(x) + B_2(x)\right)\left(x^2-1\right)\phi^2(x)dx,\\ C_{ivar} &= \int g^\mathsf{T}\left(x;\beta_0\right)Pg\left(x;\beta_0\right)\phi^2(x)dx.\end{aligned}$$

Remark also that for $X \sim N(0,\sigma^2)$, $r = 0,1,2,\ldots$, when r is even $\operatorname{E}[X^r] = \sigma^r\operatorname{OF}(r)$, and when r is odd, $\operatorname{E}[X^r] = 0$. Also, $\phi(x)^r = (2\pi)^{(1-r)/2}\phi_{r^{-1/2}}(x)\,/r^{1/2}$. Hence, if r is even,

$$\int x^r\phi^2(x)dx = \left((2\pi)^{-1/2}\Big/\sqrt{2}\right)\int x^r\phi_{2^{-1/2}}(x)dx = \frac{\operatorname{OF}(r)}{2^{(r+2)/2}\sqrt{\pi}},$$

and 0 if r is odd.

4.1 Known mean

$g(x,\beta) = x$. $\Omega = \operatorname{E}\left[X^2\right] = 1$, and $\Psi = \operatorname{E}\left[X^3\right] = 0$. Hence $B_1(x) = 1 - x^2$ and $B_2(x) = 0$, and

$$C_{isb} = -k_\rho\int(1-x^2)^2\phi^2(x)dx = -k_\rho\left[\frac{1}{2\sqrt{\pi}} - 2\frac{1}{4\sqrt{\pi}} + \frac{3}{8\sqrt{\pi}}\right] = -k_\rho\frac{3}{8\sqrt{\pi}} \approx -0.2116k_\rho.$$

$$C_{ivar} = \int x^2\phi^2(x)dx = \frac{1}{4\sqrt{\pi}} = 0.1410.$$

4.2 Known mean and variance

$$g(x,\beta) = \begin{bmatrix} x \\ x^2-1\end{bmatrix}; \qquad \Omega = \operatorname{E}\begin{bmatrix} x^2 & x^3 - x \\ x^3 - x & (x^2-1)^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix};$$

$$\Psi = \operatorname{E}\begin{bmatrix} x^3 & x^4 - x^2 & x^4 - x^2 & x^5 - 2x^3 + x \\ x^4 - x^2 & x^5 - 2x^3 + x & x^5 - 2x^3 + x & (x^2-1)^3\end{bmatrix} = \begin{bmatrix}0 & 2 & 2 & 0\\ 2 & 0 & 0 & 8\end{bmatrix}$$

$$\begin{aligned}B_1(x) &= -\begin{bmatrix}x & x^2-1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}x\\ x^2-1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0 & 1/2\end{bmatrix}\begin{bmatrix}0 & 2\\ 2 & 0\\ 2 & 0\\ 0 & 8\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}\begin{bmatrix}x\\ x^2-1\end{bmatrix} + 2\\ &= -\frac{1}{2}x^4 + 3x^2 - 1\frac{1}{2}.\end{aligned}$$

$$\begin{aligned}C_{isb} &= k_\rho\int\left(-\frac{1}{2}x^4 + 3x^2 - 1\frac{1}{2}\right)(x^2-1)\phi^2(x)dx = k_\rho\frac{1}{2}\int\left(-x^6 + 7x^4 - 9x^2 + 3\right)\phi^2(x)dx\\ &= k_\rho\frac{1}{2}\left[-\frac{15}{16\sqrt{\pi}} + 7\frac{3}{8\sqrt{\pi}} - 9\frac{1}{4\sqrt{\pi}} + 3\frac{1}{2\sqrt{\pi}}\right] = k_\rho\frac{15}{32\sqrt{\pi}} \approx 0.2645k_\rho.\end{aligned}$$

$$C_{ivar} = \int\frac{1}{2}\left(x^4+1\right)\phi^2(x)dx = \frac{1}{2}\left[\frac{3}{8\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}}\right] = \frac{7}{16\sqrt{\pi}} \approx 0.2468.$$

4.3 Known mean and third moment

$$g(x,\beta) = \begin{bmatrix} x \\ x^3 \end{bmatrix}; \qquad \Omega = \mathrm{E}\begin{bmatrix} x^2 & x^4 \\ x^4 & x^6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix}, \qquad \Omega^{-1} = \frac{1}{6}\begin{bmatrix} 15 & -3 \\ -3 & 1 \end{bmatrix}.$$

$$\Psi = \mathrm{E}\begin{bmatrix} x^3 & x^5 & x^5 & x^7 \\ x^5 & x^7 & x^7 & x^9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$B_1(x) = -\frac{1}{6}\begin{bmatrix} x & x^3 \end{bmatrix}\begin{bmatrix} 15 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ x^3 \end{bmatrix} + 2 = -\frac{1}{6}\left(x^6 - 6x^4 + 15x^2\right) + 2.$$

$$C_{isb} = k_\rho \frac{1}{6}\int \left(-x^8 + 7x^6 - 21x^4 + 27x^2 - 12\right)\phi^2(x)dx = k_\rho \frac{1}{6}\left[-\frac{105}{32\sqrt{\pi}} + 7\frac{15}{16\sqrt{\pi}} - 21\frac{3}{8\sqrt{\pi}} + 27\frac{1}{4\sqrt{\pi}} - 12\frac{1}{2\sqrt{\pi}}\right]$$
$$= -k_\rho \frac{41}{64\sqrt{\pi}} \approx -0.3614k_\rho.$$

$$C_{ivar} = \frac{1}{6}\int \left(x^6 - 6x^4 + 15x^2\right)\phi^2(x)dx = \frac{1}{6}\left[\frac{15}{16\sqrt{\pi}} - 6\frac{3}{8\sqrt{\pi}} + 15\frac{1}{4\sqrt{\pi}}\right] = \frac{13}{32\sqrt{\pi}} \approx 0.2292.$$

4.4 Unknown mean and known variance

$$g(x,\beta) = \begin{bmatrix} x-\beta \\ (x-\beta)^2 - 1 \end{bmatrix}; \quad \beta_0 = 0; \quad \mathscr{D}_\beta g(x,\beta_0) = \begin{bmatrix} -1 \\ -2x \end{bmatrix}; \quad \mathscr{H}_\beta^\circ g(x,\beta_0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$$\Omega = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 8 \end{bmatrix}, \quad \Sigma = 1, \quad Q = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix},$$

$$\text{and} \quad T = \mathrm{E}\begin{bmatrix} -x & -x^2+1 \\ -2x^2 & -2x^3+2x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}.$$

$$B_1(x) = -\begin{bmatrix} x & x^2-1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ x^2-1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 2 & 0 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ x^2-1 \end{bmatrix} + 2 - 1 = -\frac{1}{2}x^4 + 3x^2 - 1\frac{1}{2}.$$

$$B_2(x) = \begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ x^2-1 \end{bmatrix} - \frac{1}{2}1 \cdot \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ x^2-1 \end{bmatrix} = \frac{1}{2}(x^2-1).$$

$$C_{isb} = \int \left[k_\rho\left(-\frac{1}{2}x^4 + 3x^2 - 1\frac{1}{2}\right) + \frac{1}{2}(x^2-1)\right](x^2-1)\phi^2(x)dx$$
$$= k_\rho \int \left(-\frac{1}{2}x^4 + 3x^2 - 1\frac{1}{2}\right)(x^2-1)\phi^2(x)dx + \frac{1}{2}\int (x^2-1)^2\phi^2(x)dx = \frac{15}{32\sqrt{\pi}}k_\rho + \frac{3}{16\sqrt{\pi}} \approx 0.2645k_\rho + 0.1058.$$

$$C_{ivar} = \frac{1}{2}\int (x^2-1)^2\phi^2(x)dx = \frac{3}{16\sqrt{\pi}} \approx 0.1058.$$

4.5 Unknown mean and known third central moment

$$g(x,\beta) = \begin{bmatrix} x-\beta \\ (x-\beta)^3 \end{bmatrix}; \quad \beta_0 = 0; \quad \mathscr{D}_\beta g(x,\beta_0) = \begin{bmatrix} -1 \\ -3x^2 \end{bmatrix}; \quad \mathscr{H}_\beta^\circ g(x,\beta_0) = \begin{bmatrix} 0 \\ 6x \end{bmatrix}.$$

$$\Omega = \begin{bmatrix} 1 & 3 \\ 3 & 15 \end{bmatrix}, \quad \Omega^{-1} = \frac{1}{6}\begin{bmatrix} 15 & -3 \\ -3 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and } T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $B_2(x) = 0$.

$$\Sigma = 1, \quad Q = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad P = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}.$$

$$B_1(x) = -\frac{1}{6}\begin{bmatrix} x & x^3 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ x^3 \end{bmatrix} + 2 - 1 = -\frac{1}{6}\left(x^6 - 6x^4 + 9x^2 - 6\right).$$

$$C_{isb} = -k_\rho \frac{1}{6}\int \left(x^8 - 7x^6 + 15x^4 - 15x^2 + 6\right)\phi^2(x)dx = -k_\rho \frac{1}{6}\left[\frac{105}{32\sqrt{\pi}} - 7\frac{15}{16\sqrt{\pi}} + 15\frac{3}{8\sqrt{\pi}} - 15\frac{1}{4\sqrt{\pi}} + 6\frac{1}{2\sqrt{\pi}}\right]$$
$$= -k_\rho \frac{17}{64\sqrt{\pi}} \approx -0.1499k_\rho.$$

$$C_{ivar} = \frac{1}{6}\int \left(x^6 - 6x^4 + 9x^2\right)\phi^2(x)dx = \frac{1}{6}\left[\frac{15}{16\sqrt{\pi}} - 6\frac{3}{8\sqrt{\pi}} + 9\frac{1}{4\sqrt{\pi}}\right] = \frac{5}{32\sqrt{\pi}} \approx 0.0882.$$

5 Example 4

In this example, X_i, $i=1,\dots,n$, are i.i.d. with p.d.f $f(x)=\sum_{j=1}^k w_j\phi_{\sigma_j}(x-\mu_j)$, where $w_j>0$, $j=1,\dots,k$, $\sum_{j=1}^k w_j=1$, and $\phi_{\sigma_j}(x-\mu_j)$ denotes the density of a normal random variable with mean μ_j and variance σ_j^2. Noting that $\phi_\sigma^{(r)}(x-\mu)=(-1)^r H_r\left(\frac{x-\mu}{\sigma}\right)\phi\left(\frac{x-\mu}{\sigma}\right)/\sigma^{r+1}$, where $H_r(x)$ is the r-th Hermite polynomial, $H_2=x^2-1$, and using (Aldershof et al., 1995, Corollary 4.2), gives

$$f^2(x)=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\phi_{\bar\sigma_{jl}}\left(x-\mu_{jl}^*\right) \tag{15a}$$

where $\sigma_{jl}^*=\sqrt{\sigma_j^2+\sigma_l^2}$, $\bar\sigma_{jl}=\sigma_j\sigma_l/\sigma_{jl}^*$, and $\mu_{jl}^*=\left(\mu_j\sigma_l^2+\mu_l\sigma_j^2\right)/\left(\sigma_j^2+\sigma_l^2\right)$;

$$f(x)f^{(2)}(x)=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\left[\frac{(x-\mu_l)^2}{\sigma_l^4}-\frac{1}{\sigma_l^2}\right]\phi_{\bar\sigma_{jl}}\left(x-\mu_{jl}^*\right), \tag{15b}$$

and

$$\left[f^{(2)}(x)\right]^2=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\left[\frac{(x-\mu_j)^2}{\sigma_j^4}-\frac{1}{\sigma_j^2}\right]\left[\frac{(x-\mu_l)^2}{\sigma_l^4}-\frac{1}{\sigma_l^2}\right]\phi_{\bar\sigma_{jl}}\left(x-\mu_{jl}^*\right). \tag{15c}$$

From (15a), one immediately obtains that

$$R(f)=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)=\frac{1}{2\sqrt{\pi}}\left[\sum_{j=1}^k\frac{w_j^2}{\sigma_j}+4\sum_{j=1}^k\sum_{l>j}^k\frac{w_j w_l}{\sqrt{2(\sigma_j^2+\sigma_l^2)}}\exp\left(-\frac{(\mu_j-\mu_l)^2}{2(\sigma_j^2+\sigma_l^2)}\right)\right]$$

Noting that the product $\left[(x-\mu_j)^2-\sigma_j^2\right]\cdot\left[(x-\mu_l)^2-\sigma_l^2\right]/\left(\sigma_j^4\sigma_l^4\right)$ in (15c) can be written as

$$\frac{1}{\sigma_j^4\sigma_l^4}(x-\mu_{jl}^*)^4+a_3(x-\mu_{jl}^*)^3+\frac{U_{jl,j}+U_{jl,l}}{\sigma_j^4\sigma_l^4}(x-\mu_{jl}^*)^2+a_1(x-\mu_{jl}^*)+\frac{U_{jl,j}U_{jl,l}}{\sigma_j^4\sigma_l^4},$$

where $U_{jl,j}=(\mu_{jl}^*-\mu_j)^2-\sigma_j^2$, $U_{jl,l}=(\mu_{jl}^*-\mu_l)^2-\sigma_l^2$, and for some a_1 and a_3, one obtains

$$\begin{aligned}
R\left(f^{(2)}\right)&=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\left[\frac{3\bar\sigma_{jl}^4}{\sigma_j^4\sigma_l^4}+(U_{jl,j}+U_{jl,l})\frac{\bar\sigma_{jl}^2}{\sigma_j^4\sigma_l^4}+\frac{U_{jl,j}U_{jl,l}}{\sigma_j^4\sigma_l^4}\right]\\
&=\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\left[\frac{(\mu_j-\mu_l)^4}{(\sigma_j^2+\sigma_l^2)^4}-\frac{2(\mu_j-\mu_l)^2}{(\sigma_j^2+\sigma_l^2)^3}+\frac{3}{(\sigma_j^2+\sigma_l^2)^2}\right]\\
&=\frac{3}{8\sqrt{\pi}}\left[\sum_{j=1}^k\frac{w_j^2}{\sigma_j^5}+\frac{8\sqrt{2}}{3}\sum_{j=1}^k\sum_{l>j}^k\frac{w_j w_l}{(\sigma_j^2+\sigma_l^2)^{5/2}}\left(\frac{(\mu_j-\mu_l)^4}{(\sigma_j^2+\sigma_l^2)^2}-\frac{2(\mu_j-\mu_l)^2}{(\sigma_j^2+\sigma_l^2)}+3\right)\exp\left(-\frac{(\mu_j-\mu_l)^2}{2(\sigma_j^2+\sigma_l^2)}\right)\right].
\end{aligned}$$

In this example the mean of X_1, $\mu=\sum_{j=1}^k w_j\mu_j$, is known, i.e. $g(x,\mu)=x-\mu$. Then $\Omega=\mathrm{E}\left[(X-\mu)^2\right]=\sum_{j=1}^k w_j\left[\sigma_j^2+(\mu_j-\mu)^2\right]\equiv\sigma^2$ and $\Psi=\mathrm{E}\left[(X-\mu)^3\right]=\sum_{j=1}^k w_j\left[(\mu_j-\mu)^3+3\sigma_j^2(\mu_j-\mu)\right]\equiv m_3$. Thus, using (15a), gives

$$C_{ivar}=\frac{1}{\sigma^2}\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\left[\bar\sigma_{jl}^2+\left(\mu_{jl}^*-\mu\right)^2\right].$$

Furthermore, $B_1(x)=-\frac{1}{\sigma^2}(x-\mu)^2+\frac{m_3}{\sigma^4}(x-\mu)+1$, and one can write

$$B_1(x)\left[\frac{(x-\mu_l)^2}{\sigma_l^4}-\frac{1}{\sigma_l^2}\right]=-\frac{1}{\sigma^2}\frac{1}{\sigma_l^4}\left[(x-\mu_{jl}^*)^4+a_3(x-\mu_{jl}^*)^3+(U_{jl,l}+V_{jl})(x-\mu_{jl}^*)^2+a_1(x-\mu_{jl}^*)+U_{jl,l}V_{jl}\right],$$

where $V_{jl}=(\mu_{jl}^*-\mu)^2-\frac{m_3}{\sigma^2}(\mu_{jl}^*-\mu)-\sigma^2$, and for some a_1 and a_3. This yields

$$C_{isb}=-\frac{1}{\sigma^2}\sum_{j=1}^k\sum_{l=1}^k w_j w_l \phi_{\sigma_{jl}^*}(\mu_j-\mu_l)\frac{1}{\sigma_l^4}\left[3\bar\sigma_{jl}^4+\bar\sigma_{jl}^2(U_{jl,l}+V_{jl})+U_{jl,l}V_{jl}\right].$$

6 Monte-Carlo study

This simulation study investigates the small sample performance of GELKDE in situations presented in Examples 3 and 4. The sample sizes were chosen to be approximately equispaced on a common logarithm scale. The exact MISE of the unweighted KDE is evaluated analytically, and the exact MISE-minimising bandwidth, h_{MISE}, is used throughout; the values are given in Table 1.

Table 1: Exact MISE-minimising bandwidth

n	Density			
	$N(0,1)$	NM_1	NM_2	NM_3
25	0.6094	0.4251	0.1481	0.0646
32	0.5755	0.3999	0.1316	0.0609
40	0.5469	0.3788	0.1190	0.0577
50	0.5199	0.3591	0.1082	0.0548
63	0.4936	0.3400	0.0984	0.0520
80	0.4680	0.3215	0.0898	0.0492
100	0.4455	0.3054	0.0827	0.0468
126	0.4235	0.2897	0.0762	0.0445
159	0.4025	0.2749	0.0705	0.0423
200	0.3830	0.2611	0.0654	0.0402
250	0.3651	0.2485	0.0610	0.0383
316	0.3472	0.2360	0.0569	0.0364
398	0.3306	0.2244	0.0532	0.0346
500	0.3150	0.2136	0.0499	0.0330
631	0.2999	0.2032	0.0468	0.0314
795	0.2857	0.1934	0.0440	0.0299
1000	0.2723	0.1841	0.0414	0.0285

All results are based on 500,000 replications. Multiple-segment trapezoidal rule numerical integration over a grid of 1,000 points on the $[-4, 4]$ interval is used to obtain the ISB, IVar and MISE of GELKDE. The simulated values for the unweighted KDE are checked against the exact values. For the Gaussian density, the maximum absolute difference between the exact and simulated MISE is $1.6 \cdot 10^{-5}$; for the mixture densities NM_1, NM_2, and NM_3 these are $1.8 \cdot 10^{-5}$, $4.1 \cdot 10^{-5}$, and $1.2 \cdot 10^{-4}$, respectively. For NM_3, the largest error is for $n = 25$; for $n > 25$ the error is less than 10^{-4}. All comparisons are based on the simulated values.

In all cases, implied probabilities are shrunk where necessary to ensure $\tilde{f}$ is nonnegative. EL and ET implied probabilities are non-negative by construction. CUE, however, may produce negative 'probabilities'; Table 2 shows the percentage of times these were shrunk (— corresponds to no such cases, whereas 0.00 means that the proportion is smaller than 0.005%).

Table 2: Percentage of replications in which CUE implied probabilities were shrunk

n	Example 3					Example 4		
	Case 1	Case 2	Case 3	Case 4	Case 5	NM_1	NM_2	NM_3
25	2.92	5.56	27.07	2.85	16.71	3.70	1.28	22.04
32	1.83	3.64	24.45	1.86	16.88	2.44	0.71	20.60
40	1.11	2.73	22.35	1.43	16.46	1.55	0.42	18.76
50	0.63	2.17	20.26	1.18	15.71	0.91	0.21	16.57
63	0.31	1.74	18.21	0.95	14.67	0.54	0.11	14.13
80	0.13	1.38	16.18	0.79	13.39	0.27	0.06	11.48
100	0.06	1.12	14.59	0.64	12.15	0.13	0.03	9.15
126	0.02	0.86	12.83	0.50	10.93	0.06	0.01	7.02
159	0.01	0.62	11.10	0.37	9.73	0.02	0.00	5.15
200	0.00	0.44	9.67	0.25	8.54	0.01	0.00	3.60
250	—	0.33	8.36	0.19	7.45	0.00	—	2.40
316	0.00	0.20	7.02	0.12	6.29	0.00	—	1.48
398	—	0.13	5.90	0.08	5.29	—	—	0.90
500	—	0.09	4.85	0.05	4.43	—	—	0.48
631	—	0.04	3.90	0.03	3.60	—	—	0.22
795	—	0.02	3.08	0.01	2.86	—	—	0.11
1000	—	0.01	2.45	0.01	2.26	—	—	0.04

Finally, in a small proportion of replications in Case 2 of Example 3 zero was not in the convex hull of the data, and an adjusted-GEL solution was taken (EL and ET only). The proportion of such cases was 0.084% with $n = 25$, 0.006% with $n = 32$, and in one out $500,000$ replications with $n = 40$. There was also one such occurence in Case 3 with $n = 25$.

Tables 3–10 show simulation results for scenarios examined in Examples 3 and 4. For each scenario, the first table (a) gives the ISB, IVar, and MISE of the (unweighted) KDE, EL-, ET-, and CUE-based KDEs, and their bias-corrected (bc) versions (bias-correction for EL-based KDE is only done in cases 4 and 5 in Example 3 as otherwise the $1/n$ bias term is zero). Δ denotes the difference between ISB (IVar, MISE) of a GEL-based KDE and the unweighted KDE, and $\Delta\%$ denotes the relative difference. In all tables, the values of $\Delta\%$ ISB (IVar, MISE) are measured in percents; whereas the values of ISB, IVar, and MISE, and the absolute differences, Δ, are multiplied by $10,000$. Figures in section 5 of the main text display the relative differences.

The second table (b) shows the rescaled absolute and relative differences, viz. $nh^{-2}\Delta\,\mathrm{ISB}$, $nh^{2}\Delta\%\,\mathrm{ISB}$, $n\Delta\,\mathrm{IVar}$, $h^{-1}\Delta\%\,\mathrm{IVar}$, $n\Delta\,\mathrm{MISE}$, and $h^{-1}\Delta\%\,\mathrm{MISE}$. The last row in each case ($n = \infty$) shows the respective limiting values. In all examples, the values with $n = 1,000$ are reasonably close to their respective limits. Remark that nh^{-2} can be quite large, which accounts for the lack of precision of the rescaled differences in ISB.

It should be noted that in very small samples bias-correction can perform extremely poorly. An example of a 'bad case' is shown in Figure 1. Here, 25 observations are drawn from a standard normal distribution (shown as small spikes on the horizontal axes in panels (a) and (c)); the sample mean and variance are 0.039 and 0.4, respectively. The extra information used by the GEL-based estimators is that the mean and variance are known to be zero and unity, respectively. As the sample variance is much smaller than one, GEL implied probabilities, shown on panel (b), put most of the weight on the extreme values, producing a bimodal density estimates (CUE probabilities were shrunk). The $1/n$ bias term in this case is a fourth-order polynomial in x multiplying the density estimate, and hence the bias-corrected estimates, shown on panel (c), place most of the mass at what should be the tails of the distribution. This is especially evident with ET (EL in this case is not bias-corrected). Such cases are the reason behind the dramatic increase in MISE of the bias-corrected ET- and CUE-based estimators in very small sample sizes.

Table 3a: Example 1, case 1: Known mean

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%
25	48.45	46.16	-2.29	-4.73	38.48	-9.98	-20.59	32.26	-16.20	-33.43	51.16	2.70	5.57	57.89	9.43	19.47
32	40.10	38.99	-1.11	-2.77	32.97	-7.13	-17.79	27.95	-12.16	-30.32	41.99	1.89	4.70	45.96	5.86	14.61
40	33.77	33.24	-0.53	-1.56	28.55	-5.22	-15.45	24.54	-9.23	-27.34	35.19	1.42	4.19	37.71	3.94	11.67
50	28.41	28.14	-0.27	-0.93	24.54	-3.86	-13.59	21.39	-7.01	-24.68	29.42	1.01	3.56	31.04	2.63	9.27
63	23.77	23.65	-0.13	-0.53	20.95	-2.83	-11.88	18.53	-5.24	-22.05	24.49	0.72	3.02	25.53	1.75	7.38
80	19.74	19.68	-0.06	-0.32	17.69	-2.05	-10.39	15.88	-3.87	-19.58	20.24	0.49	2.49	20.90	1.16	5.86
100	16.47	16.44	-0.04	-0.21	14.95	-1.52	-9.25	13.57	-2.90	-17.60	16.81	0.34	2.05	17.24	0.77	4.69
126	13.81	13.79	-0.02	-0.13	12.69	-1.12	-8.13	11.66	-2.16	-15.61	14.04	0.23	1.67	14.33	0.52	3.73
159	11.49	11.48	-0.01	-0.06	10.67	-0.82	-7.11	9.90	-1.58	-13.79	11.65	0.16	1.40	11.83	0.35	3.04
200	9.55	9.55	-0.00	-0.04	8.95	-0.60	-6.30	8.38	-1.17	-12.27	9.66	0.11	1.13	9.78	0.23	2.42
250	8.02	8.01	-0.00	-0.02	7.57	-0.45	-5.56	7.14	-0.87	-10.89	8.09	0.07	0.93	8.17	0.16	1.96
316	6.65	6.65	-0.00	-0.03	6.32	-0.33	-4.93	6.01	-0.64	-9.66	6.70	0.05	0.71	6.75	0.10	1.51
398	5.55	5.55	-0.00	-0.01	5.31	-0.24	-4.32	5.07	-0.47	-8.50	5.58	0.03	0.59	5.61	0.07	1.22
500	4.62	4.62	-0.00	-0.00	4.44	-0.18	-3.80	4.27	-0.35	-7.50	4.64	0.02	0.48	4.66	0.05	1.00
631	3.83	3.83	0.00	0.01	3.71	-0.13	-3.33	3.58	-0.25	-6.59	3.85	0.02	0.40	3.86	0.03	0.80
795	3.19	3.19	-0.00	-0.01	3.10	-0.09	-2.94	3.01	-0.19	-5.81	3.20	0.01	0.30	3.21	0.02	0.62
1000	2.66	2.66	-0.00	-0.00	2.59	-0.07	-2.57	2.52	-0.14	-5.10	2.67	0.01	0.25	2.67	0.01	0.51
n	IVar	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%
25	88.72	57.52	-31.20	-35.17	56.79	-31.93	-35.99	57.16	-31.56	-35.57	58.73	-29.98	-33.80	60.90	-27.82	-31.36
32	76.83	50.81	-26.02	-33.86	50.27	-26.56	-34.57	50.61	-26.22	-34.13	51.58	-25.25	-32.87	53.14	-23.69	-30.83
40	67.11	45.17	-21.93	-32.68	44.76	-22.35	-33.30	45.04	-22.07	-32.89	45.66	-21.45	-31.96	46.79	-20.31	-30.27
50	58.45	40.10	-18.35	-31.39	39.80	-18.64	-31.90	40.01	-18.44	-31.54	40.42	-18.03	-30.85	41.21	-17.24	-29.50
63	50.54	35.32	-15.22	-30.11	35.12	-15.43	-30.52	35.27	-15.28	-30.23	35.53	-15.02	-29.71	36.06	-14.48	-28.65
80	43.31	30.85	-12.45	-28.76	30.73	-12.57	-29.04	30.83	-12.47	-28.80	30.99	-12.31	-28.43	31.35	-11.95	-27.60
100	37.58	27.27	-10.31	-27.44	27.19	-10.39	-27.65	27.25	-10.32	-27.47	27.36	-10.22	-27.19	27.60	-9.98	-26.56
126	32.23	23.78	-8.45	-26.22	23.74	-8.49	-26.36	23.78	-8.45	-26.22	23.85	-8.38	-26.00	24.01	-8.23	-25.52
159	27.61	20.70	-6.90	-25.01	20.68	-6.93	-25.09	20.71	-6.89	-24.97	20.75	-6.85	-24.82	20.86	-6.75	-24.45
200	23.66	18.02	-5.64	-23.83	18.01	-5.65	-23.87	18.03	-5.63	-23.79	18.06	-5.60	-23.67	18.12	-5.53	-23.40
250	20.30	15.69	-4.61	-22.70	15.69	-4.61	-22.71	15.70	-4.60	-22.65	15.72	-4.58	-22.56	15.76	-4.54	-22.35
316	17.27	13.55	-3.72	-21.53	13.55	-3.72	-21.52	13.56	-3.71	-21.48	13.57	-3.70	-21.41	13.60	-3.67	-21.25
398	14.73	11.72	-3.01	-20.45	11.72	-3.01	-20.42	11.73	-3.00	-20.39	11.74	-3.00	-20.34	11.75	-2.98	-20.22
500	12.53	10.10	-2.43	-19.40	10.11	-2.43	-19.37	10.11	-2.42	-19.35	10.11	-2.42	-19.31	10.12	-2.41	-19.22
631	10.64	8.68	-1.96	-18.46	8.68	-1.96	-18.43	8.68	-1.96	-18.41	8.68	-1.96	-18.38	8.69	-1.95	-18.32
795	9.00	7.43	-1.57	-17.45	7.44	-1.57	-17.42	7.44	-1.57	-17.40	7.44	-1.56	-17.38	7.44	-1.56	-17.33
1000	7.64	6.38	-1.26	-16.49	6.38	-1.26	-16.46	6.38	-1.26	-16.45	6.38	-1.25	-16.43	6.38	-1.25	-16.39
n	MISE	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%
25	137.17	103.68	-33.49	-24.41	95.27	-41.90	-30.55	89.42	-47.75	-34.81	109.89	-27.28	-19.89	118.78	-18.39	-13.40
32	116.93	89.81	-27.13	-23.20	83.24	-33.70	-28.82	78.55	-38.38	-32.82	93.57	-23.37	-19.98	99.11	-17.83	-15.24
40	100.88	78.42	-22.46	-22.27	73.31	-27.57	-27.33	69.57	-31.30	-31.03	80.85	-20.03	-19.85	84.51	-16.37	-16.23
50	86.85	68.24	-18.61	-21.43	64.35	-22.50	-25.91	61.41	-25.45	-29.30	69.83	-17.02	-19.59	72.24	-14.61	-16.82
63	74.32	58.97	-15.34	-20.65	56.07	-18.25	-24.56	53.80	-20.52	-27.61	60.02	-14.30	-19.24	61.59	-12.72	-17.12
80	63.05	50.53	-12.52	-19.85	48.42	-14.63	-23.20	46.71	-16.34	-25.91	51.23	-11.82	-18.74	52.25	-10.80	-17.12
100	54.05	43.70	-10.35	-19.14	42.13	-11.91	-22.04	40.83	-13.22	-24.46	44.17	-9.88	-18.28	44.84	-9.21	-17.03
126	46.04	37.58	-8.47	-18.39	36.43	-9.62	-20.89	35.44	-10.61	-23.03	37.89	-8.15	-17.70	38.33	-7.71	-16.75
159	39.09	32.18	-6.91	-17.68	31.35	-7.74	-19.81	30.61	-8.48	-21.69	32.40	-6.69	-17.12	32.69	-6.40	-16.37
200	33.21	27.56	-5.64	-16.99	26.96	-6.25	-18.82	26.41	-6.80	-20.48	27.71	-5.49	-16.54	27.90	-5.30	-15.97
250	28.31	23.70	-4.61	-16.28	23.26	-5.05	-17.85	22.84	-5.47	-19.32	23.81	-4.50	-15.91	23.93	-4.38	-15.46
316	23.92	20.20	-3.72	-15.55	19.88	-4.04	-16.91	19.57	-4.35	-18.19	20.27	-3.65	-15.26	20.35	-3.57	-14.92
398	20.28	17.27	-3.01	-14.86	17.03	-3.25	-16.02	16.80	-3.48	-17.14	17.31	-2.96	-14.62	17.37	-2.91	-14.36
500	17.15	14.72	-2.43	-14.18	14.55	-2.60	-15.18	14.38	-2.77	-16.16	14.75	-2.40	-13.98	14.79	-2.36	-13.78
631	14.47	12.51	-1.96	-13.57	12.39	-2.09	-14.43	12.26	-2.21	-15.28	12.53	-1.94	-13.41	12.56	-1.92	-13.25
795	12.20	10.62	-1.57	-12.89	10.53	-1.66	-13.63	10.44	-1.75	-14.37	10.64	-1.56	-12.75	10.66	-1.54	-12.63
1000	10.30	9.04	-1.26	-12.23	8.97	-1.33	-12.87	8.90	-1.39	-13.52	9.05	-1.25	-12.12	9.06	-1.24	-12.03

Table 3b: Example 1, case 1: Known mean

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	-0.0154	-43.9	-0.0672	-191.1	-0.1090	-310.3	0.0182	51.7	0.0635	180.8
32	-0.0107	-29.4	-0.0689	-188.5	-0.1175	-321.3	0.0182	49.8	0.0566	154.9
40	-0.0071	-18.7	-0.0698	-184.8	-0.1235	-327.1	0.0189	50.2	0.0527	139.6
50	-0.0049	-12.6	-0.0714	-183.7	-0.1297	-333.5	0.0187	48.2	0.0487	125.2
63	-0.0032	-8.1	-0.0731	-182.4	-0.1356	-338.4	0.0185	46.3	0.0454	113.2
80	-0.0023	-5.6	-0.0750	-182.1	-0.1412	-343.1	0.0180	43.7	0.0422	102.6
100	-0.0018	-4.2	-0.0767	-183.5	-0.1461	-349.2	0.0170	40.6	0.0389	93.1
126	-0.0013	-3.0	-0.0789	-183.6	-0.1515	-352.7	0.0162	37.7	0.0362	84.2
159	-0.0006	-1.5	-0.0802	-183.3	-0.1554	-355.3	0.0158	36.2	0.0343	78.3
200	-0.0005	-1.2	-0.0820	-184.9	-0.1598	-360.2	0.0147	33.1	0.0315	71.0
250	-0.0003	-0.8	-0.0837	-185.4	-0.1637	-362.7	0.0139	30.9	0.0295	65.3
316	-0.0006	-1.3	-0.0860	-187.9	-0.1684	-368.1	0.0123	26.9	0.0263	57.4
398	-0.0001	-0.3	-0.0872	-187.8	-0.1717	-369.8	0.0118	25.5	0.0247	53.1
500	-0.0001	-0.2	-0.0883	-188.5	-0.1743	-371.9	0.0112	23.9	0.0232	49.6
631	0.0002	0.5	-0.0895	-188.9	-0.1772	-374.0	0.0106	22.4	0.0216	45.6
795	-0.0002	-0.3	-0.0913	-190.7	-0.1807	-377.4	0.0093	19.5	0.0193	40.4
1000	-0.0001	-0.2	-0.0923	-190.8	-0.1829	-378.3	0.0089	18.4	0.0183	37.8
∞	0	0	-0.1058	-200	-0.2116	-400	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.0780	-57.7	-0.0798	-59.1	-0.0789	-58.4	-0.0750	-55.5	-0.0696	-51.5
32	-0.0833	-58.8	-0.0850	-60.1	-0.0839	-59.3	-0.0808	-57.1	-0.0758	-53.6
40	-0.0877	-59.8	-0.0894	-60.9	-0.0883	-60.1	-0.0858	-58.4	-0.0812	-55.3
50	-0.0917	-60.4	-0.0932	-61.4	-0.0922	-60.7	-0.0902	-59.3	-0.0862	-56.7
63	-0.0959	-61.0	-0.0972	-61.8	-0.0962	-61.2	-0.0946	-60.2	-0.0912	-58.0
80	-0.0996	-61.5	-0.1006	-62.0	-0.0998	-61.5	-0.0985	-60.7	-0.0956	-59.0
100	-0.1031	-61.6	-0.1039	-62.1	-0.1032	-61.7	-0.1022	-61.0	-0.0998	-59.6
126	-0.1065	-61.9	-0.1070	-62.2	-0.1065	-61.9	-0.1056	-61.4	-0.1036	-60.3
159	-0.1098	-62.1	-0.1101	-62.3	-0.1096	-62.0	-0.1090	-61.7	-0.1073	-60.7
200	-0.1127	-62.2	-0.1130	-62.3	-0.1126	-62.1	-0.1120	-61.8	-0.1107	-61.1
250	-0.1152	-62.2	-0.1152	-62.2	-0.1149	-62.0	-0.1145	-61.8	-0.1134	-61.2
316	-0.1175	-62.0	-0.1174	-62.0	-0.1172	-61.8	-0.1168	-61.7	-0.1160	-61.2
398	-0.1199	-61.8	-0.1198	-61.8	-0.1196	-61.7	-0.1193	-61.5	-0.1186	-61.2
500	-0.1216	-61.6	-0.1214	-61.5	-0.1212	-61.4	-0.1210	-61.3	-0.1204	-61.0
631	-0.1240	-61.6	-0.1237	-61.4	-0.1236	-61.4	-0.1234	-61.3	-0.1230	-61.1
795	-0.1249	-61.1	-0.1247	-60.9	-0.1246	-60.9	-0.1244	-60.8	-0.1240	-60.6
1000	-0.1259	-60.6	-0.1257	-60.4	-0.1256	-60.4	-0.1255	-60.3	-0.1252	-60.2
∞	-0.1410	-50	-0.1410	-50	-0.1410	-50	-0.1410	-50	-0.1410	-50
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	-0.0837	-40.1	-0.1048	-50.1	-0.1194	-57.1	-0.0682	-32.6	-0.0460	-22.0
32	-0.0868	-40.3	-0.1078	-50.1	-0.1228	-57.0	-0.0748	-34.7	-0.0570	-26.5
40	-0.0898	-40.7	-0.1103	-50.0	-0.1252	-56.7	-0.0801	-36.3	-0.0655	-29.7
50	-0.0931	-41.2	-0.1125	-49.8	-0.1272	-56.4	-0.0851	-37.7	-0.0730	-32.4
63	-0.0967	-41.8	-0.1150	-49.8	-0.1293	-55.9	-0.0901	-39.0	-0.0802	-34.7
80	-0.1001	-42.4	-0.1170	-49.6	-0.1307	-55.4	-0.0945	-40.1	-0.0864	-36.6
100	-0.1035	-43.0	-0.1191	-49.5	-0.1322	-54.9	-0.0988	-41.0	-0.0921	-38.2
126	-0.1067	-43.4	-0.1212	-49.3	-0.1336	-54.4	-0.1027	-41.8	-0.0972	-39.5
159	-0.1099	-43.9	-0.1231	-49.2	-0.1348	-53.9	-0.1064	-42.5	-0.1018	-40.7
200	-0.1128	-44.3	-0.1250	-49.1	-0.1360	-53.5	-0.1099	-43.2	-0.1061	-41.7
250	-0.1152	-44.6	-0.1264	-48.9	-0.1367	-52.9	-0.1126	-43.6	-0.1094	-42.4
316	-0.1176	-44.8	-0.1278	-48.7	-0.1375	-52.4	-0.1153	-43.9	-0.1128	-43.0
398	-0.1199	-44.9	-0.1293	-48.5	-0.1383	-51.8	-0.1180	-44.2	-0.1159	-43.4
500	-0.1216	-45.0	-0.1302	-48.2	-0.1385	-51.3	-0.1199	-44.4	-0.1181	-43.7
631	-0.1239	-45.2	-0.1318	-48.1	-0.1396	-50.9	-0.1225	-44.7	-0.1210	-44.2
795	-0.1249	-45.1	-0.1321	-47.7	-0.1393	-50.3	-0.1236	-44.6	-0.1225	-44.2
1000	-0.1259	-44.9	-0.1325	-47.3	-0.1392	-49.6	-0.1248	-44.5	-0.1238	-44.2
∞	-0.1410	-40	-0.1410	-40	-0.1410	-40	-0.1410	-40	-0.1410	-40

Table 4a: Example 1, case 2: Known mean and variance

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$
25	48.45	68.13	19.68	40.61	68.33	19.87	41.02	68.92	20.47	42.24	81.76	33.31	68.74	90.20	41.74	86.15
32	40.10	54.30	14.19	35.39	54.59	14.48	36.11	56.11	16.01	39.91	55.70	15.59	38.89	60.48	20.37	50.79
40	33.77	43.68	9.91	29.34	44.41	10.63	31.49	46.20	12.43	36.81	43.36	9.59	28.38	45.21	11.44	33.86
50	28.41	34.47	6.06	21.34	36.13	7.73	27.21	37.89	9.49	33.41	34.45	6.04	21.28	35.03	6.63	23.33
63	23.77	26.94	3.17	13.33	29.25	5.48	23.05	30.84	7.07	29.72	27.50	3.72	15.66	27.55	3.78	15.89
80	19.74	21.53	1.79	9.07	23.56	3.82	19.35	24.91	5.17	26.16	21.99	2.25	11.37	21.85	2.11	10.69
100	16.47	17.59	1.12	6.79	19.24	2.77	16.83	20.37	3.90	23.68	17.92	1.44	8.77	17.75	1.28	7.77
126	13.81	14.34	0.53	3.80	15.69	1.88	13.58	16.60	2.79	20.16	14.61	0.80	5.78	14.45	0.64	4.64
159	11.49	11.78	0.30	2.61	12.80	1.32	11.45	13.52	2.04	17.72	11.95	0.47	4.06	11.82	0.34	2.94
200	9.55	9.75	0.20	2.09	10.52	0.97	10.18	11.08	1.54	16.08	9.86	0.31	3.27	9.76	0.21	2.21
250	8.02	8.07	0.05	0.62	8.69	0.67	8.37	9.12	1.11	13.82	8.18	0.16	2.03	8.10	0.09	1.08
316	6.65	6.66	0.01	0.16	7.13	0.48	7.18	7.46	0.81	12.20	6.75	0.09	1.41	6.69	0.04	0.59
398	5.55	5.54	-0.00	-0.04	5.88	0.34	6.04	6.13	0.59	10.62	5.59	0.05	0.85	5.55	0.01	0.16
500	4.62	4.60	-0.01	-0.26	4.85	0.24	5.10	5.04	0.43	9.25	4.64	0.02	0.43	4.61	-0.01	-0.15
631	3.83	3.83	-0.00	-0.11	4.01	0.17	4.50	4.15	0.32	8.24	3.85	0.01	0.34	3.83	-0.01	-0.13
795	3.19	3.19	-0.00	-0.04	3.31	0.12	3.86	3.42	0.23	7.20	3.20	0.01	0.17	3.18	-0.01	-0.23
1000	2.66	2.66	0.00	0.07	2.75	0.09	3.42	2.83	0.17	6.41	2.66	0.00	0.15	2.66	-0.00	-0.17
n	IVar	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$
25	88.72	53.76	-34.95	-39.40	51.11	-37.61	-42.39	56.02	-32.70	-36.86	51686.96	51598.24	58161.32	74.03	-14.69	-16.56
32	76.83	47.54	-29.29	-38.12	43.58	-33.25	-43.28	48.44	-28.39	-36.95	103.06	26.23	34.15	51.08	-25.75	-33.52
40	67.11	41.56	-25.55	-38.07	37.65	-29.46	-43.90	41.59	-25.52	-38.03	37.68	-29.43	-43.86	41.87	-25.24	-37.61
50	58.45	36.11	-22.33	-38.21	32.73	-25.72	-44.00	35.65	-22.80	-39.01	32.60	-25.85	-44.23	35.43	-23.02	-39.39
63	50.54	31.44	-19.10	-37.80	28.51	-22.04	-43.60	30.51	-20.03	-39.63	28.36	-22.18	-43.88	30.23	-20.31	-40.19
80	43.31	26.55	-16.76	-38.70	24.73	-18.58	-42.90	26.06	-17.24	-39.81	24.61	-18.70	-43.17	25.83	-17.47	-40.35
100	37.58	23.12	-14.46	-38.47	21.83	-15.74	-41.89	22.75	-14.83	-39.47	21.74	-15.83	-42.13	22.57	-15.01	-39.94
126	32.23	19.99	-12.24	-37.97	19.08	-13.15	-40.81	19.68	-12.55	-38.93	19.01	-13.22	-41.01	19.55	-12.68	-39.33
159	27.61	17.26	-10.34	-37.46	16.69	-10.91	-39.54	17.09	-10.52	-38.10	16.64	-10.96	-39.70	17.00	-10.61	-38.43
200	23.66	15.01	-8.65	-36.55	14.64	-9.01	-38.10	14.90	-8.75	-37.01	14.61	-9.05	-38.24	14.84	-8.82	-37.28
250	20.30	13.17	-7.12	-35.09	12.87	-7.42	-36.59	13.04	-7.25	-35.73	12.85	-7.45	-36.70	13.00	-7.30	-35.95
316	17.27	11.46	-5.81	-33.66	11.20	-6.07	-35.14	11.31	-5.95	-34.48	11.18	-6.08	-35.23	11.28	-5.98	-34.66
398	14.73	9.98	-4.75	-32.27	9.78	-4.95	-33.63	9.85	-4.88	-33.13	9.77	-4.96	-33.70	9.83	-4.90	-33.27
500	12.53	8.58	-3.95	-31.50	8.51	-4.02	-32.11	8.56	-3.98	-31.72	8.50	-4.03	-32.17	8.54	-3.99	-31.84
631	10.64	7.42	-3.22	-30.25	7.37	-3.27	-30.74	7.40	-3.24	-30.44	7.37	-3.28	-30.78	7.39	-3.25	-30.54
795	9.00	6.40	-2.60	-28.90	6.37	-2.63	-29.24	6.39	-2.61	-29.01	6.37	-2.64	-29.28	6.39	-2.62	-29.09
1000	7.64	5.51	-2.12	-27.81	5.51	-2.12	-27.79	5.53	-2.11	-27.62	5.51	-2.12	-27.82	5.52	-2.11	-27.67
n	MISE	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$
25	137.17	121.89	-15.28	-11.14	119.43	-17.74	-12.93	124.94	-12.23	-8.91	51768.72	51631.55	37640.36	164.23	27.06	19.72
32	116.93	101.84	-15.10	-12.91	98.16	-18.77	-16.05	104.56	-12.38	-10.59	158.76	41.83	35.77	111.55	-5.38	-4.60
40	100.88	85.24	-15.64	-15.50	82.05	-18.83	-18.66	87.79	-13.09	-12.97	81.03	-19.84	-19.67	87.08	-13.80	-13.68
50	86.85	70.58	-16.27	-18.73	68.86	-17.99	-20.71	73.54	-13.31	-15.33	67.05	-19.80	-22.80	70.46	-16.39	-18.87
63	74.32	58.38	-15.94	-21.44	57.76	-16.56	-22.28	61.35	-12.97	-17.45	55.86	-18.46	-24.84	57.78	-16.54	-22.25
80	63.05	48.08	-14.97	-23.74	48.29	-14.76	-23.41	50.97	-12.08	-19.15	46.60	-16.45	-26.09	47.69	-15.36	-24.37
100	54.05	40.71	-13.34	-24.68	41.08	-12.97	-24.00	43.12	-10.93	-20.22	39.66	-14.39	-26.62	40.32	-13.73	-25.40
126	46.04	34.33	-11.71	-25.44	34.77	-11.28	-24.49	36.28	-9.76	-21.20	33.62	-12.42	-26.97	34.01	-12.04	-26.14
159	39.09	29.05	-10.04	-25.69	29.49	-9.60	-24.55	30.61	-8.48	-21.70	28.60	-10.49	-26.84	28.82	-10.27	-26.28
200	33.21	24.76	-8.45	-25.44	25.16	-8.04	-24.22	25.99	-7.22	-21.74	24.47	-8.73	-26.30	24.60	-8.61	-25.92
250	28.31	21.24	-7.07	-24.98	21.56	-6.75	-23.86	22.17	-6.14	-21.70	21.03	-7.29	-25.73	21.10	-7.21	-25.47
316	23.92	18.12	-5.80	-24.26	18.33	-5.59	-23.37	18.78	-5.14	-21.50	17.93	-5.99	-25.04	17.97	-5.95	-24.86
398	20.28	15.52	-4.76	-23.46	15.66	-4.62	-22.78	15.99	-4.29	-21.16	15.36	-4.92	-24.25	15.38	-4.89	-24.13
500	17.15	13.19	-3.96	-23.09	13.36	-3.79	-22.10	13.60	-3.55	-20.70	13.14	-4.01	-23.40	13.15	-4.00	-23.31
631	14.47	11.25	-3.22	-22.27	11.38	-3.10	-21.40	11.55	-2.92	-20.20	11.21	-3.26	-22.54	11.22	-3.25	-22.48
795	12.20	9.59	-2.60	-21.34	9.69	-2.51	-20.58	9.81	-2.38	-19.54	9.57	-2.63	-21.57	9.57	-2.63	-21.53
1000	10.30	8.17	-2.12	-20.61	8.26	-2.03	-19.72	8.36	-1.94	-18.83	8.18	-2.12	-20.59	8.18	-2.12	-20.57

Table 4b: Example 1, case 2: Known mean and variance

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	0.1325	377.0	0.1338	380.8	0.1378	392.2	0.2242	638.2	0.2810	799.8
32	0.1371	375.2	0.1399	382.8	0.1546	423.1	0.1507	412.2	0.1968	538.4
40	0.1325	351.0	0.1422	376.7	0.1663	440.4	0.1282	339.5	0.1530	405.1
50	0.1121	288.4	0.1430	367.7	0.1755	451.4	0.1118	287.6	0.1226	315.3
63	0.0819	204.5	0.1417	353.8	0.1827	456.1	0.0963	240.4	0.0977	243.9
80	0.0654	158.9	0.1395	339.0	0.1887	458.4	0.0820	199.3	0.0771	187.3
100	0.0563	134.7	0.1397	334.0	0.1965	469.8	0.0728	174.0	0.0645	154.2
126	0.0369	86.0	0.1318	306.8	0.1957	455.6	0.0561	130.6	0.0451	104.9
159	0.0294	67.1	0.1291	295.0	0.1997	456.6	0.0458	104.7	0.0331	75.7
200	0.0272	61.3	0.1325	298.6	0.2093	471.8	0.0426	95.9	0.0288	64.8
250	0.0094	20.8	0.1258	278.8	0.2078	460.6	0.0304	67.5	0.0162	35.9
316	0.0027	6.0	0.1252	273.6	0.2127	465.0	0.0245	53.6	0.0102	22.4
398	-0.0009	-1.9	0.1221	263.0	0.2144	461.9	0.0171	36.8	0.0032	6.9
500	-0.0060	-12.7	0.1185	252.9	0.2150	458.8	0.0099	21.2	-0.0035	-7.5
631	-0.0029	-6.2	0.1211	255.6	0.2215	467.6	0.0092	19.4	-0.0036	-7.6
795	-0.0011	-2.3	0.1200	250.7	0.2239	467.6	0.0052	10.9	-0.0071	-14.7
1000	0.0027	5.5	0.1228	254.0	0.2298	475.3	0.0055	11.5	-0.0060	-12.4
∞	0	0	0.1322	250	0.2645	500	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.0874	-64.7	-0.0940	-69.6	-0.0817	-60.5	128.9956	95443.1	-0.0367	-27.2
32	-0.0937	-66.2	-0.1064	-75.2	-0.0908	-64.2	0.0840	59.3	-0.0824	-58.2
40	-0.1022	-69.6	-0.1178	-80.3	-0.1021	-69.5	-0.1177	-80.2	-0.1009	-68.8
50	-0.1117	-73.5	-0.1286	-84.6	-0.1140	-75.0	-0.1292	-85.1	-0.1151	-75.8
63	-0.1204	-76.6	-0.1388	-88.3	-0.1262	-80.3	-0.1397	-88.9	-0.1280	-81.4
80	-0.1341	-82.7	-0.1486	-91.7	-0.1379	-85.1	-0.1496	-92.3	-0.1398	-86.2
100	-0.1446	-86.4	-0.1574	-94.0	-0.1483	-88.6	-0.1583	-94.6	-0.1501	-89.7
126	-0.1542	-89.7	-0.1657	-96.4	-0.1581	-91.9	-0.1666	-96.8	-0.1597	-92.9
159	-0.1644	-93.1	-0.1735	-98.2	-0.1672	-94.7	-0.1743	-98.6	-0.1687	-95.5
200	-0.1729	-95.4	-0.1803	-99.5	-0.1751	-96.6	-0.1809	-99.8	-0.1764	-97.3
250	-0.1780	-96.1	-0.1856	-100.2	-0.1813	-97.9	-0.1862	-100.5	-0.1824	-98.5
316	-0.1837	-96.9	-0.1917	-101.2	-0.1881	-99.3	-0.1922	-101.5	-0.1891	-99.8
398	-0.1892	-97.6	-0.1972	-101.7	-0.1942	-100.2	-0.1976	-101.9	-0.1951	-100.6
500	-0.1974	-100.0	-0.2012	-101.9	-0.1988	-100.7	-0.2016	-102.1	-0.1995	-101.1
631	-0.2031	-100.9	-0.2064	-102.5	-0.2044	-101.5	-0.2067	-102.6	-0.2050	-101.8
795	-0.2069	-101.1	-0.2093	-102.3	-0.2077	-101.5	-0.2096	-102.5	-0.2082	-101.8
1000	-0.2123	-102.1	-0.2122	-102.0	-0.2109	-101.4	-0.2124	-102.1	-0.2113	-101.6
∞	-0.2468	-87.5	-0.2468	-87.5	-0.2468	-87.5	-0.2468	-87.5	-0.2468	-87.5
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	-0.0382	-18.3	-0.0443	-21.2	-0.0306	-14.6	129.0789	61768.1	0.0676	32.4
32	-0.0483	-22.4	-0.0601	-27.9	-0.0396	-18.4	0.1339	62.2	-0.0172	-8.0
40	-0.0626	-28.3	-0.0753	-34.1	-0.0523	-23.7	-0.0794	-36.0	-0.0552	-25.0
50	-0.0814	-36.0	-0.0899	-39.8	-0.0666	-29.5	-0.0990	-43.9	-0.0820	-36.3
63	-0.1004	-43.4	-0.1043	-45.1	-0.0817	-35.4	-0.1163	-50.3	-0.1042	-45.1
80	-0.1197	-50.7	-0.1181	-50.0	-0.0966	-40.9	-0.1316	-55.8	-0.1229	-52.1
100	-0.1334	-55.4	-0.1297	-53.9	-0.1093	-45.4	-0.1439	-59.8	-0.1373	-57.0
126	-0.1476	-60.1	-0.1421	-57.8	-0.1230	-50.1	-0.1565	-63.7	-0.1516	-61.7
159	-0.1597	-63.8	-0.1526	-61.0	-0.1349	-53.9	-0.1668	-66.7	-0.1633	-65.3
200	-0.1689	-66.4	-0.1608	-63.2	-0.1444	-56.8	-0.1747	-68.7	-0.1721	-67.7
250	-0.1768	-68.4	-0.1689	-65.3	-0.1536	-59.4	-0.1821	-70.5	-0.1802	-69.8
316	-0.1834	-69.9	-0.1767	-67.3	-0.1625	-61.9	-0.1893	-72.1	-0.1879	-71.6
398	-0.1893	-71.0	-0.1838	-68.9	-0.1708	-64.0	-0.1957	-73.4	-0.1947	-73.0
500	-0.1980	-73.3	-0.1895	-70.1	-0.1775	-65.7	-0.2006	-74.3	-0.1999	-74.0
631	-0.2034	-74.2	-0.1955	-71.4	-0.1845	-67.3	-0.2059	-75.2	-0.2054	-75.0
795	-0.2069	-74.7	-0.1995	-72.0	-0.1894	-68.4	-0.2091	-75.5	-0.2088	-75.4
1000	-0.2121	-75.7	-0.2031	-72.4	-0.1938	-69.1	-0.2120	-75.6	-0.2117	-75.5
∞	-0.2468	-70	-0.2468	-70	-0.2468	-70	-0.2468	-70	-0.2468	-70

Table 5a: Example 1, case 3: Known mean and third moment

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%
25	48.45	35.72	-12.74	-26.29	24.50	-23.96	-49.44	22.39	-26.06	-53.79	240.37	191.91	396.06	504.36	455.90	940.88
32	40.10	32.93	-7.18	-17.90	22.97	-17.13	-42.72	20.39	-19.71	-49.15	84.83	44.73	111.53	186.20	146.09	364.28
40	33.77	29.74	-4.03	-11.93	21.32	-12.46	-36.88	18.64	-15.13	-44.80	49.93	16.16	47.85	93.75	59.97	177.59
50	28.41	26.22	-2.18	-7.69	19.37	-9.03	-31.80	16.84	-11.57	-40.72	35.03	6.62	23.31	55.67	27.26	95.98
63	23.77	22.70	-1.08	-4.53	17.34	-6.43	-27.07	15.07	-8.70	-36.62	26.68	2.90	12.22	36.89	13.12	55.19
80	19.74	19.25	-0.49	-2.48	15.21	-4.54	-22.98	13.29	-6.46	-32.71	21.08	1.34	6.79	26.35	6.61	33.48
100	16.47	16.26	-0.21	-1.27	13.20	-3.27	-19.86	11.61	-4.86	-29.52	17.17	0.69	4.21	20.12	3.65	22.16
126	13.81	13.74	-0.07	-0.51	11.47	-2.34	-16.94	10.19	-3.63	-26.26	14.19	0.38	2.75	15.87	2.06	14.92
159	11.49	11.47	-0.01	-0.09	9.82	-1.66	-14.47	8.81	-2.68	-23.33	11.71	0.23	1.98	12.69	1.20	10.46
200	9.55	9.56	0.01	0.13	8.36	-1.19	-12.48	7.56	-1.99	-20.81	9.69	0.14	1.47	10.27	0.72	7.50
250	8.02	8.03	0.02	0.19	7.15	-0.87	-10.83	6.53	-1.49	-18.53	8.11	0.09	1.11	8.46	0.44	5.51
316	6.65	6.67	0.01	0.21	6.03	-0.62	-9.36	5.56	-1.09	-16.43	6.71	0.06	0.85	6.92	0.27	4.01
398	5.55	5.56	0.01	0.21	5.10	-0.45	-8.08	4.74	-0.80	-14.51	5.58	0.04	0.69	5.71	0.17	2.99
500	4.62	4.62	0.01	0.17	4.29	-0.32	-7.03	4.02	-0.59	-12.83	4.64	0.03	0.57	4.72	0.11	2.28
631	3.83	3.84	0.01	0.17	3.60	-0.23	-6.09	3.40	-0.43	-11.31	3.85	0.02	0.46	3.90	0.07	1.72
795	3.19	3.19	0.00	0.10	3.02	-0.17	-5.30	2.87	-0.32	-9.96	3.20	0.01	0.37	3.23	0.04	1.30
1000	2.66	2.66	0.00	0.09	2.54	-0.12	-4.62	2.43	-0.23	-8.78	2.67	0.01	0.30	2.69	0.03	0.98
n	IVar	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%
25	88.72	46.28	-42.44	-47.84	46.56	-42.16	-47.52	44.74	-43.97	-49.57	365988.16	365899.44	412440.30	512.94	424.23	478.19
32	76.83	40.20	-36.63	-47.67	40.45	-36.38	-47.35	39.71	-37.12	-48.31	284.44	207.61	270.22	125.23	48.40	63.00
40	67.11	35.54	-31.56	-47.03	35.60	-31.51	-46.96	35.45	-31.66	-47.18	43.23	-23.87	-35.58	58.38	-8.73	-13.00
50	58.45	31.75	-26.69	-45.67	31.64	-26.81	-45.87	31.74	-26.71	-45.70	34.01	-24.43	-41.80	39.75	-18.69	-31.98
63	50.54	28.24	-22.30	-44.13	28.02	-22.53	-44.57	28.22	-22.33	-44.18	29.51	-21.03	-41.61	31.93	-18.62	-36.83
80	43.31	24.97	-18.34	-42.35	24.70	-18.61	-42.97	24.89	-18.41	-42.52	25.84	-17.47	-40.33	27.11	-16.20	-37.40
100	37.58	22.31	-15.26	-40.62	22.05	-15.53	-41.33	22.21	-15.37	-40.90	22.94	-14.64	-38.96	23.77	-13.81	-36.74
126	32.23	19.72	-12.51	-38.81	19.49	-12.74	-39.54	19.60	-12.64	-39.20	20.16	-12.07	-37.44	20.75	-11.48	-35.61
159	27.61	17.38	-10.23	-37.05	17.18	-10.43	-37.77	17.25	-10.36	-37.52	17.68	-9.92	-35.95	18.11	-9.50	-34.40
200	23.66	15.27	-8.38	-35.43	15.11	-8.54	-36.11	15.15	-8.50	-35.94	15.48	-8.18	-34.57	15.78	-7.87	-33.27
250	20.30	13.42	-6.88	-33.88	13.30	-7.00	-34.48	13.31	-6.98	-34.39	13.56	-6.73	-33.18	13.78	-6.51	-32.09
316	17.27	11.70	-5.56	-32.22	11.61	-5.66	-32.78	11.61	-5.66	-32.76	11.80	-5.47	-31.69	11.95	-5.32	-30.80
398	14.73	10.20	-4.53	-30.77	10.13	-4.60	-31.25	10.13	-4.61	-31.26	10.26	-4.47	-30.36	10.37	-4.37	-29.63
500	12.53	8.86	-3.67	-29.31	8.81	-3.72	-29.72	8.80	-3.73	-29.77	8.90	-3.63	-28.98	8.97	-3.56	-28.41
631	10.64	7.76	-2.88	-27.08	7.63	-3.01	-28.29	7.62	-3.02	-28.37	7.69	-2.95	-27.69	7.74	-2.90	-27.25
795	9.00	6.61	-2.39	-26.56	6.59	-2.41	-26.81	6.58	-2.42	-26.90	6.63	-2.37	-26.33	6.66	-2.34	-25.98
1000	7.64	5.75	-1.88	-24.67	5.69	-1.95	-25.52	5.68	-1.95	-25.60	5.72	-1.92	-25.13	5.74	-1.90	-24.86
n	MISE	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%
25	137.17	81.99	-55.18	-40.23	71.06	-66.12	-48.20	67.13	-70.04	-51.06	366228.53	366091.36	266887.43	1017.30	880.13	641.63
32	116.93	73.13	-43.81	-37.46	63.42	-53.51	-45.76	60.11	-56.83	-48.60	369.28	252.34	215.80	311.43	194.50	166.33
40	100.88	65.29	-35.59	-35.28	56.91	-43.97	-43.58	54.09	-46.79	-46.38	93.16	-7.71	-7.65	152.13	51.25	50.80
50	86.85	57.97	-28.88	-33.25	51.01	-35.84	-41.27	48.58	-38.28	-44.07	69.04	-17.81	-20.51	95.42	8.57	9.87
63	74.32	50.94	-23.38	-31.46	45.35	-28.96	-38.97	43.28	-31.03	-41.76	56.19	-18.12	-24.39	68.82	-5.50	-7.40
80	63.05	44.22	-18.83	-29.86	39.91	-23.14	-36.71	38.18	-24.87	-39.45	46.92	-16.12	-25.57	53.46	-9.58	-15.20
100	54.05	38.58	-15.47	-28.62	35.25	-18.80	-34.78	33.82	-20.23	-37.43	40.10	-13.95	-25.80	43.89	-10.16	-18.79
126	46.04	33.46	-12.58	-27.32	30.96	-15.08	-32.76	29.78	-16.26	-35.32	34.36	-11.69	-25.38	36.63	-9.42	-20.45
159	39.09	28.85	-10.24	-26.19	27.00	-12.09	-30.93	26.05	-13.04	-33.35	29.39	-9.70	-24.81	30.80	-8.30	-21.22
200	33.21	24.84	-8.37	-25.21	23.47	-9.74	-29.32	22.72	-10.49	-31.59	25.17	-8.04	-24.21	26.05	-7.16	-21.55
250	28.31	21.45	-6.86	-24.23	20.44	-7.87	-27.79	19.85	-8.47	-29.90	21.67	-6.64	-23.47	22.24	-6.07	-21.44
316	23.92	18.37	-5.55	-23.20	17.64	-6.28	-26.26	17.17	-6.75	-28.21	18.51	-5.41	-22.64	18.87	-5.05	-21.12
398	20.28	15.76	-4.52	-22.30	15.23	-5.05	-24.92	14.87	-5.41	-26.68	15.84	-4.43	-21.87	16.08	-4.20	-20.71
500	17.15	13.48	-3.67	-21.38	13.10	-4.05	-23.61	12.83	-4.32	-25.21	13.54	-3.61	-21.03	13.69	-3.46	-20.15
631	14.47	11.60	-2.87	-19.86	11.23	-3.24	-22.41	11.02	-3.45	-23.85	11.55	-2.93	-20.24	11.64	-2.83	-19.58
795	12.20	9.81	-2.39	-19.58	9.61	-2.58	-21.18	9.46	-2.74	-22.46	9.84	-2.36	-19.34	9.90	-2.30	-18.84
1000	10.30	8.41	-1.88	-18.27	8.22	-2.07	-20.12	8.11	-2.19	-21.25	8.38	-1.91	-18.56	8.42	-1.87	-18.18

Table 5b: Example 1, case 3: Known mean and third moment

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	-0.0858	-244.1	-0.1613	-459.0	-0.1755	-499.4	1.2920	3676.9	3.0692	8734.8
32	-0.0693	-189.7	-0.1655	-452.8	-0.1904	-521.0	0.4321	1182.2	1.4114	3861.2
40	-0.0539	-142.7	-0.1666	-441.2	-0.2024	-536.0	0.2162	572.4	0.8022	2124.4
50	-0.0404	-103.9	-0.1671	-429.7	-0.2140	-550.2	0.1225	315.0	0.5044	1297.1
63	-0.0278	-69.5	-0.1664	-415.4	-0.2251	-561.9	0.0751	187.5	0.3393	847.0
80	-0.0179	-43.4	-0.1657	-402.7	-0.2359	-573.1	0.0490	119.0	0.2415	586.6
100	-0.0105	-25.2	-0.1648	-394.0	-0.2450	-585.8	0.0350	83.6	0.1840	439.8
126	-0.0050	-11.6	-0.1644	-382.7	-0.2549	-593.4	0.0267	62.1	0.1448	337.2
159	-0.0010	-2.4	-0.1631	-372.8	-0.2630	-601.1	0.0223	51.0	0.1179	269.6
200	0.0017	3.8	-0.1625	-366.3	-0.2709	-610.6	0.0191	43.0	0.0976	220.1
250	0.0029	6.4	-0.1628	-360.8	-0.2786	-617.5	0.0167	37.1	0.0828	183.5
316	0.0036	7.8	-0.1631	-356.6	-0.2864	-625.9	0.0149	32.5	0.0699	152.8
398	0.0043	9.3	-0.1632	-351.6	-0.2931	-631.4	0.0140	30.1	0.0603	129.9
500	0.0039	8.3	-0.1634	-348.6	-0.2985	-636.8	0.0132	28.2	0.0530	113.0
631	0.0047	9.8	-0.1638	-345.7	-0.3040	-641.7	0.0125	26.3	0.0463	97.6
795	0.0031	6.5	-0.1646	-343.8	-0.3094	-646.3	0.0116	24.2	0.0405	84.6
1000	0.0033	6.9	-0.1657	-342.7	-0.3148	-651.1	0.0106	21.9	0.0352	72.8
∞	0	0	-0.1807	-341.6	-0.3614	-683.3	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.1061	-78.5	-0.1054	-78.0	-0.1099	-81.3	914.7486	676817.3	1.0606	784.7
32	-0.1172	-82.8	-0.1164	-82.3	-0.1188	-83.9	0.6644	469.5	0.1549	109.5
40	-0.1262	-86.0	-0.1260	-85.9	-0.1266	-86.3	-0.0955	-65.1	-0.0349	-23.8
50	-0.1335	-87.9	-0.1340	-88.2	-0.1335	-87.9	-0.1222	-80.4	-0.0935	-61.5
63	-0.1405	-89.4	-0.1419	-90.3	-0.1407	-89.5	-0.1325	-84.3	-0.1173	-74.6
80	-0.1467	-90.5	-0.1489	-91.8	-0.1473	-90.9	-0.1397	-86.2	-0.1296	-79.9
100	-0.1526	-91.2	-0.1553	-92.8	-0.1537	-91.8	-0.1464	-87.5	-0.1381	-82.5
126	-0.1576	-91.6	-0.1606	-93.4	-0.1592	-92.6	-0.1520	-88.4	-0.1446	-84.1
159	-0.1626	-92.0	-0.1658	-93.8	-0.1647	-93.2	-0.1578	-89.3	-0.1510	-85.5
200	-0.1676	-92.5	-0.1709	-94.3	-0.1701	-93.8	-0.1636	-90.3	-0.1574	-86.9
250	-0.1719	-92.8	-0.1750	-94.5	-0.1745	-94.2	-0.1683	-90.9	-0.1628	-87.9
316	-0.1758	-92.8	-0.1788	-94.4	-0.1787	-94.3	-0.1729	-91.3	-0.1681	-88.7
398	-0.1804	-93.1	-0.1832	-94.5	-0.1833	-94.5	-0.1780	-91.8	-0.1737	-89.6
500	-0.1837	-93.0	-0.1862	-94.3	-0.1865	-94.5	-0.1816	-92.0	-0.1780	-90.2
631	-0.1818	-90.3	-0.1900	-94.3	-0.1905	-94.6	-0.1859	-92.3	-0.1830	-90.9
795	-0.1901	-92.9	-0.1920	-93.8	-0.1925	-94.1	-0.1885	-92.1	-0.1860	-90.9
1000	-0.1884	-90.6	-0.1948	-93.7	-0.1955	-94.0	-0.1918	-92.3	-0.1898	-91.3
∞	-0.2292	-81.25	-0.2292	-81.25	-0.2292	-81.25	-0.2292	-81.25	-0.2292	-81.25
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	-0.1379	-66.0	-0.1653	-79.1	-0.1751	-83.8	915.2284	437964.1	2.2003	1052.9
32	-0.1402	-65.1	-0.1712	-79.5	-0.1818	-84.4	0.8075	374.9	0.6224	289.0
40	-0.1424	-64.5	-0.1759	-79.7	-0.1872	-84.8	-0.0309	-14.0	0.2050	92.9
50	-0.1444	-64.0	-0.1792	-79.4	-0.1914	-84.8	-0.0891	-39.4	0.0429	19.0
63	-0.1473	-63.7	-0.1825	-79.0	-0.1955	-84.6	-0.1142	-49.4	-0.0346	-15.0
80	-0.1506	-63.8	-0.1852	-78.4	-0.1990	-84.3	-0.1290	-54.6	-0.0767	-32.5
100	-0.1547	-64.3	-0.1880	-78.1	-0.2023	-84.0	-0.1395	-57.9	-0.1016	-42.2
126	-0.1585	-64.5	-0.1901	-77.4	-0.2049	-83.4	-0.1473	-59.9	-0.1187	-48.3
159	-0.1628	-65.1	-0.1922	-76.8	-0.2073	-82.9	-0.1542	-61.6	-0.1319	-52.7
200	-0.1674	-65.8	-0.1947	-76.5	-0.2098	-82.5	-0.1608	-63.2	-0.1431	-56.3
250	-0.1715	-66.4	-0.1967	-76.1	-0.2116	-81.9	-0.1661	-64.3	-0.1518	-58.7
316	-0.1754	-66.8	-0.1985	-75.6	-0.2133	-81.3	-0.1711	-65.2	-0.1597	-60.8
398	-0.1800	-67.4	-0.2011	-75.4	-0.2153	-80.7	-0.1765	-66.1	-0.1672	-62.6
500	-0.1833	-67.9	-0.2024	-74.9	-0.2162	-80.0	-0.1803	-66.7	-0.1728	-64.0
631	-0.1814	-66.2	-0.2047	-74.7	-0.2178	-79.5	-0.1848	-67.5	-0.1788	-65.3
795	-0.1899	-68.5	-0.2054	-74.1	-0.2178	-78.6	-0.1875	-67.7	-0.1827	-65.9
1000	-0.1881	-67.1	-0.2071	-73.9	-0.2188	-78.0	-0.1911	-68.1	-0.1872	-66.8
∞	-0.2292	-65	-0.2292	-65	-0.2292	-65	-0.2292	-65	-0.2292	-65

Table 6a: Example 1, case 4: Uknown mean and known variance

n	KDE	EL-KDE			ET-KDE			CUE-KDE			EL-KDE (bc)			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$
25	48.45	74.55	26.10	53.86	78.57	30.11	62.14	82.64	34.19	70.56	56.03	7.57	15.63	60.85	12.39	25.57	67.92	19.47	40.18
32	40.10	58.30	18.20	45.38	61.53	21.42	53.42	65.50	25.39	63.32	46.14	6.04	15.05	47.04	6.93	17.28	49.75	9.64	24.04
40	33.77	46.29	12.52	37.07	49.36	15.59	46.16	52.83	19.06	56.45	37.86	4.08	12.09	38.04	4.27	12.65	39.04	5.27	15.61
50	28.41	36.23	7.82	27.53	39.74	11.34	39.92	42.64	14.24	50.12	30.35	1.95	6.86	31.04	2.64	9.29	31.31	2.91	10.23
63	23.77	28.43	4.65	19.58	31.87	8.10	34.07	34.21	10.44	43.90	24.35	0.58	2.44	25.31	1.54	6.46	25.26	1.49	6.27
80	19.74	22.59	2.85	14.43	25.47	5.72	28.99	27.30	7.56	38.28	19.76	0.02	0.10	20.60	0.86	4.35	20.45	0.71	3.60
100	16.47	18.37	1.90	11.51	20.66	4.19	25.45	22.11	5.64	34.24	16.34	-0.13	-0.78	17.01	0.54	3.29	16.86	0.39	2.35
126	13.81	14.94	1.13	8.16	16.74	2.93	21.20	17.86	4.04	29.27	13.50	-0.31	-2.25	14.03	0.22	1.60	13.89	0.08	0.57
159	11.49	12.27	0.79	6.85	13.58	2.09	18.23	14.43	2.94	25.62	11.25	-0.24	-2.09	11.58	0.10	0.86	11.47	-0.02	-0.16
200	9.56	10.16	0.61	6.38	11.08	1.53	15.97	11.73	2.17	22.73	9.43	-0.13	-1.36	9.61	0.05	0.55	9.52	-0.04	-0.39
250	8.02	8.50	0.48	6.00	9.15	1.12	13.99	9.64	1.61	20.13	7.96	-0.06	-0.73	8.05	0.03	0.33	7.98	-0.04	-0.51
316	6.65	6.99	0.34	5.16	7.44	0.79	11.94	7.81	1.16	17.44	6.61	-0.04	-0.64	6.65	-0.00	-0.06	6.60	-0.05	-0.81
398	5.54	5.79	0.25	4.56	6.12	0.58	10.49	6.39	0.85	15.43	5.51	-0.03	-0.47	5.53	-0.00	-0.06	5.50	-0.04	-0.69
500	4.62	4.80	0.18	3.88	5.04	0.41	8.96	5.24	0.62	13.37	4.60	-0.02	-0.49	4.61	-0.01	-0.30	4.58	-0.04	-0.85
631	3.84	3.96	0.13	3.34	4.13	0.29	7.69	4.28	0.45	11.61	3.82	-0.02	-0.44	3.82	-0.02	-0.43	3.80	-0.03	-0.88
795	3.19	3.29	0.10	3.17	3.41	0.22	6.89	3.52	0.33	10.37	3.18	-0.00	-0.13	3.18	-0.01	-0.23	3.17	-0.02	-0.60
1000	2.66	2.73	0.07	2.81	2.82	0.16	5.96	2.90	0.24	9.05	2.66	-0.00	-0.05	2.65	-0.01	-0.26	2.64	-0.01	-0.56
n	IVar	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$
25	88.72	81.42	-7.30	-8.22	74.94	-13.78	-15.53	75.38	-13.33	-15.03	80.47	-8.24	-9.29	5233.90	5145.18	5799.63	84.31	-4.40	-4.96
32	76.83	71.66	-5.17	-6.73	64.65	-12.18	-15.85	65.43	-11.40	-14.84	71.08	-5.75	-7.49	68.93	-7.90	-10.28	68.54	-8.29	-10.79
40	67.11	63.10	-4.01	-5.97	56.36	-10.75	-16.02	57.11	-10.00	-14.89	62.73	-4.38	-6.52	56.79	-10.32	-15.38	58.96	-8.14	-12.13
50	58.45	54.33	-4.12	-7.04	49.03	-9.42	-16.12	49.59	-8.85	-15.15	54.12	-4.32	-7.40	49.38	-9.07	-15.52	50.83	-7.62	-13.03
63	50.54	46.49	-4.05	-8.02	42.37	-8.17	-16.16	42.77	-7.77	-15.38	46.37	-4.17	-8.25	42.64	-7.90	-15.64	43.61	-6.93	-13.71
80	43.31	38.92	-4.39	-10.13	36.34	-6.97	-16.09	36.60	-6.70	-15.48	38.84	-4.46	-10.31	36.53	-6.78	-15.65	37.17	-6.14	-14.17
100	37.58	33.40	-4.18	-11.12	31.58	-5.99	-15.95	31.76	-5.82	-15.48	33.34	-4.23	-11.26	31.72	-5.85	-15.58	32.15	-5.43	-14.44
126	32.23	28.52	-3.71	-11.52	27.18	-5.05	-15.67	27.30	-4.94	-15.31	28.48	-3.75	-11.63	27.28	-4.95	-15.37	27.56	-4.67	-14.50
159	27.61	24.43	-3.18	-11.51	23.38	-4.23	-15.31	23.45	-4.15	-15.05	24.40	-3.20	-11.60	23.44	-4.16	-15.08	23.62	-3.98	-14.42
200	23.66	20.87	-2.78	-11.77	20.14	-3.52	-14.87	20.18	-3.47	-14.67	20.85	-2.80	-11.85	20.18	-3.47	-14.69	20.30	-3.36	-14.19
250	20.34	17.83	-2.51	-12.35	17.41	-2.93	-14.41	17.44	-2.90	-14.26	17.81	-2.52	-12.41	17.43	-2.90	-14.27	17.51	-2.82	-13.88
316	17.24	15.22	-2.02	-11.72	14.84	-2.40	-13.93	14.86	-2.38	-13.81	15.21	-2.03	-11.77	14.86	-2.38	-13.83	14.91	-2.33	-13.54
398	14.73	12.95	-1.78	-12.09	12.75	-1.98	-13.43	12.76	-1.97	-13.35	12.94	-1.79	-12.13	12.76	-1.97	-13.35	12.79	-1.94	-13.14
500	12.53	11.21	-1.32	-10.50	10.91	-1.62	-12.95	10.91	-1.61	-12.88	11.21	-1.32	-10.53	10.91	-1.61	-12.88	10.93	-1.59	-12.72
631	10.64	9.50	-1.14	-10.70	9.32	-1.32	-12.42	9.32	-1.32	-12.37	9.50	-1.14	-10.73	9.32	-1.32	-12.37	9.33	-1.30	-12.25
795	9.00	8.08	-0.92	-10.22	7.93	-1.07	-11.91	7.93	-1.07	-11.87	8.08	-0.92	-10.24	7.93	-1.07	-11.88	7.94	-1.06	-11.78
1000	7.64	6.91	-0.73	-9.55	6.76	-0.87	-11.44	6.77	-0.87	-11.41	6.91	-0.73	-9.56	6.76	-0.87	-11.42	6.77	-0.87	-11.35
n	MISE	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$
25	137.17	155.97	18.80	13.71	153.50	16.33	11.91	158.03	20.86	15.20	136.50	-0.67	-0.49	5294.74	5157.57	3759.97	152.24	15.07	10.98
32	116.93	129.96	13.03	11.14	126.18	9.24	7.90	130.92	13.99	11.96	117.22	0.28	0.24	115.97	-0.97	-0.83	118.29	1.35	1.16
40	100.88	109.39	8.51	8.44	105.72	4.84	4.80	109.95	9.07	8.99	100.59	-0.29	-0.29	94.83	-6.05	-5.99	98.01	-2.87	-2.85
50	86.85	90.56	3.71	4.27	88.77	1.92	2.21	92.23	5.38	6.20	84.48	-2.37	-2.73	80.42	-6.43	-7.41	82.14	-4.71	-5.43
63	74.32	74.92	0.60	0.81	74.25	-0.07	-0.09	76.98	2.67	3.59	70.72	-3.59	-4.83	67.95	-6.37	-8.57	68.88	-5.44	-7.32
80	63.05	61.51	-1.54	-2.43	61.80	-1.24	-1.97	63.90	0.85	1.35	58.61	-4.44	-7.05	57.13	-5.92	-9.39	57.62	-5.43	-8.61
100	54.05	51.77	-2.28	-4.22	52.25	-1.80	-3.33	53.87	-0.18	-0.33	49.69	-4.36	-8.07	48.74	-5.31	-9.83	49.01	-5.04	-9.32
126	46.04	43.46	-2.59	-5.62	43.92	-2.12	-4.61	45.15	-0.89	-1.94	41.98	-4.06	-8.82	41.31	-4.73	-10.28	41.45	-4.60	-9.98
159	39.09	36.70	-2.39	-6.11	36.96	-2.13	-5.46	37.88	-1.21	-3.10	35.65	-3.44	-8.81	35.03	-4.06	-10.40	35.09	-4.00	-10.23
200	33.21	31.04	-2.17	-6.55	31.22	-1.99	-6.00	31.91	-1.30	-3.91	30.28	-2.93	-8.83	29.79	-3.42	-10.30	29.82	-3.39	-10.22
250	28.36	26.33	-2.03	-7.16	26.55	-1.81	-6.38	27.07	-1.28	-4.53	25.78	-2.58	-9.11	25.48	-2.87	-10.14	25.49	-2.86	-10.10
316	23.89	22.22	-1.68	-7.02	22.29	-1.61	-6.73	22.67	-1.22	-5.12	21.82	-2.07	-8.67	21.51	-2.39	-10.00	21.51	-2.39	-9.99
398	20.27	18.74	-1.53	-7.54	18.87	-1.40	-6.90	19.15	-1.11	-5.49	18.45	-1.81	-8.94	18.30	-1.97	-9.72	18.29	-1.97	-9.74
500	17.15	16.02	-1.14	-6.63	15.94	-1.21	-7.04	16.16	-1.00	-5.81	15.81	-1.34	-7.82	15.52	-1.63	-9.49	15.52	-1.63	-9.52
631	14.47	13.46	-1.01	-6.98	13.45	-1.03	-7.09	13.60	-0.87	-6.01	13.31	-1.16	-8.00	13.14	-1.33	-9.21	13.14	-1.34	-9.24
795	12.19	11.37	-0.82	-6.72	11.34	-0.85	-7.00	11.45	-0.74	-6.06	11.26	-0.93	-7.60	11.11	-1.08	-8.83	11.11	-1.08	-8.86
1000	10.29	9.64	-0.65	-6.36	9.58	-0.72	-6.95	9.66	-0.63	-6.13	9.56	-0.73	-7.11	9.42	-0.88	-8.54	9.41	-0.88	-8.56

Table 6b: Example 1, case 4: Uknown mean and known variance

n	EL-KDE		ET-KDE		CUE-KDE		EL-KDE (bc)		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	0.1757	500.0	0.2027	576.9	0.2302	655.0	0.0510	145.1	0.0834	237.4	0.1311	373.0
32	0.1758	481.0	0.2070	566.2	0.2453	671.1	0.0583	159.6	0.0670	183.2	0.0932	254.9
40	0.1674	443.4	0.2085	552.1	0.2550	675.2	0.0546	144.7	0.0571	151.3	0.0705	186.7
50	0.1447	372.1	0.2098	539.4	0.2634	677.3	0.0360	92.7	0.0488	125.5	0.0537	138.2
63	0.1204	300.5	0.2095	522.9	0.2699	673.8	0.0150	37.4	0.0397	99.1	0.0385	96.2
80	0.1041	252.9	0.2091	508.0	0.2760	670.6	0.0007	1.8	0.0314	76.2	0.0259	63.0
100	0.0955	228.3	0.2113	505.1	0.2842	679.4	-0.0064	-15.4	0.0273	65.3	0.0195	46.6
126	0.0792	184.3	0.2058	479.0	0.2841	661.2	-0.0219	-50.9	0.0155	36.2	0.0055	12.9
159	0.0772	176.6	0.2054	469.6	0.2887	660.0	-0.0235	-53.8	0.0097	22.1	-0.0018	-4.1
200	0.0831	187.3	0.2079	468.5	0.2961	667.0	-0.0177	-39.8	0.0072	16.1	-0.0050	-11.3
250	0.0903	199.9	0.2105	466.2	0.3029	670.7	-0.0109	-24.2	0.0050	11.1	-0.0076	-16.9
316	0.0900	196.8	0.2081	454.9	0.3040	664.5	-0.0112	-24.4	-0.0011	-2.4	-0.0141	-30.8
398	0.0920	198.5	0.2115	456.4	0.3111	671.3	-0.0095	-20.4	-0.0013	-2.7	-0.0140	-30.2
500	0.0903	192.3	0.2089	444.8	0.3114	663.1	-0.0114	-24.2	-0.0071	-15.0	-0.0198	-42.2
631	0.0900	189.8	0.2068	436.3	0.3122	658.8	-0.0118	-25.0	-0.0115	-24.3	-0.0237	-50.0
795	0.0983	205.6	0.2137	446.9	0.3217	672.8	-0.0039	-8.1	-0.0070	-14.7	-0.0187	-39.2
1000	0.1007	208.4	0.2136	442.1	0.3245	671.5	-0.0018	-3.8	-0.0093	-19.3	-0.0202	-41.7
∞	0.1058	200	0.2380	450	0.3702	700	0	0	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.0182	-13.5	-0.0345	-25.5	-0.0333	-24.7	-0.0206	-15.2	12.8630	9517.2	-0.0110	-8.1
32	-0.0166	-11.7	-0.0390	-27.5	-0.0365	-25.8	-0.0184	-13.0	-0.0253	-17.9	-0.0265	-18.7
40	-0.0160	-10.9	-0.0430	-29.3	-0.0400	-27.2	-0.0175	-11.9	-0.0413	-28.1	-0.0326	-22.2
50	-0.0206	-13.5	-0.0471	-31.0	-0.0443	-29.1	-0.0216	-14.2	-0.0453	-29.8	-0.0381	-25.1
63	-0.0255	-16.2	-0.0515	-32.8	-0.0490	-31.2	-0.0263	-16.7	-0.0498	-31.7	-0.0437	-27.8
80	-0.0351	-21.6	-0.0558	-34.4	-0.0536	-33.1	-0.0357	-22.0	-0.0542	-33.4	-0.0491	-30.3
100	-0.0418	-25.0	-0.0599	-35.8	-0.0582	-34.7	-0.0423	-25.3	-0.0585	-35.0	-0.0543	-32.4
126	-0.0468	-27.2	-0.0636	-37.0	-0.0622	-36.2	-0.0472	-27.5	-0.0624	-36.3	-0.0589	-34.2
159	-0.0505	-28.6	-0.0672	-38.0	-0.0661	-37.4	-0.0509	-28.8	-0.0662	-37.5	-0.0633	-35.8
200	-0.0557	-30.7	-0.0704	-38.8	-0.0694	-38.3	-0.0561	-30.9	-0.0695	-38.3	-0.0671	-37.1
250	-0.0628	-33.8	-0.0733	-39.5	-0.0725	-39.0	-0.0631	-34.0	-0.0725	-39.1	-0.0706	-38.0
316	-0.0639	-33.7	-0.0759	-40.1	-0.0753	-39.8	-0.0641	-33.9	-0.0753	-39.8	-0.0738	-39.0
398	-0.0709	-36.6	-0.0788	-40.6	-0.0783	-40.4	-0.0711	-36.7	-0.0783	-40.4	-0.0770	-39.7
500	-0.0658	-33.3	-0.0811	-41.1	-0.0807	-40.9	-0.0660	-33.4	-0.0807	-40.9	-0.0797	-40.4
631	-0.0718	-35.7	-0.0833	-41.4	-0.0830	-41.2	-0.0720	-35.8	-0.0830	-41.2	-0.0822	-40.8
795	-0.0732	-35.8	-0.0853	-41.7	-0.0850	-41.6	-0.0733	-35.8	-0.0850	-41.6	-0.0844	-41.2
1000	-0.0729	-35.1	-0.0874	-42.0	-0.0872	-41.9	-0.0730	-35.1	-0.0872	-41.9	-0.0867	-41.7
∞	-0.1058	-37.5	-0.1058	-37.5	-0.1058	-37.5	-0.1058	-37.5	-0.1058	-37.5	-0.1058	-37.5
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	0.0470	22.5	0.0408	19.5	0.0521	25.0	-0.0017	-0.8	12.8939	6170.1	0.0377	18.0
32	0.0417	19.4	0.0296	13.7	0.0448	20.8	0.0009	0.4	-0.0031	-1.4	0.0043	2.0
40	0.0340	15.4	0.0194	8.8	0.0363	16.4	-0.0012	-0.5	-0.0242	-11.0	-0.0115	-5.2
50	0.0185	8.2	0.0096	4.2	0.0269	11.9	-0.0119	-5.3	-0.0322	-14.2	-0.0236	-10.4
63	0.0038	1.6	-0.0004	-0.2	0.0168	7.3	-0.0226	-9.8	-0.0401	-17.4	-0.0343	-14.8
80	-0.0123	-5.2	-0.0100	-4.2	0.0068	2.9	-0.0355	-15.1	-0.0473	-20.1	-0.0434	-18.4
100	-0.0228	-9.5	-0.0180	-7.5	-0.0018	-0.7	-0.0436	-18.1	-0.0531	-22.1	-0.0504	-20.9
126	-0.0326	-13.3	-0.0267	-10.9	-0.0112	-4.6	-0.0512	-20.8	-0.0596	-24.3	-0.0579	-23.6
159	-0.0380	-15.2	-0.0339	-13.6	-0.0193	-7.7	-0.0547	-21.9	-0.0646	-25.8	-0.0636	-25.4
200	-0.0435	-17.1	-0.0398	-15.7	-0.0260	-10.2	-0.0587	-23.1	-0.0684	-26.9	-0.0679	-26.7
250	-0.0507	-19.6	-0.0452	-17.5	-0.0321	-12.4	-0.0646	-24.9	-0.0719	-27.8	-0.0716	-27.7
316	-0.0530	-20.2	-0.0508	-19.4	-0.0386	-14.7	-0.0655	-25.0	-0.0755	-28.8	-0.0755	-28.8
398	-0.0608	-22.8	-0.0556	-20.9	-0.0443	-16.6	-0.0721	-27.1	-0.0784	-29.4	-0.0786	-29.5
500	-0.0568	-21.0	-0.0604	-22.3	-0.0498	-18.4	-0.0671	-24.8	-0.0814	-30.1	-0.0817	-30.2
631	-0.0638	-23.3	-0.0648	-23.6	-0.0549	-20.0	-0.0731	-26.7	-0.0841	-30.7	-0.0843	-30.8
795	-0.0651	-23.5	-0.0678	-24.5	-0.0587	-21.2	-0.0736	-26.6	-0.0856	-30.9	-0.0859	-31.0
1000	-0.0655	-23.3	-0.0715	-25.5	-0.0631	-22.5	-0.0732	-26.1	-0.0879	-31.3	-0.0882	-31.4
∞	-0.1058	-30	-0.1058	-30	-0.1058	-30	-0.1058	-30	-0.1058	-30	-0.1058	-30

Table 7a: Example 1, case 5: Uknown mean and known third central moment

n	KDE	EL-KDE			ET-KDE			CUE-KDE			EL-KDE (bc)			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$	ISB	Δ	$\Delta\%$
25	48.45	55.36	6.90	14.25	45.72	-2.73	-5.64	42.08	-6.37	-13.16	55.37	6.92	14.28	90.95	42.50	87.70	176.08	127.62	263.38
32	40.10	44.52	4.41	11.00	37.17	-2.94	-7.32	33.67	-6.43	-16.04	44.55	4.44	11.07	57.62	17.51	43.67	89.89	49.79	124.14
40	33.77	36.70	2.93	8.66	31.16	-2.61	-7.72	28.05	-5.73	-16.96	36.72	2.95	8.73	42.32	8.55	25.31	56.62	22.85	67.66
50	28.36	30.26	1.91	6.73	26.17	-2.19	-7.72	23.58	-4.78	-16.85	30.28	1.92	6.78	32.74	4.38	15.45	39.35	10.99	38.77
63	23.72	24.97	1.25	5.28	21.98	-1.74	-7.34	19.92	-3.80	-16.03	24.98	1.26	5.31	26.02	2.30	9.70	29.09	5.37	22.64
80	19.67	20.46	0.79	4.03	18.33	-1.34	-6.80	16.78	-2.89	-14.68	20.46	0.80	4.04	20.89	1.23	6.23	22.36	2.69	13.68
100	16.51	17.04	0.53	3.20	15.49	-1.03	-6.21	14.33	-2.19	-13.24	17.04	0.53	3.21	17.22	0.71	4.28	17.98	1.47	8.92
126	13.76	14.10	0.34	2.46	12.98	-0.78	-5.68	12.12	-1.64	-11.92	14.10	0.34	2.46	14.16	0.40	2.89	14.56	0.80	5.81
159	11.44	11.67	0.22	1.96	10.86	-0.58	-5.06	10.23	-1.21	-10.56	11.67	0.22	1.96	11.68	0.24	2.06	11.89	0.45	3.95
200	9.56	9.70	0.14	1.48	9.12	-0.43	-4.54	8.66	-0.90	-9.37	9.70	0.14	1.48	9.69	0.14	1.45	9.81	0.26	2.72
250	8.02	8.12	0.10	1.20	7.70	-0.32	-4.02	7.36	-0.67	-8.30	8.12	0.10	1.20	8.11	0.09	1.09	8.18	0.16	1.93
316	6.67	6.73	0.06	0.88	6.43	-0.24	-3.57	6.18	-0.49	-7.28	6.73	0.06	0.88	6.72	0.05	0.77	6.76	0.09	1.39
398	5.54	5.58	0.04	0.66	5.37	-0.18	-3.16	5.19	-0.36	-6.42	5.58	0.04	0.66	5.57	0.03	0.56	5.60	0.06	1.00
500	4.62	4.64	0.02	0.51	4.49	-0.13	-2.80	4.36	-0.26	-5.65	4.64	0.02	0.51	4.64	0.02	0.42	4.65	0.03	0.74
631	3.84	3.85	0.01	0.39	3.75	-0.09	-2.45	3.65	-0.19	-4.95	3.85	0.01	0.39	3.85	0.01	0.31	3.86	0.02	0.55
795	3.19	3.20	0.01	0.34	3.12	-0.07	-2.16	3.05	-0.14	-4.35	3.20	0.01	0.34	3.20	0.01	0.23	3.20	0.01	0.41
1000	2.66	2.66	0.01	0.23	2.61	-0.05	-1.90	2.56	-0.10	-3.81	2.66	0.01	0.23	2.66	0.00	0.17	2.67	0.01	0.30
n	IVar	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$	IVar	Δ	$\Delta\%$
25	88.72	93.36	4.64	5.23	95.17	6.45	7.27	97.00	8.28	9.34	94.13	5.41	6.10	145.13	56.41	63.58	165.03	76.31	86.02
32	76.83	77.29	0.46	0.60	78.01	1.18	1.53	79.94	3.11	4.05	77.86	1.03	1.34	78.64	1.81	2.36	89.15	12.32	16.04
40	67.11	65.14	-1.96	-2.92	65.22	-1.89	-2.82	67.06	-0.04	-0.06	65.55	-1.56	-2.32	65.39	-1.72	-2.56	68.43	1.32	1.97
50	58.43	55.25	-3.18	-5.44	54.97	-3.46	-5.92	56.44	-1.98	-3.40	55.53	-2.90	-4.97	55.65	-2.77	-4.75	56.99	-1.44	-2.46
63	50.54	46.90	-3.65	-7.21	46.45	-4.09	-8.10	47.48	-3.06	-6.06	47.07	-3.47	-6.87	47.37	-3.18	-6.29	48.23	-2.31	-4.57
80	43.39	39.69	-3.70	-8.52	39.23	-4.16	-9.59	39.91	-3.48	-8.02	39.80	-3.59	-8.28	40.13	-3.25	-7.50	40.85	-2.54	-5.85
100	37.56	34.08	-3.48	-9.27	33.66	-3.90	-10.39	34.10	-3.47	-9.23	34.15	-3.42	-9.10	34.44	-3.12	-8.31	35.04	-2.52	-6.72
126	32.21	29.09	-3.12	-9.70	28.72	-3.49	-10.83	28.98	-3.23	-10.04	29.12	-3.09	-9.59	29.36	-2.85	-8.85	29.84	-2.37	-7.37
159	27.60	24.88	-2.71	-9.84	24.58	-3.01	-10.91	24.72	-2.87	-10.41	24.90	-2.70	-9.77	25.08	-2.52	-9.13	25.44	-2.15	-7.81
200	23.66	21.29	-2.36	-9.98	21.07	-2.58	-10.93	21.14	-2.52	-10.64	21.30	-2.35	-9.94	21.44	-2.22	-9.36	21.71	-1.94	-8.21
250	20.34	18.31	-2.02	-9.95	18.13	-2.20	-10.83	18.16	-2.17	-10.69	18.32	-2.02	-9.92	18.41	-1.93	-9.48	18.61	-1.73	-8.49
316	17.25	15.57	-1.69	-9.78	15.44	-1.82	-10.52	15.44	-1.81	-10.49	15.57	-1.68	-9.76	15.64	-1.62	-9.37	15.78	-1.47	-8.54
398	14.71	13.30	-1.41	-9.60	13.20	-1.51	-10.24	13.20	-1.51	-10.28	13.30	-1.41	-9.59	13.34	-1.36	-9.27	13.44	-1.26	-8.59
500	12.53	11.40	-1.13	-9.02	11.29	-1.24	-9.92	11.28	-1.26	-10.02	11.40	-1.13	-9.01	11.39	-1.14	-9.12	11.46	-1.08	-8.60
631	10.61	9.67	-0.94	-8.84	9.59	-1.01	-9.56	9.58	-1.03	-9.68	9.67	-0.94	-8.84	9.66	-0.94	-8.90	9.71	-0.90	-8.48
795	9.00	8.39	-0.61	-6.82	8.17	-0.83	-9.26	8.16	-0.85	-9.39	8.39	-0.61	-6.82	8.22	-0.79	-8.73	8.25	-0.76	-8.40
1000	7.64	7.02	-0.61	-8.04	6.96	-0.68	-8.88	6.95	-0.69	-9.02	7.02	-0.61	-8.04	6.99	-0.65	-8.45	7.01	-0.63	-8.21
n	MISE	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$	MISE	Δ	$\Delta\%$
25	137.17	148.72	11.55	8.42	140.89	3.72	2.71	139.08	1.91	1.39	149.50	12.33	8.99	236.08	98.90	72.10	341.10	203.93	148.67
32	116.93	121.81	4.88	4.17	115.18	-1.76	-1.50	113.62	-3.32	-2.84	122.41	5.47	4.68	136.26	19.33	16.53	179.05	62.11	53.12
40	100.88	101.84	0.96	0.96	96.38	-4.50	-4.46	95.11	-5.77	-5.72	102.27	1.39	1.38	107.71	6.83	6.77	125.05	24.17	23.96
50	86.79	85.52	-1.27	-1.46	81.14	-5.65	-6.51	80.02	-6.76	-7.79	85.80	-0.98	-1.13	88.39	1.61	1.85	96.34	9.56	11.01
63	74.27	71.87	-2.39	-3.22	68.43	-5.83	-7.86	67.40	-6.86	-9.24	72.05	-2.21	-2.98	73.39	-0.88	-1.18	77.33	3.06	4.12
80	63.06	60.15	-2.90	-4.60	57.56	-5.50	-8.72	56.69	-6.37	-10.10	60.26	-2.80	-4.43	61.03	-2.03	-3.22	63.21	0.15	0.24
100	54.08	51.12	-2.95	-5.46	49.15	-4.93	-9.12	48.42	-5.65	-10.45	51.19	-2.89	-5.34	51.66	-2.41	-4.46	53.03	-1.05	-1.94
126	45.97	43.18	-2.78	-6.06	41.70	-4.27	-9.29	41.09	-4.87	-10.60	43.22	-2.75	-5.98	43.51	-2.45	-5.34	44.39	-1.57	-3.42
159	39.04	36.55	-2.49	-6.38	35.45	-3.59	-9.20	34.96	-4.08	-10.45	36.57	-2.47	-6.33	36.75	-2.28	-5.85	37.33	-1.70	-4.36
200	33.21	30.99	-2.22	-6.68	30.19	-3.02	-9.09	29.80	-3.41	-10.27	31.00	-2.21	-6.65	31.13	-2.08	-6.25	31.53	-1.68	-5.07
250	28.36	26.43	-1.93	-6.79	25.83	-2.53	-8.90	25.52	-2.84	-10.01	26.44	-1.92	-6.77	26.52	-1.84	-6.49	26.79	-1.57	-5.54
316	23.92	22.30	-1.63	-6.81	21.87	-2.05	-8.58	21.63	-2.30	-9.60	22.30	-1.63	-6.79	22.36	-1.57	-6.54	22.54	-1.38	-5.78
398	20.25	18.87	-1.38	-6.79	18.57	-1.68	-8.30	18.38	-1.87	-9.22	18.87	-1.37	-6.79	18.92	-1.33	-6.58	19.04	-1.21	-5.97
500	17.15	16.04	-1.11	-6.45	15.78	-1.37	-8.00	15.63	-1.52	-8.84	16.04	-1.11	-6.45	16.03	-1.12	-6.56	16.11	-1.04	-6.08
631	14.44	13.52	-0.92	-6.39	13.34	-1.11	-7.67	13.23	-1.22	-8.42	13.52	-0.92	-6.39	13.51	-0.93	-6.45	13.57	-0.88	-6.08
795	12.19	11.59	-0.60	-4.95	11.29	-0.90	-7.41	11.21	-0.98	-8.07	11.59	-0.60	-4.95	11.42	-0.78	-6.39	11.45	-0.74	-6.10
1000	10.30	9.69	-0.61	-5.91	9.57	-0.73	-7.08	9.51	-0.79	-7.67	9.69	-0.61	-5.91	9.65	-0.64	-6.23	9.68	-0.62	-6.01

Table 7b: Example 1, case 5: Uknown mean and known third central moment

n	EL-KDE		ET-KDE		CUE-KDE		EL-KDE (bc)		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	0.0465	132.3	-0.0184	-52.3	-0.0429	-122.1	0.0466	132.6	0.2861	814.2	0.8592	2445.1
32	0.0426	116.6	-0.0284	-77.6	-0.0621	-170.0	0.0429	117.4	0.1692	462.8	0.4810	1315.9
40	0.0391	103.6	-0.0349	-92.4	-0.0766	-202.8	0.0394	104.5	0.1143	302.8	0.3056	809.3
50	0.0353	90.9	-0.0405	-104.4	-0.0884	-227.7	0.0356	91.6	0.0810	208.7	0.2034	523.9
63	0.0324	81.0	-0.0450	-112.6	-0.0983	-245.9	0.0326	81.5	0.0595	148.8	0.1389	347.5
80	0.0289	70.5	-0.0489	-119.2	-0.1054	-257.1	0.0290	70.8	0.0448	109.2	0.0982	239.6
100	0.0266	63.5	-0.0517	-123.2	-0.1102	-262.8	0.0267	63.7	0.0356	84.9	0.0742	176.9
126	0.0238	55.6	-0.0549	-128.4	-0.1152	-269.3	0.0238	55.7	0.0279	65.3	0.0562	131.3
159	0.0220	50.6	-0.0568	-130.3	-0.1185	-271.9	0.0221	50.6	0.0231	53.1	0.0444	101.9
200	0.0193	43.4	-0.0591	-133.2	-0.1221	-275.0	0.0193	43.4	0.0189	42.5	0.0354	79.8
250	0.0181	40.1	-0.0604	-133.8	-0.1249	-276.5	0.0181	40.1	0.0163	36.2	0.0291	64.4
316	0.0153	33.4	-0.0624	-136.0	-0.1273	-277.5	0.0153	33.4	0.0135	29.4	0.0243	52.9
398	0.0134	28.9	-0.0638	-137.6	-0.1296	-279.3	0.0134	28.8	0.0114	24.6	0.0203	43.7
500	0.0119	25.4	-0.0651	-138.8	-0.1315	-280.5	0.0119	25.4	0.0097	20.7	0.0172	36.7
631	0.0105	22.1	-0.0661	-139.3	-0.1333	-281.0	0.0105	22.1	0.0084	17.7	0.0147	31.0
795	0.0105	22.0	-0.0671	-140.2	-0.1350	-282.0	0.0105	22.0	0.0072	15.0	0.0126	26.3
1000	0.0083	17.2	-0.0680	-140.7	-0.1366	-282.9	0.0083	17.2	0.0062	12.9	0.0107	22.2
∞	0	0	-0.0749	-141.7	-0.1499	-283.3	0	0	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	0.0116	8.6	0.0161	11.9	0.0207	15.3	0.0135	10.0	0.1410	104.3	0.1908	141.2
32	0.0015	1.1	0.0038	2.7	0.0100	7.0	0.0033	2.3	0.0058	4.1	0.0394	27.9
40	-0.0078	-5.3	-0.0076	-5.1	-0.0002	-0.1	-0.0062	-4.2	-0.0069	-4.7	0.0053	3.6
50	-0.0159	-10.5	-0.0173	-11.4	-0.0099	-6.5	-0.0145	-9.6	-0.0139	-9.1	-0.0072	-4.7
63	-0.0230	-14.6	-0.0258	-16.4	-0.0193	-12.3	-0.0219	-13.9	-0.0200	-12.7	-0.0146	-9.3
80	-0.0296	-18.2	-0.0333	-20.5	-0.0278	-17.1	-0.0287	-17.7	-0.0260	-16.0	-0.0203	-12.5
100	-0.0348	-20.8	-0.0390	-23.3	-0.0347	-20.7	-0.0342	-20.4	-0.0312	-18.6	-0.0252	-15.1
126	-0.0393	-22.9	-0.0439	-25.6	-0.0407	-23.7	-0.0389	-22.6	-0.0359	-20.9	-0.0299	-17.4
159	-0.0432	-24.4	-0.0479	-27.1	-0.0457	-25.9	-0.0429	-24.3	-0.0401	-22.7	-0.0342	-19.4
200	-0.0472	-26.1	-0.0517	-28.5	-0.0503	-27.8	-0.0470	-25.9	-0.0443	-24.4	-0.0388	-21.4
250	-0.0506	-27.2	-0.0551	-29.7	-0.0543	-29.3	-0.0504	-27.2	-0.0482	-26.0	-0.0431	-23.2
316	-0.0533	-28.2	-0.0574	-30.3	-0.0572	-30.2	-0.0532	-28.1	-0.0511	-27.0	-0.0466	-24.6
398	-0.0562	-29.0	-0.0599	-31.0	-0.0602	-31.1	-0.0562	-29.0	-0.0543	-28.0	-0.0503	-26.0
500	-0.0565	-28.6	-0.0622	-31.5	-0.0628	-31.8	-0.0565	-28.6	-0.0572	-29.0	-0.0539	-27.3
631	-0.0592	-29.5	-0.0640	-31.9	-0.0648	-32.3	-0.0592	-29.5	-0.0596	-29.7	-0.0568	-28.3
795	-0.0488	-23.9	-0.0663	-32.4	-0.0672	-32.9	-0.0488	-23.9	-0.0625	-30.6	-0.0602	-29.4
1000	-0.0614	-29.5	-0.0679	-32.6	-0.0689	-33.1	-0.0614	-29.5	-0.0646	-31.0	-0.0627	-30.1
∞	-0.0882	-31.25	-0.0882	-31.25	-0.0882	-31.25	-0.0882	-31.25	-0.0882	-31.25	-0.0882	-31.25
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	0.0289	13.8	0.0093	4.5	0.0048	2.3	0.0308	14.8	0.2473	118.3	0.5098	244.0
32	0.0156	7.2	-0.0056	-2.6	-0.0106	-4.9	0.0175	8.1	0.0618	28.7	0.1988	92.3
40	0.0039	1.7	-0.0180	-8.2	-0.0231	-10.5	0.0056	2.5	0.0273	12.4	0.0967	43.8
50	-0.0063	-2.8	-0.0282	-12.5	-0.0338	-15.0	-0.0049	-2.2	0.0080	3.6	0.0478	21.2
63	-0.0151	-6.5	-0.0368	-15.9	-0.0432	-18.7	-0.0139	-6.0	-0.0055	-2.4	0.0193	8.4
80	-0.0232	-9.8	-0.0440	-18.6	-0.0509	-21.6	-0.0224	-9.5	-0.0162	-6.9	0.0012	0.5
100	-0.0295	-12.3	-0.0493	-20.5	-0.0565	-23.5	-0.0289	-12.0	-0.0241	-10.0	-0.0105	-4.4
126	-0.0351	-14.3	-0.0538	-21.9	-0.0614	-25.0	-0.0346	-14.1	-0.0309	-12.6	-0.0198	-8.1
159	-0.0396	-15.8	-0.0571	-22.8	-0.0649	-26.0	-0.0393	-15.7	-0.0363	-14.5	-0.0271	-10.8
200	-0.0444	-17.4	-0.0604	-23.7	-0.0682	-26.8	-0.0442	-17.4	-0.0415	-16.3	-0.0336	-13.2
250	-0.0482	-18.6	-0.0631	-24.4	-0.0710	-27.4	-0.0480	-18.6	-0.0460	-17.8	-0.0393	-15.2
316	-0.0515	-19.6	-0.0649	-24.7	-0.0726	-27.6	-0.0514	-19.6	-0.0495	-18.8	-0.0437	-16.6
398	-0.0548	-20.5	-0.0669	-25.1	-0.0743	-27.9	-0.0547	-20.5	-0.0530	-19.9	-0.0481	-18.0
500	-0.0553	-20.5	-0.0686	-25.4	-0.0758	-28.1	-0.0553	-20.5	-0.0562	-20.8	-0.0522	-19.3
631	-0.0582	-21.3	-0.0699	-25.6	-0.0768	-28.1	-0.0582	-21.3	-0.0588	-21.5	-0.0554	-20.3
795	-0.0480	-17.3	-0.0718	-25.9	-0.0782	-28.2	-0.0479	-17.3	-0.0619	-22.4	-0.0591	-21.3
1000	-0.0608	-21.7	-0.0729	-26.0	-0.0790	-28.2	-0.0608	-21.7	-0.0641	-22.9	-0.0619	-22.1
∞	-0.0882	-25	-0.0882	-25	-0.0882	-25	-0.0882	-25	-0.0882	-25	-0.0882	-25

Table 8a: Skewed unimodal density

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%
25	74.73	74.80	0.07	0.09	65.92	-8.81	-11.79	58.97	-15.76	-21.09	78.69	3.96	5.30	84.56	9.83	13.15
32	61.48	61.81	0.33	0.53	55.19	-6.29	-10.23	49.85	-11.62	-18.91	64.11	2.63	4.28	67.45	5.98	9.72
40	51.64	52.03	0.39	0.76	47.08	-4.56	-8.83	42.95	-8.69	-16.83	53.55	1.91	3.71	55.62	3.98	7.71
50	43.55	43.79	0.24	0.54	40.11	-3.44	-7.91	36.95	-6.60	-15.15	44.83	1.28	2.94	46.16	2.61	5.99
63	36.20	36.34	0.14	0.38	33.65	-2.56	-7.06	31.29	-4.92	-13.59	37.03	0.82	2.28	37.87	1.66	4.60
80	30.05	30.16	0.11	0.35	28.20	-1.85	-6.14	26.46	-3.59	-11.95	30.61	0.56	1.86	31.14	1.09	3.63
100	25.05	25.12	0.07	0.28	23.69	-1.37	-5.45	22.38	-2.67	-10.66	25.43	0.38	1.51	25.78	0.73	2.92
126	20.94	21.00	0.06	0.30	19.94	-1.00	-4.78	18.97	-1.97	-9.41	21.20	0.26	1.25	21.43	0.49	2.35
159	17.38	17.44	0.06	0.33	16.66	-0.73	-4.18	15.94	-1.44	-8.29	17.56	0.18	1.04	17.71	0.33	1.91
200	14.48	14.52	0.04	0.28	13.94	-0.54	-3.72	13.41	-1.07	-7.37	14.60	0.12	0.81	14.70	0.22	1.50
250	12.11	12.13	0.02	0.15	11.69	-0.42	-3.44	11.30	-0.81	-6.66	12.17	0.06	0.50	12.24	0.13	1.06
316	10.04	10.07	0.03	0.29	9.75	-0.29	-2.92	9.46	-0.58	-5.78	10.09	0.05	0.50	10.13	0.09	0.94
398	8.36	8.39	0.03	0.30	8.15	-0.22	-2.58	7.94	-0.43	-5.10	8.40	0.03	0.38	8.43	0.06	0.75
500	6.97	6.99	0.02	0.28	6.80	-0.16	-2.33	6.65	-0.32	-4.55	6.98	0.02	0.24	7.00	0.04	0.53
631	5.78	5.80	0.02	0.42	5.67	-0.11	-1.95	5.55	-0.23	-3.90	5.80	0.02	0.29	5.81	0.03	0.53
795	4.83	4.85	0.02	0.40	4.75	-0.09	-1.76	4.67	-0.17	-3.48	4.84	0.01	0.18	4.85	0.02	0.37
1000	4.00	4.02	0.02	0.45	3.94	-0.06	-1.52	3.88	-0.12	-3.03	4.01	0.01	0.17	4.01	0.01	0.32
n	IVar	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%
25	136.54	102.29	-34.25	-25.08	100.73	-35.81	-26.22	100.58	-35.96	-26.33	104.10	-32.44	-23.76	106.86	-29.68	-21.74
32	118.38	89.38	-29.00	-24.50	88.19	-30.19	-25.50	88.37	-30.01	-25.35	90.37	-28.01	-23.66	92.52	-25.86	-21.85
40	103.49	78.84	-24.65	-23.82	77.94	-25.55	-24.68	78.22	-25.27	-24.42	79.39	-24.09	-23.28	81.01	-22.48	-21.72
50	90.12	69.33	-20.79	-23.07	68.73	-21.39	-23.74	69.00	-21.12	-23.44	69.68	-20.44	-22.68	70.85	-19.27	-21.39
63	77.97	60.59	-17.37	-22.29	60.18	-17.78	-22.81	60.41	-17.56	-22.52	60.80	-17.17	-22.02	61.60	-16.36	-20.98
80	66.93	52.59	-14.34	-21.42	52.35	-14.58	-21.78	52.54	-14.39	-21.51	52.74	-14.19	-21.21	53.29	-13.64	-20.38
100	57.79	45.92	-11.88	-20.55	45.75	-12.04	-20.83	45.87	-11.92	-20.62	46.00	-11.79	-20.40	46.37	-11.43	-19.77
126	49.73	39.95	-9.78	-19.67	39.84	-9.89	-19.88	39.92	-9.81	-19.72	40.00	-9.73	-19.56	40.24	-9.49	-19.09
159	42.59	34.58	-8.01	-18.81	34.51	-8.08	-18.97	34.57	-8.02	-18.84	34.61	-7.98	-18.73	34.76	-7.83	-18.37
200	36.42	29.91	-6.51	-17.87	29.88	-6.54	-17.96	29.92	-6.50	-17.86	29.94	-6.48	-17.79	30.04	-6.38	-17.51
250	31.23	25.91	-5.32	-17.05	25.88	-5.35	-17.13	25.91	-5.33	-17.05	25.93	-5.31	-17.00	25.99	-5.25	-16.79
316	26.52	22.23	-4.29	-16.18	22.21	-4.31	-16.24	22.23	-4.29	-16.18	22.24	-4.28	-16.15	22.28	-4.24	-15.99
398	22.61	19.13	-3.47	-15.37	19.12	-3.49	-15.42	19.13	-3.47	-15.37	19.14	-3.47	-15.34	19.17	-3.44	-15.23
500	19.21	16.42	-2.79	-14.54	16.41	-2.80	-14.58	16.41	-2.79	-14.54	16.42	-2.79	-14.52	16.44	-2.77	-14.44
631	16.27	14.03	-2.25	-13.81	14.02	-2.25	-13.83	14.03	-2.25	-13.80	14.03	-2.24	-13.79	14.04	-2.23	-13.72
795	13.79	11.98	-1.81	-13.10	11.98	-1.81	-13.11	11.98	-1.81	-13.09	11.98	-1.80	-13.08	11.99	-1.80	-13.03
1000	11.68	10.24	-1.44	-12.35	10.24	-1.44	-12.37	10.24	-1.44	-12.35	10.24	-1.44	-12.35	10.24	-1.44	-12.31
n	MISE	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%
25	211.27	177.09	-34.18	-16.18	166.66	-44.62	-21.12	159.55	-51.72	-24.48	182.79	-28.48	-13.48	191.42	-19.85	-9.40
32	179.86	151.19	-28.67	-15.94	143.38	-36.48	-20.28	138.22	-41.64	-23.15	154.48	-25.38	-14.11	159.97	-19.89	-11.06
40	155.13	130.87	-24.26	-15.64	125.02	-30.11	-19.41	121.17	-33.96	-21.89	132.95	-22.18	-14.30	136.63	-18.50	-11.92
50	133.67	113.12	-20.55	-15.38	108.84	-24.84	-18.58	105.95	-27.72	-20.74	114.51	-19.16	-14.34	117.01	-16.67	-12.47
63	114.17	96.93	-17.24	-15.10	93.83	-20.34	-17.82	91.69	-22.48	-19.69	97.82	-16.34	-14.32	99.47	-14.70	-12.87
80	96.98	82.75	-14.23	-14.67	80.56	-16.42	-16.94	79.00	-17.98	-18.54	83.35	-13.63	-14.06	84.43	-12.55	-12.94
100	82.84	71.04	-11.81	-14.25	69.44	-13.40	-16.18	68.26	-14.59	-17.61	71.44	-11.41	-13.77	72.15	-10.69	-12.91
126	70.67	60.95	-9.72	-13.75	59.78	-10.89	-15.40	58.89	-11.78	-16.66	61.20	-9.47	-13.39	61.67	-9.00	-12.74
159	59.97	52.02	-7.95	-13.26	51.17	-8.80	-14.68	50.51	-9.46	-15.78	52.18	-7.80	-13.00	52.48	-7.49	-12.49
200	50.90	44.43	-6.47	-12.70	43.82	-7.08	-13.91	43.33	-7.57	-14.87	44.54	-6.36	-12.50	44.74	-6.16	-12.10
250	43.34	38.04	-5.31	-12.24	37.58	-5.77	-13.30	37.21	-6.13	-14.15	38.10	-5.25	-12.11	38.23	-5.12	-11.81
316	36.56	32.30	-4.26	-11.66	31.96	-4.60	-12.58	31.69	-4.87	-13.33	32.33	-4.23	-11.57	32.41	-4.15	-11.34
398	30.97	27.52	-3.45	-11.14	27.27	-3.70	-11.95	27.07	-3.90	-12.60	27.54	-3.44	-11.10	27.59	-3.38	-10.91
500	26.18	23.40	-2.77	-10.60	23.21	-2.96	-11.32	23.06	-3.11	-11.88	23.40	-2.77	-10.59	23.44	-2.74	-10.45
631	22.05	19.83	-2.22	-10.08	19.69	-2.36	-10.72	19.58	-2.47	-11.21	19.83	-2.23	-10.10	19.85	-2.20	-9.99
795	18.62	16.83	-1.79	-9.59	16.73	-1.89	-10.17	16.65	-1.97	-10.60	16.83	-1.79	-9.64	16.84	-1.78	-9.56
1000	15.68	14.25	-1.43	-9.09	14.17	-1.51	-9.60	14.12	-1.56	-9.97	14.24	-1.44	-9.15	14.25	-1.42	-9.09

Table 8b: Skewed unimodal density

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	0.0009	0.4	-0.1219	-53.3	-0.2181	-95.3	0.0547	23.9	0.1360	59.4
32	0.0066	2.7	-0.1259	-52.4	-0.2326	-96.8	0.0526	21.9	0.1195	49.8
40	0.0109	4.3	-0.1271	-50.7	-0.2423	-96.6	0.0533	21.3	0.1109	44.2
50	0.0091	3.5	-0.1335	-51.0	-0.2559	-97.7	0.0496	18.9	0.1012	38.6
63	0.0075	2.8	-0.1394	-51.4	-0.2681	-98.9	0.0450	16.6	0.0907	33.5
80	0.0082	2.9	-0.1428	-50.8	-0.2778	-98.8	0.0433	15.4	0.0843	30.0
100	0.0076	2.6	-0.1465	-50.9	-0.2864	-99.4	0.0407	14.1	0.0784	27.2
126	0.0095	3.2	-0.1502	-50.5	-0.2958	-99.5	0.0393	13.2	0.0740	24.9
159	0.0120	3.9	-0.1528	-50.2	-0.3033	-99.6	0.0380	12.5	0.0698	22.9
200	0.0120	3.9	-0.1581	-50.8	-0.3130	-100.5	0.0342	11.0	0.0639	20.5
250	0.0073	2.3	-0.1686	-53.1	-0.3267	-102.9	0.0245	7.7	0.0521	16.4
316	0.0166	5.1	-0.1664	-51.4	-0.3291	-101.7	0.0283	8.7	0.0537	16.6
398	0.0201	6.1	-0.1708	-51.8	-0.3369	-102.2	0.0254	7.7	0.0494	15.0
500	0.0215	6.4	-0.1781	-53.2	-0.3474	-103.8	0.0187	5.6	0.0408	12.2
631	0.0371	10.9	-0.1723	-50.8	-0.3447	-101.6	0.0259	7.6	0.0471	13.9
795	0.0409	11.8	-0.1809	-52.3	-0.3574	-103.3	0.0188	5.4	0.0375	10.9
1000	0.0528	15.2	-0.1793	-51.5	-0.3572	-102.7	0.0201	5.8	0.0376	10.8
∞	0	0	-0.2281	-46.5	-0.4562	-93.0	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.0856	-59.0	-0.0895	-61.7	-0.0899	-61.9	-0.0811	-55.9	-0.0742	-51.1
32	-0.0928	-61.3	-0.0966	-63.8	-0.0960	-63.4	-0.0896	-59.2	-0.0828	-54.6
40	-0.0986	-62.9	-0.1022	-65.2	-0.1011	-64.5	-0.0964	-61.5	-0.0899	-57.3
50	-0.1039	-64.2	-0.1070	-66.1	-0.1056	-65.3	-0.1022	-63.2	-0.0964	-59.6
63	-0.1095	-65.6	-0.1120	-67.1	-0.1106	-66.2	-0.1082	-64.8	-0.1031	-61.7
80	-0.1147	-66.6	-0.1166	-67.7	-0.1152	-66.9	-0.1136	-66.0	-0.1091	-63.4
100	-0.1188	-67.3	-0.1204	-68.2	-0.1192	-67.5	-0.1179	-66.8	-0.1143	-64.7
126	-0.1233	-67.9	-0.1246	-68.6	-0.1236	-68.1	-0.1226	-67.5	-0.1196	-65.9
159	-0.1273	-68.4	-0.1284	-69.0	-0.1276	-68.5	-0.1268	-68.1	-0.1244	-66.8
200	-0.1301	-68.4	-0.1308	-68.8	-0.1301	-68.4	-0.1296	-68.1	-0.1276	-67.1
250	-0.1331	-68.6	-0.1338	-68.9	-0.1332	-68.6	-0.1327	-68.4	-0.1311	-67.6
316	-0.1356	-68.6	-0.1361	-68.8	-0.1356	-68.6	-0.1353	-68.4	-0.1340	-67.8
398	-0.1383	-68.5	-0.1387	-68.7	-0.1383	-68.5	-0.1381	-68.4	-0.1370	-67.8
500	-0.1397	-68.1	-0.1400	-68.2	-0.1397	-68.1	-0.1395	-68.0	-0.1387	-67.6
631	-0.1418	-68.0	-0.1420	-68.1	-0.1417	-67.9	-0.1416	-67.9	-0.1409	-67.5
795	-0.1435	-67.7	-0.1437	-67.8	-0.1435	-67.7	-0.1434	-67.7	-0.1429	-67.4
1000	-0.1443	-67.1	-0.1445	-67.2	-0.1443	-67.1	-0.1442	-67.0	-0.1438	-66.8
∞	-0.1600	-56.7	-0.1600	-56.7	-0.1600	-56.7	-0.1600	-56.7	-0.1600	-56.7
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	-0.0855	-38.1	-0.1115	-49.7	-0.1293	-57.6	-0.0712	-31.7	-0.0496	-22.1
32	-0.0918	-39.9	-0.1167	-50.7	-0.1332	-57.9	-0.0812	-35.3	-0.0636	-27.6
40	-0.0970	-41.3	-0.1204	-51.2	-0.1358	-57.8	-0.0887	-37.7	-0.0740	-31.5
50	-0.1028	-42.8	-0.1242	-51.7	-0.1386	-57.8	-0.0958	-39.9	-0.0833	-34.7
63	-0.1086	-44.4	-0.1282	-52.4	-0.1416	-57.9	-0.1030	-42.1	-0.0926	-37.9
80	-0.1139	-45.6	-0.1314	-52.7	-0.1439	-57.7	-0.1091	-43.7	-0.1004	-40.3
100	-0.1181	-46.7	-0.1340	-53.0	-0.1459	-57.7	-0.1141	-45.1	-0.1069	-42.3
126	-0.1225	-47.5	-0.1372	-53.2	-0.1484	-57.5	-0.1193	-46.2	-0.1134	-44.0
159	-0.1264	-48.2	-0.1400	-53.4	-0.1505	-57.4	-0.1240	-47.3	-0.1191	-45.5
200	-0.1293	-48.7	-0.1416	-53.3	-0.1514	-57.0	-0.1272	-47.9	-0.1232	-46.3
250	-0.1327	-49.3	-0.1442	-53.5	-0.1534	-56.9	-0.1312	-48.7	-0.1279	-47.5
316	-0.1347	-49.4	-0.1454	-53.3	-0.1540	-56.5	-0.1337	-49.0	-0.1310	-48.1
398	-0.1373	-49.6	-0.1473	-53.2	-0.1553	-56.1	-0.1368	-49.4	-0.1345	-48.6
500	-0.1387	-49.6	-0.1481	-53.0	-0.1555	-55.6	-0.1386	-49.6	-0.1368	-48.9
631	-0.1403	-49.6	-0.1492	-52.8	-0.1560	-55.2	-0.1406	-49.7	-0.1390	-49.2
795	-0.1420	-49.6	-0.1505	-52.6	-0.1569	-54.8	-0.1427	-49.9	-0.1414	-49.4
1000	-0.1425	-49.4	-0.1505	-52.1	-0.1564	-54.2	-0.1435	-49.7	-0.1425	-49.3
∞	-0.1600	-45.4	-0.1600	-45.4	-0.1600	-45.4	-0.1600	-45.4	-0.1600	-45.4

Table 9a: Strongly skewed density

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%
25	447.64	455.93	8.29	1.85	470.65	23.02	5.14	488.66	41.02	9.16	459.34	11.70	2.61	463.72	16.08	3.59
32	368.14	372.28	4.15	1.13	383.09	14.95	4.06	396.24	28.10	7.63	374.39	6.26	1.70	377.32	9.18	2.49
40	307.03	309.17	2.13	0.69	317.30	10.27	3.34	327.00	19.96	6.50	310.54	3.50	1.14	312.44	5.41	1.76
50	255.71	256.99	1.28	0.50	263.19	7.48	2.93	270.27	14.56	5.70	257.96	2.25	0.88	259.13	3.43	1.34
63	211.31	211.75	0.44	0.21	216.57	5.26	2.49	221.67	10.36	4.90	212.60	1.30	0.61	213.31	2.00	0.95
80	173.52	173.24	-0.28	-0.16	177.12	3.60	2.07	180.77	7.25	4.18	174.15	0.63	0.36	174.57	1.05	0.60
100	143.70	143.52	-0.18	-0.12	146.40	2.70	1.88	149.05	5.35	3.72	144.17	0.47	0.33	144.42	0.72	0.50
126	118.69	118.59	-0.09	-0.08	120.62	1.94	1.63	122.55	3.87	3.26	118.95	0.27	0.23	119.11	0.42	0.36
159	97.77	97.77	-0.00	-0.00	99.17	1.40	1.44	100.56	2.79	2.86	97.94	0.17	0.18	98.03	0.27	0.27
200	80.72	80.70	-0.02	-0.02	81.69	0.97	1.20	82.69	1.98	2.45	80.77	0.06	0.07	80.83	0.11	0.14
250	66.92	66.94	0.02	0.03	67.65	0.73	1.09	68.39	1.46	2.19	66.97	0.05	0.08	67.01	0.08	0.12
316	55.03	55.05	0.01	0.02	55.55	0.52	0.94	56.08	1.05	1.90	55.06	0.02	0.05	55.08	0.04	0.08
398	45.43	45.45	0.02	0.05	45.82	0.39	0.85	46.20	0.77	1.70	45.46	0.03	0.06	45.47	0.04	0.08
500	37.54	37.57	0.03	0.09	37.84	0.29	0.78	38.11	0.57	1.52	37.57	0.03	0.07	37.58	0.03	0.09
631	30.91	30.91	0.00	0.01	31.10	0.19	0.61	31.30	0.39	1.27	30.90	-0.01	-0.02	30.91	-0.00	-0.00
795	25.54	25.54	0.00	0.01	25.67	0.13	0.51	25.81	0.27	1.07	25.53	-0.01	-0.05	25.53	-0.01	-0.04
1000	21.09	21.11	0.02	0.08	21.20	0.10	0.49	21.30	0.21	1.00	21.09	0.00	0.01	21.10	0.00	0.01
n	IVar	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%
25	584.48	548.37	-36.11	-6.18	552.15	-32.33	-5.53	560.47	-24.01	-4.11	548.58	-35.90	-6.14	552.34	-32.13	-5.50
32	527.24	485.34	-41.90	-7.95	488.35	-38.89	-7.38	496.25	-30.99	-5.88	485.62	-41.62	-7.89	490.32	-36.92	-7.00
40	475.94	435.14	-40.79	-8.57	437.38	-38.56	-8.10	443.19	-32.74	-6.88	435.33	-40.61	-8.53	438.84	-37.10	-7.80
50	426.88	390.33	-36.56	-8.56	391.90	-34.98	-8.19	395.95	-30.93	-7.25	390.41	-36.48	-8.55	392.81	-34.08	-7.98
63	378.18	346.32	-31.86	-8.43	347.02	-31.16	-8.24	349.64	-28.54	-7.55	345.98	-32.20	-8.52	347.47	-30.71	-8.12
80	331.58	305.07	-26.51	-7.99	305.09	-26.49	-7.99	306.77	-24.81	-7.48	304.37	-27.21	-8.21	305.30	-26.28	-7.93
100	291.48	268.92	-22.57	-7.74	268.92	-22.57	-7.74	270.01	-21.47	-7.37	268.42	-23.06	-7.91	268.99	-22.49	-7.72
126	253.99	235.15	-18.84	-7.42	235.20	-18.80	-7.40	235.90	-18.09	-7.12	234.86	-19.13	-7.53	235.22	-18.77	-7.39
159	219.75	204.13	-15.62	-7.11	204.23	-15.52	-7.06	204.67	-15.08	-6.86	204.01	-15.74	-7.16	204.23	-15.52	-7.06
200	189.77	176.97	-12.80	-6.75	177.08	-12.69	-6.69	177.37	-12.40	-6.53	176.94	-12.83	-6.76	177.08	-12.69	-6.69
250	164.21	153.66	-10.55	-6.43	153.75	-10.46	-6.37	153.93	-10.28	-6.26	153.65	-10.56	-6.43	153.74	-10.47	-6.38
316	140.53	131.98	-8.55	-6.09	132.03	-8.50	-6.05	132.15	-8.38	-5.96	131.97	-8.56	-6.09	132.02	-8.51	-6.05
398	120.06	113.16	-6.90	-5.75	113.19	-6.87	-5.72	113.27	-6.79	-5.66	113.15	-6.91	-5.75	113.19	-6.87	-5.72
500	102.59	97.03	-5.57	-5.43	97.05	-5.55	-5.41	97.09	-5.50	-5.36	97.02	-5.57	-5.43	97.04	-5.55	-5.41
631	87.16	82.65	-4.51	-5.17	82.66	-4.50	-5.16	82.69	-4.46	-5.12	82.64	-4.51	-5.18	82.66	-4.50	-5.16
795	73.91	70.33	-3.58	-4.85	70.33	-3.58	-4.84	70.35	-3.56	-4.81	70.32	-3.59	-4.85	70.33	-3.58	-4.84
1000	62.69	59.80	-2.89	-4.61	59.80	-2.89	-4.61	59.82	-2.88	-4.59	59.80	-2.90	-4.62	59.80	-2.89	-4.61
n	MISE	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%
25	1032.11	1004.29	-27.82	-2.70	1022.80	-9.31	-0.90	1049.13	17.02	1.65	1007.92	-24.20	-2.34	1016.06	-16.06	-1.56
32	895.38	857.62	-37.75	-4.22	871.44	-23.94	-2.67	892.49	-2.89	-0.32	860.02	-35.36	-3.95	867.64	-27.74	-3.10
40	782.97	744.31	-38.66	-4.94	754.68	-28.29	-3.61	770.19	-12.78	-1.63	745.86	-37.11	-4.74	751.28	-31.70	-4.05
50	682.59	647.32	-35.27	-5.17	655.09	-27.50	-4.03	666.22	-16.37	-2.40	648.36	-34.23	-5.01	651.94	-30.65	-4.49
63	589.49	558.06	-31.43	-5.33	563.59	-25.89	-4.39	571.31	-18.18	-3.08	558.58	-30.91	-5.24	560.77	-28.71	-4.87
80	505.10	478.31	-26.79	-5.30	482.21	-22.89	-4.53	487.54	-17.56	-3.48	478.53	-26.57	-5.26	479.86	-25.24	-5.00
100	435.18	412.44	-22.74	-5.23	415.31	-19.87	-4.57	419.06	-16.12	-3.70	412.59	-22.59	-5.19	413.41	-21.77	-5.00
126	372.68	353.74	-18.93	-5.08	355.82	-16.86	-4.52	358.45	-14.22	-3.82	353.82	-18.86	-5.06	354.33	-18.34	-4.92
159	317.52	301.90	-15.62	-4.92	303.40	-14.12	-4.45	305.23	-12.28	-3.87	301.95	-15.57	-4.90	302.26	-15.26	-4.80
200	270.49	257.67	-12.82	-4.74	258.77	-11.72	-4.33	260.06	-10.42	-3.85	257.71	-12.77	-4.72	257.91	-12.58	-4.65
250	231.13	220.60	-10.53	-4.56	221.40	-9.74	-4.21	222.32	-8.81	-3.81	220.62	-10.51	-4.55	220.75	-10.39	-4.49
316	195.56	187.02	-8.54	-4.37	187.58	-7.98	-4.08	188.23	-7.33	-3.75	187.03	-8.53	-4.36	187.10	-8.46	-4.33
398	165.49	158.61	-6.88	-4.16	159.01	-6.48	-3.92	159.47	-6.02	-3.64	158.61	-6.88	-4.16	158.65	-6.84	-4.13
500	140.14	134.60	-5.53	-3.95	134.88	-5.25	-3.75	135.21	-4.93	-3.52	134.59	-5.55	-3.96	134.62	-5.52	-3.94
631	118.07	113.56	-4.50	-3.81	113.76	-4.31	-3.65	113.99	-4.07	-3.45	113.55	-4.52	-3.83	113.57	-4.50	-3.81
795	99.45	95.87	-3.58	-3.60	96.00	-3.45	-3.47	96.17	-3.28	-3.30	95.85	-3.60	-3.62	95.86	-3.59	-3.61
1000	83.78	80.91	-2.87	-3.43	81.00	-2.78	-3.32	81.12	-2.67	-3.18	80.89	-2.89	-3.45	80.90	-2.89	-3.45

Table 9b: Strongly skewed density

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^{2}\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^{2}\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^{2}\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^{2}\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^{2}\Delta\%$ ISB
25	0.9450	1.0	2.6238	2.8	4.6768	5.0	1.3341	1.4	1.8329	2.0
32	0.7657	0.6	2.7614	2.3	5.1905	4.2	1.1556	0.9	1.6950	1.4
40	0.6025	0.4	2.9004	1.9	5.6384	3.7	0.9889	0.6	1.5266	1.0
50	0.5478	0.3	3.1989	1.7	6.2251	3.3	0.9613	0.5	1.4644	0.8
63	0.2842	0.1	3.4215	1.5	6.7332	3.0	0.8427	0.4	1.2999	0.6
80	-0.2827	-0.1	3.5721	1.3	7.1935	2.7	0.6274	0.2	1.0397	0.4
100	-0.2573	-0.1	3.9467	1.3	7.8297	2.5	0.6846	0.2	1.0540	0.3
126	-0.2044	-0.1	4.2012	1.2	8.3924	2.4	0.5824	0.2	0.9174	0.3
159	-0.0018	-0.0	4.4922	1.1	8.9432	2.3	0.5563	0.1	0.8494	0.2
200	-0.0793	-0.0	4.5240	1.0	9.2279	2.1	0.2694	0.1	0.5134	0.1
250	0.1226	0.0	4.8790	1.0	9.8140	2.0	0.3405	0.1	0.5475	0.1
316	0.1239	0.0	5.0607	1.0	10.2300	1.9	0.2427	0.0	0.4202	0.1
398	0.3319	0.1	5.4575	1.0	10.8675	1.9	0.3574	0.1	0.5081	0.1
500	0.6473	0.1	5.8908	1.0	11.5043	1.9	0.5485	0.1	0.6746	0.1
631	0.0955	0.0	5.4273	0.8	11.2751	1.7	-0.1611	-0.0	-0.0343	-0.0
795	0.0775	0.0	5.3014	0.8	11.2716	1.7	-0.4857	-0.1	-0.4206	-0.1
1000	0.9946	0.1	6.0726	0.8	12.2548	1.7	0.0830	0.0	0.1678	0.0
∞	0	0	6.9122	0.4	13.8243	0.7	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	-0.0903	-41.7	-0.0808	-37.3	-0.0600	-27.7	-0.0898	-41.5	-0.0803	-37.1
32	-0.1341	-60.4	-0.1245	-56.0	-0.0992	-44.7	-0.1332	-60.0	-0.1181	-53.2
40	-0.1632	-72.0	-0.1542	-68.1	-0.1310	-57.8	-0.1624	-71.7	-0.1484	-65.5
50	-0.1828	-79.2	-0.1749	-75.8	-0.1547	-67.0	-0.1824	-79.0	-0.1704	-73.8
63	-0.2007	-85.6	-0.1963	-83.7	-0.1798	-76.7	-0.2029	-86.5	-0.1935	-82.5
80	-0.2121	-89.1	-0.2119	-89.0	-0.1985	-83.4	-0.2177	-91.4	-0.2103	-88.3
100	-0.2257	-93.6	-0.2257	-93.6	-0.2147	-89.1	-0.2306	-95.7	-0.2249	-93.3
126	-0.2374	-97.3	-0.2368	-97.1	-0.2280	-93.5	-0.2410	-98.8	-0.2365	-96.9
159	-0.2483	-100.9	-0.2468	-100.2	-0.2397	-97.4	-0.2503	-101.6	-0.2468	-100.2
200	-0.2560	-103.1	-0.2538	-102.2	-0.2479	-99.8	-0.2566	-103.3	-0.2538	-102.2
250	-0.2638	-105.3	-0.2616	-104.4	-0.2569	-102.5	-0.2640	-105.4	-0.2618	-104.5
316	-0.2702	-107.0	-0.2685	-106.3	-0.2648	-104.9	-0.2705	-107.1	-0.2688	-106.4
398	-0.2747	-108.1	-0.2733	-107.6	-0.2704	-106.4	-0.2749	-108.2	-0.2736	-107.6
500	-0.2783	-108.8	-0.2773	-108.4	-0.2749	-107.5	-0.2786	-108.9	-0.2775	-108.5
631	-0.2844	-110.5	-0.2837	-110.2	-0.2817	-109.5	-0.2847	-110.6	-0.2838	-110.3
795	-0.2848	-110.2	-0.2844	-110.0	-0.2829	-109.5	-0.2853	-110.4	-0.2846	-110.1
1000	-0.2892	-111.3	-0.2889	-111.2	-0.2876	-110.7	-0.2896	-111.5	-0.2890	-111.3
∞	-0.3135	-111.1	-0.3135	-111.1	-0.3135	-111.1	-0.3135	-111.1	-0.3135	-111.1
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	-0.0696	-18.2	-0.0233	-6.1	0.0425	11.1	-0.0605	-15.8	-0.0401	-10.5
32	-0.1208	-32.0	-0.0766	-20.3	-0.0092	-2.4	-0.1131	-30.0	-0.0888	-23.5
40	-0.1546	-41.5	-0.1132	-30.4	-0.0511	-13.7	-0.1484	-39.8	-0.1268	-34.0
50	-0.1764	-47.8	-0.1375	-37.2	-0.0818	-22.2	-0.1712	-46.4	-0.1533	-41.5
63	-0.1980	-54.2	-0.1631	-44.6	-0.1145	-31.3	-0.1947	-53.3	-0.1809	-49.5
80	-0.2143	-59.1	-0.1831	-50.5	-0.1405	-38.7	-0.2126	-58.6	-0.2019	-55.7
100	-0.2274	-63.2	-0.1987	-55.2	-0.1612	-44.8	-0.2259	-62.8	-0.2177	-60.5
126	-0.2386	-66.7	-0.2124	-59.3	-0.1792	-50.1	-0.2376	-66.4	-0.2311	-64.6
159	-0.2483	-69.8	-0.2245	-63.1	-0.1953	-54.9	-0.2475	-69.6	-0.2426	-68.2
200	-0.2563	-72.4	-0.2344	-66.2	-0.2084	-58.9	-0.2555	-72.2	-0.2516	-71.1
250	-0.2633	-74.7	-0.2434	-69.0	-0.2204	-62.5	-0.2627	-74.5	-0.2597	-73.6
316	-0.2698	-76.8	-0.2522	-71.7	-0.2317	-65.9	-0.2697	-76.7	-0.2674	-76.1
398	-0.2737	-78.1	-0.2579	-73.6	-0.2396	-68.4	-0.2739	-78.2	-0.2721	-77.7
500	-0.2767	-79.2	-0.2627	-75.2	-0.2463	-70.5	-0.2773	-79.3	-0.2759	-78.9
631	-0.2842	-81.5	-0.2718	-78.0	-0.2570	-73.7	-0.2851	-81.8	-0.2839	-81.4
795	-0.2847	-81.9	-0.2742	-78.8	-0.2611	-75.1	-0.2862	-82.3	-0.2854	-82.1
1000	-0.2875	-82.8	-0.2785	-80.2	-0.2666	-76.8	-0.2894	-83.4	-0.2887	-83.2
∞	-0.3135	-88.9	-0.3135	-88.9	-0.3135	-88.9	-0.3135	-88.9	-0.3135	-88.9

Table 10a: Outlier density

n	KDE	EL-KDE			ET-KDE			CUE-KDE			ET-KDE (bc)			CUE-KDE (bc)		
	ISB	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%	ISB	Δ	Δ%
25	472.58	474.14	1.56	0.33	377.91	-94.67	-20.03	363.84	-108.74	-23.01	464.71	-7.87	-1.66	543.06	70.48	14.91
32	390.93	397.61	6.68	1.71	325.02	-65.91	-16.86	312.50	-78.43	-20.06	384.02	-6.91	-1.77	433.04	42.12	10.77
40	328.92	336.02	7.10	2.16	282.01	-46.91	-14.26	270.75	-58.17	-17.68	323.86	-5.06	-1.54	355.63	26.71	8.12
50	276.70	282.08	5.39	1.95	243.01	-33.69	-12.18	233.03	-43.67	-15.78	272.81	-3.89	-1.40	293.11	16.41	5.93
63	229.66	233.30	3.64	1.59	205.88	-23.78	-10.35	197.20	-32.45	-14.13	226.95	-2.71	-1.18	239.45	9.79	4.26
80	191.07	193.17	2.10	1.10	174.41	-16.66	-8.72	167.07	-24.00	-12.56	189.30	-1.78	-0.93	196.76	5.69	2.98
100	159.79	160.92	1.13	0.71	147.83	-11.95	-7.48	141.69	-18.10	-11.33	158.63	-1.16	-0.73	163.13	3.34	2.09
126	133.36	133.95	0.59	0.44	124.87	-8.48	-6.36	119.84	-13.52	-10.14	132.67	-0.69	-0.52	135.26	1.91	1.43
159	110.86	111.14	0.28	0.25	104.81	-6.05	-5.46	100.80	-10.06	-9.07	110.45	-0.42	-0.38	111.92	1.06	0.95
200	92.36	92.50	0.13	0.15	88.03	-4.33	-4.69	84.88	-7.49	-8.11	92.14	-0.23	-0.25	92.96	0.60	0.64
250	77.30	77.36	0.06	0.08	74.15	-3.14	-4.07	71.71	-5.59	-7.23	77.17	-0.13	-0.17	77.64	0.34	0.44
316	64.18	64.21	0.03	0.05	61.94	-2.24	-3.49	60.08	-4.10	-6.39	64.12	-0.06	-0.09	64.38	0.20	0.31
398	53.36	53.38	0.01	0.03	51.74	-1.62	-3.03	50.35	-3.01	-5.65	53.33	-0.03	-0.05	53.48	0.11	0.22
500	44.48	44.49	0.01	0.02	43.30	-1.18	-2.65	42.25	-2.23	-5.02	44.46	-0.02	-0.04	44.54	0.06	0.12
631	36.98	36.98	0.00	0.01	36.13	-0.85	-2.29	35.36	-1.62	-4.39	36.97	-0.01	-0.02	37.02	0.04	0.11
795	30.87	30.87	0.00	0.00	30.25	-0.62	-2.00	29.68	-1.19	-3.85	30.86	-0.00	-0.01	30.89	0.02	0.08
1000	25.58	25.57	-0.00	-0.00	25.13	-0.45	-1.75	24.71	-0.87	-3.39	25.57	-0.00	-0.00	25.59	0.01	0.05
n	IVar	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%	IVar	Δ	Δ%
25	950.57	997.29	46.73	4.92	895.72	-54.84	-5.77	876.54	-74.03	-7.79	1011.55	60.99	6.42	1051.10	100.53	10.58
32	815.20	864.94	49.74	6.10	787.95	-27.25	-3.34	768.55	-46.65	-5.72	868.63	53.43	6.55	889.82	74.62	9.15
40	708.62	752.72	44.10	6.22	695.29	-13.33	-1.88	677.56	-31.05	-4.38	751.31	42.69	6.02	762.05	53.43	7.54
50	613.76	648.85	35.09	5.72	607.90	-5.86	-0.96	592.84	-20.91	-3.41	645.17	31.41	5.12	649.73	35.97	5.86
63	527.77	552.17	24.40	4.62	524.63	-3.13	-0.59	512.98	-14.79	-2.80	548.11	20.34	3.85	549.69	21.92	4.15
80	450.14	464.69	14.55	3.23	447.62	-2.52	-0.56	439.46	-10.68	-2.37	461.61	11.47	2.55	462.06	11.92	2.65
100	387.43	394.90	7.47	1.93	384.50	-2.92	-0.75	379.06	-8.37	-2.16	392.92	5.49	1.42	393.18	5.76	1.49
126	331.89	334.43	2.54	0.77	328.54	-3.35	-1.01	325.20	-6.69	-2.01	333.43	1.55	0.47	333.77	1.88	0.57
159	282.55	282.44	-0.11	-0.04	279.22	-3.33	-1.18	277.29	-5.27	-1.86	282.05	-0.51	-0.18	282.43	-0.12	-0.04
200	241.55	239.98	-1.57	-0.65	238.28	-3.27	-1.35	237.21	-4.34	-1.80	239.93	-1.62	-0.67	240.31	-1.24	-0.51
250	206.69	204.69	-1.99	-0.96	203.74	-2.94	-1.42	203.13	-3.56	-1.72	204.74	-1.95	-0.94	205.05	-1.64	-0.79
316	175.01	172.88	-2.13	-1.22	172.37	-2.64	-1.51	172.05	-2.97	-1.70	172.96	-2.05	-1.17	173.20	-1.82	-1.04
398	148.43	146.49	-1.95	-1.31	146.21	-2.23	-1.50	146.03	-2.41	-1.62	146.57	-1.87	-1.26	146.73	-1.70	-1.15
500	126.12	124.41	-1.72	-1.36	124.26	-1.87	-1.48	124.15	-1.98	-1.57	124.48	-1.65	-1.31	124.59	-1.54	-1.22
631	106.56	105.09	-1.47	-1.38	105.02	-1.54	-1.45	104.96	-1.60	-1.50	105.15	-1.41	-1.32	105.23	-1.33	-1.25
795	90.19	88.97	-1.22	-1.36	88.94	-1.26	-1.39	88.90	-1.29	-1.43	89.02	-1.17	-1.30	89.07	-1.12	-1.24
1000	76.36	75.33	-1.02	-1.34	75.32	-1.04	-1.36	75.30	-1.06	-1.38	75.37	-0.98	-1.29	75.40	-0.95	-1.25
n	MISE	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%	MISE	Δ	Δ%
25	1423.15	1471.43	48.28	3.39	1273.63	-149.52	-10.51	1240.37	-182.77	-12.84	1476.27	53.12	3.73	1594.16	171.02	12.02
32	1206.13	1262.55	56.42	4.68	1112.97	-93.16	-7.72	1081.05	-125.08	-10.37	1252.65	46.52	3.86	1322.86	116.73	9.68
40	1037.54	1088.75	51.21	4.94	977.30	-60.24	-5.81	948.32	-89.22	-8.60	1075.16	37.63	3.63	1117.68	80.14	7.72
50	890.45	930.93	40.48	4.55	850.90	-39.55	-4.44	825.87	-64.59	-7.25	917.98	27.52	3.09	942.84	52.38	5.88
63	757.42	785.47	28.04	3.70	730.51	-26.91	-3.55	710.18	-47.24	-6.24	775.06	17.63	2.33	789.14	31.72	4.19
80	641.21	657.86	16.64	2.60	622.03	-19.19	-2.99	606.53	-34.69	-5.41	650.91	9.70	1.51	658.82	17.61	2.75
100	547.21	555.81	8.60	1.57	532.34	-14.88	-2.72	520.75	-26.47	-4.84	551.54	4.33	0.79	556.31	9.10	1.66
126	465.24	468.37	3.13	0.67	453.41	-11.83	-2.54	445.04	-20.20	-4.34	466.10	0.86	0.18	469.03	3.79	0.81
159	393.41	393.58	0.17	0.04	384.03	-9.38	-2.38	378.09	-15.32	-3.90	392.49	-0.92	-0.23	394.35	0.93	0.24
200	333.91	332.48	-1.44	-0.43	326.31	-7.60	-2.28	322.09	-11.83	-3.54	332.07	-1.85	-0.55	333.27	-0.64	-0.19
250	283.98	282.05	-1.93	-0.68	277.90	-6.09	-2.14	274.83	-9.15	-3.22	281.91	-2.08	-0.73	282.68	-1.30	-0.46
316	239.19	237.09	-2.10	-0.88	234.31	-4.89	-2.04	232.13	-7.07	-2.95	237.08	-2.11	-0.88	237.57	-1.62	-0.68
398	201.80	199.86	-1.93	-0.96	197.95	-3.84	-1.90	196.38	-5.42	-2.69	199.90	-1.90	-0.94	200.21	-1.59	-0.79
500	170.61	168.90	-1.71	-1.00	167.56	-3.05	-1.79	166.40	-4.21	-2.47	168.94	-1.67	-0.98	169.12	-1.48	-0.87
631	143.54	142.07	-1.47	-1.02	141.15	-2.39	-1.66	140.32	-3.22	-2.24	142.13	-1.41	-0.98	142.25	-1.29	-0.90
795	121.06	119.84	-1.22	-1.01	119.19	-1.87	-1.55	118.58	-2.48	-2.05	119.89	-1.17	-0.97	119.96	-1.10	-0.91
1000	101.93	100.91	-1.03	-1.01	100.45	-1.48	-1.45	100.01	-1.92	-1.89	100.95	-0.98	-0.97	100.99	-0.94	-0.92

Table 10b: Outlier density

n	EL-KDE		ET-KDE		CUE-KDE		ET-KDE (bc)		CUE-KDE (bc)	
	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB	$nh^{-2}\Delta$ ISB	$nh^2\Delta\%$ ISB
25	0.9336	0.0	-56.7964	-2.1	-65.2373	-2.4	-4.7187	-0.2	42.2825	1.6
32	5.7687	0.2	-56.9492	-2.0	-67.7686	-2.4	-5.9670	-0.2	36.3885	1.3
40	8.5234	0.3	-56.2764	-1.9	-69.7820	-2.4	-6.0724	-0.2	32.0480	1.1
50	8.9614	0.3	-56.0454	-1.8	-72.6515	-2.4	-6.4665	-0.2	27.3032	0.9
63	8.4884	0.3	-55.4302	-1.8	-75.6432	-2.4	-6.3168	-0.2	22.8216	0.7
80	6.9251	0.2	-54.9742	-1.7	-79.1996	-2.4	-5.8610	-0.2	18.7760	0.6
100	5.1484	0.2	-54.5104	-1.6	-82.5332	-2.5	-5.2855	-0.2	15.2416	0.5
126	3.7376	0.1	-54.0370	-1.6	-86.0833	-2.5	-4.3960	-0.1	12.1400	0.4
159	2.4837	0.1	-53.8870	-1.5	-89.5897	-2.6	-3.7122	-0.1	9.4076	0.3
200	1.6695	0.0	-53.6672	-1.5	-92.7430	-2.6	-2.8360	-0.1	7.3714	0.2
250	1.0569	0.0	-53.6322	-1.5	-95.4157	-2.6	-2.2043	-0.1	5.8005	0.2
316	0.7398	0.0	-53.5127	-1.5	-97.8258	-2.7	-1.4196	-0.0	4.7015	0.1
398	0.4612	0.0	-53.6720	-1.4	-100.0024	-2.7	-0.9631	-0.0	3.8086	0.1
500	0.3068	0.0	-54.2176	-1.4	-102.5929	-2.7	-0.8824	-0.0	2.5335	0.1
631	0.1339	0.0	-54.2760	-1.4	-103.8975	-2.7	-0.3609	-0.0	2.5177	0.1
795	0.0057	0.0	-54.7920	-1.4	-105.7551	-2.7	-0.1900	-0.0	2.1590	0.1
1000	-0.0716	-0.0	-55.0083	-1.4	-106.8954	-2.8	-0.1081	-0.0	1.7335	0.0
∞	0	0	-60.1020	-1.4	-120.2040	-2.8	0	0	0	0
n	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar	$n\Delta$ IVar	$h^{-1}\Delta\%$ IVar
25	0.1168	76.1	-0.1371	-89.4	-0.1851	-120.6	0.1525	99.4	0.2513	163.8
32	0.1592	100.3	-0.0872	-54.9	-0.1493	-94.0	0.1710	107.7	0.2388	150.4
40	0.1764	107.8	-0.0533	-32.6	-0.1242	-75.9	0.1708	104.3	0.2137	130.6
50	0.1755	104.3	-0.0293	-17.4	-0.1046	-62.2	0.1570	93.3	0.1799	106.9
63	0.1537	88.9	-0.0197	-11.4	-0.0932	-53.9	0.1282	74.1	0.1381	79.9
80	0.1164	65.6	-0.0202	-11.4	-0.0855	-48.2	0.0918	51.8	0.0953	53.8
100	0.0747	41.2	-0.0292	-16.1	-0.0837	-46.1	0.0549	30.3	0.0576	31.7
126	0.0320	17.2	-0.0422	-22.7	-0.0842	-45.3	0.0195	10.5	0.0237	12.8
159	-0.0017	-0.9	-0.0530	-27.9	-0.0837	-44.1	-0.0080	-4.2	-0.0020	-1.0
200	-0.0314	-16.2	-0.0654	-33.7	-0.0868	-44.7	-0.0324	-16.7	-0.0248	-12.8
250	-0.0498	-25.2	-0.0736	-37.2	-0.0889	-45.0	-0.0487	-24.6	-0.0410	-20.7
316	-0.0673	-33.5	-0.0835	-41.5	-0.0938	-46.6	-0.0649	-32.2	-0.0574	-28.5
398	-0.0775	-37.9	-0.0886	-43.3	-0.0959	-46.9	-0.0743	-36.3	-0.0678	-33.1
500	-0.0858	-41.3	-0.0934	-44.9	-0.0988	-47.5	-0.0823	-39.6	-0.0769	-37.0
631	-0.0927	-43.9	-0.0973	-46.1	-0.1009	-47.8	-0.0888	-42.0	-0.0839	-39.8
795	-0.0972	-45.3	-0.0998	-46.6	-0.1025	-47.8	-0.0931	-43.4	-0.0892	-41.6
1000	-0.1025	-47.1	-0.1036	-47.6	-0.1055	-48.5	-0.0984	-45.2	-0.0952	-43.8
∞	-0.1242	-44.0	-0.1242	-44.0	-0.1242	-44.0	-0.1242	-44.0	-0.1242	-44.0
n	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE	$n\Delta$ MISE	$h^{-1}\Delta\%$ MISE
25	0.1207	52.6	-0.3738	-162.7	-0.4569	-198.9	0.1328	57.8	0.4275	186.1
32	0.1805	76.9	-0.2981	-126.9	-0.4003	-170.4	0.1489	63.4	0.3735	159.0
40	0.2048	85.5	-0.2410	-100.5	-0.3569	-148.9	0.1505	62.8	0.3206	133.8
50	0.2024	82.9	-0.1978	-81.0	-0.3229	-132.3	0.1376	56.4	0.2619	107.3
63	0.1767	71.2	-0.1696	-68.4	-0.2976	-120.0	0.1111	44.8	0.1998	80.5
80	0.1332	52.7	-0.1535	-60.8	-0.2775	-109.9	0.0776	30.7	0.1409	55.8
100	0.0860	33.6	-0.1488	-58.1	-0.2647	-103.3	0.0433	16.9	0.0910	35.5
126	0.0394	15.1	-0.1491	-57.2	-0.2546	-97.6	0.0108	4.1	0.0477	18.3
159	0.0027	1.0	-0.1492	-56.4	-0.2437	-92.2	-0.0147	-5.6	0.0148	5.6
200	-0.0287	-10.7	-0.1520	-56.7	-0.2365	-88.2	-0.0370	-13.8	-0.0129	-4.8
250	-0.0483	-17.8	-0.1522	-56.0	-0.2287	-84.2	-0.0519	-19.1	-0.0325	-12.0
316	-0.0663	-24.1	-0.1544	-56.1	-0.2233	-81.2	-0.0668	-24.3	-0.0512	-18.6
398	-0.0770	-27.7	-0.1529	-55.0	-0.2158	-77.6	-0.0754	-27.1	-0.0632	-22.7
500	-0.0855	-30.4	-0.1524	-54.2	-0.2104	-74.8	-0.0833	-29.6	-0.0742	-26.4
631	-0.0925	-32.5	-0.1508	-53.0	-0.2032	-71.5	-0.0891	-31.3	-0.0814	-28.6
795	-0.0972	-33.8	-0.1488	-51.7	-0.1970	-68.5	-0.0933	-32.4	-0.0873	-30.3
1000	-0.1025	-35.3	-0.1482	-51.0	-0.1923	-66.2	-0.0984	-33.9	-0.0938	-32.3
∞	-0.1242	-35.2	-0.1242	-35.2	-0.1242	-35.2	-0.1242	-35.2	-0.1242	-35.2

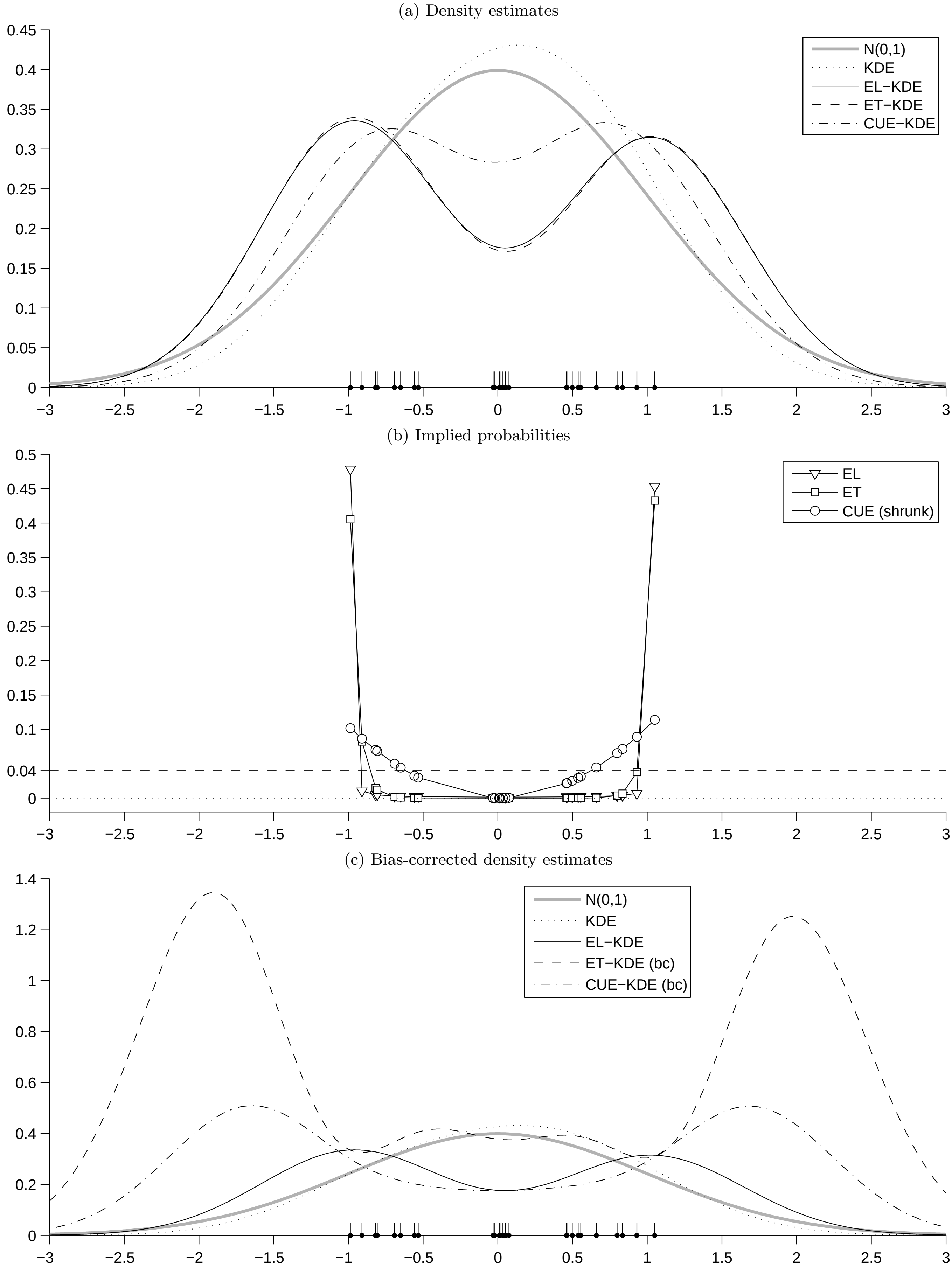

Figure 1: A ‘bad case’ (Gaussian data, known mean and variance)

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