A UNIFORM LAW FOR CONVERGENCE TO THE LOCAL TIMES OF LINEAR FRACTIONAL STABLE MOTIONS

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We provide a uniform law for the weak convergence of additive functionals of partial sum processes to the local times of linear fractional stable motions, in a setting sufficiently general for statistical applications. Our results are fundamental to the analysis of the global properties of nonparametric estimators of nonlinear statistical models that involve such processes as covariates.

1. Introduction. Let $x_t = \sum_{s=1}^{t} v_s$ be the partial sum of a scalar linear process $\{v_t\}$, for which the finite-dimensional distributions of $d_n^{-1} x_{\lfloor nr \rfloor}$ converge to those of $X(r)$. Under certain regularity conditions, we then have the finite-dimensional convergence

$$\mathcal{L}_n^f(a, h_n) \coloneqq \frac{d_n}{nh_n} \sum_{t=1}^{n} f\left(\frac{x_t - d_n a}{h_n}\right) \rightsquigarrow_{\mathrm{fdd}} \mathcal{L}(a) \int_{\mathbb{R}} f \tag{1.1}$$

where $a \in \mathbb{R}$, f is Lebesgue integrable, $h_n = o(d_n)$ is a deterministic sequence, and $\mathcal{L}$ denotes the occupation density (or *local time*; see Remark 2.5 below) associated to X. Convergence results of this kind are particularly well documented in the case where $\{x_t\}$ is a random walk (see the monograph by Borodin and Ibragimov, 1995), and have more recently been extended to cover generating mechanisms that allow the increments of $\{x_t\}$ to exhibit significant temporal dependence (Jeganathan, 2004; Wang and Phillips, 2009a).

These more general theorems concerning (1.1) have, in turn, played a fundamental role in the study of nonparametric estimation and testing in the setting of nonlinear cointegrating models. The simplest of these models take the form

$$y_t = m_0(x_t) + u_t, \tag{1.2}$$

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where $\{x_t\}$ is as above, $\{u_t\}$ is a weakly dependent error process, and m_0 is an unknown function, assumed to possess a certain degree of smoothness (or be otherwise approximable). In a series of recent papers, (1.1) has facilitated the development of a pointwise asymptotic distribution theory for kernel regression estimators of m_0 under very general conditions: see especially Wang and Phillips (2009a,b, 2011, 2015), Kasparis and Phillips (2012) and Kasparis, Andreou, and Phillips (2012).[1]

However, there are definite limits to the range of problems that can be successfully addressed with the aid of (1.1). In particular, since it concerns only the finite-dimensional convergence of $\mathcal{L}_n^f(a, h_n)$, (1.1) is suited only to studying the *local* behaviour of a nonparametric estimator: that is, its behaviour in the vicinity of a fixed spatial point. For the purpose of obtaining uniform rates of convergence for kernel regression estimators on 'wide' domains – that is, on domains having a width of the *same* order as the range of $\{x_t\}_{t=1}^n$ – it is manifestly inadequate. (See Duffy, 2015, for a detailed account.) The situation is even worse with regard to sieve nonparametric estimation in this setting – which initially motivated the author's research on this problem – since in this case the development of even a *pointwise* asymptotic distribution theory requires a prior result on the *uniform* consistency of the estimator, over the entire domain on which estimation is to be performed.

The main purpose of this paper is thus to provide conditions under which the finite-dimensional convergence in (1.1) can be strengthened to the weak convergence

$$\mathcal{L}_n^f(a, h_n) \rightsquigarrow \mathcal{L}(a) \int_{\mathbb{R}} f \tag{1.3}$$

where $\mathcal{L}_n^f(a, h_n)$ is regarded as a process indexed by $(f, a) \in \mathscr{F} \times \mathbb{R}$, and $\{h_n\}$ may be random. Results of this kind are available in the existing literature, but only in the random walk case, which requires that the increments of $\{x_t\}$ be independent, and X to be an α-stable Lévy motion (see Borodin, 1981, 1982; Perkins, 1982; and Borodin and Ibragimov, 1995, Ch. V). In contrast, we allow the increments of $\{x_t\}$ to be serially correlated, such that the associated limiting process X may be a linear fractional stable motion,

[1] If $\{x_t\}$ is Markov, then this distribution theory may be developed by quite different arguments, without the use of (1.1): see Karlsen and Tjøstheim (2001) and Karlsen, Myklebust, and Tjøstheim (2007), which have spawned a large literature. While we consider this approach to the problem to be equally important, our results touch upon it only a little, since we work with a class of regressor processes that are typically (excepting the random walk case) non-Markov.

which subsumes the α-stable Lévy motion and fractional Brownian motion as special cases. Further, we permit the bandwidth sequence $\{h_n\}$ to be a *random* process, subject only to certain weak asymptotic growth conditions: this is of considerable utility in statistical applications, where the assumption that $\{h_n\}$ is a 'given' deterministic sequence seems quite unrealistic. Crucial to the proof of (1.3) is a novel order estimate for $\mathcal{L}_n^f(a,1)$ when $\int f = 0$, which is of interest in its own right.

The remainder of this paper is organised as follows. Our assumptions on the data generating mechanism are described in Section 2. The main result (Theorem 3.1) is discussed in Section 3. An outline of the proof follows in Section 4, together with the statement of two key auxiliary results (Propositions 4.1 and 4.2). A preliminary application of our results to the kernel nonparametric estimation of m_0 in (1.2) is given in Section 5. The proof of Theorem 3.1 appears in Section 6, followed in Section 7 by proofs of Propositions 4.1 and 4.2. A proof related to the application appears in Section 8. The final two sections (9 and 10) are of a more technical nature, detailing the proofs of two lemmas required in Section 7, and so may be skipped on a first reading.

1.1. *Notation.* For a complete listing of the notation used in this paper, see Section H of the Supplement.[2] The stochastic order notations $o_p(\cdot)$ and $O_p(\cdot)$ have the usual definitions, as given e.g. in van der Vaart (1998, Sec. 2.2). For deterministic sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$, and $a_n \asymp b_n$ if $\lim_{n\to\infty} a_n/b_n \in (-\infty,\infty)\backslash\{0\}$; for random sequences, $a_n \lesssim_p b_n$ denotes $a_n = O_p(b_n)$. $X_n \rightsquigarrow X$ denotes weak convergence in the sense of van der Vaart and Wellner (1996), and $X_n \rightsquigarrow_{\text{fdd}} X$ the convergence of finite-dimensional distributions. For a metric space (Q,d), $\ell_\infty(Q)$ (resp. $\ell_{\text{ucc}}(Q)$) denotes the space of uniformly bounded functions on Q, equipped with the topology of uniform convergence (resp. uniform convergence on compacta). For $p \geq 1$, X a random variable, and $f : \mathbb{R} \to \mathbb{R}$, $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$ and $\|f\|_p := (\int_{\mathbb{R}}|f|^p)^{1/p}$. BI denotes the space of bounded and Lebesgue integrable functions on $\mathbb{R}$. $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively denote the floor and ceiling functions. C denotes a generic constant that may take different values even at different places in the same proof; $a \lesssim b$ denotes $a \leq Cb$.

2. Model and assumptions. Our assumptions on the generating mechanism are similar to those of Jeganathan (2004) – who proves a finite-dimensional counterpart to our main theorem – and are comparable to those

[2]The Supplement is available from the Economic Working Papers section of the Nuffield College website: `www.nuffield.ox.ac.uk/Research/Economics-Group/Working-Papers`.

made on the regressor process in previous work on the estimation of nonlinear cointegrating regressions (see e.g. Park and Phillips, 2001; Wang and Phillips, 2009b, 2012, 2015; and Kasparis and Phillips, 2012).

ASSUMPTION 1.

(i) $\{\epsilon_t\}$ *is a scalar i.i.d. sequence.* ϵ_0 *lies in the domain of attraction of a strictly stable distribution with index* $\alpha \in (0, 2]$, *and has characteristic function* $\psi(\lambda) := \mathbb{E}e^{i\lambda\epsilon_0}$ *satisfying* $\psi \in L^{p_0}$ *for some* $p_0 \geq 1$.

(ii) $\{x_t\}$ *is generated according to*

$$x_t := \sum_{s=1}^{t} v_s \qquad\qquad v_t := \sum_{k=0}^{\infty} \phi_k \epsilon_{t-k}, \tag{2.1}$$

and either

(a) $\alpha \in (1, 2]$, $\sum_{k=0}^{\infty}|\phi_k| < \infty$ *and* $\phi := \sum_{k=0}^{\infty} \phi_k \neq 0$; *or*

$\phi_k \sim k^{H-1-1/\alpha}\pi_k$ *for some* $\{\pi_k\}_{k\geq 0}$ *strictly positive and slowly varying at infinity, with*

(b) $H > 1/\alpha$; *or*

(c) $H < 1/\alpha$ *and* $\sum_{k=0}^{\infty} \phi_k = 0$.

In both cases (b) *and* (c), $H \in (0, 1)$.

REMARK 2.1. Part (i) implies that there exists a slowly varying sequence $\{\varrho_k\}$ such that

$$\frac{1}{n^{1/\alpha}\varrho_n} \sum_{t=1}^{\lfloor nr \rfloor} \epsilon_t \rightsquigarrow_{\text{fdd}} Z_\alpha(r) \tag{2.2}$$

where Z_α denotes an *α-stable Lévy motion* on $\mathbb{R}$, with $Z_\alpha(0) = 0$. That is, the increments of Z_α are stationary, and for any $r_1 < r_2$ the characteristic function of $Z_\alpha(r_2) - Z_\alpha(r_1)$ has the logarithm

$$-(r_2 - r_1)c|\lambda|^\alpha \left[1 + i\beta \operatorname{sgn}(\lambda) \tan\left(\frac{\pi\alpha}{2}\right)\right]$$

where $\beta \in [-1, 1]$ and $c > 0$; following Jeganathan (2004, p. 1773), we impose the further restriction that $\beta = 0$ when $\alpha = 1$. We shall also require that $\{\varrho_k\}$ be chosen such that $c = 1$ here, which provides a convenient normalisation for the scale of Z_α.

REMARK 2.2. To permit the alternative forms of (ii) to be more concisely referenced, we shall *refer* to (a) as corresponding to the case where $H = 1/\alpha$; this designation may be justified by the manner in which the finite-dimensional limit of $d_n^{-1} x_{\lfloor nr \rfloor}$ depends on (H, α), as displayed in (2.6) below. The statement that $H < 1/\alpha$ will also be used as a shorthand for (c), that is, it will always be understood that $\sum_{k=0}^{\infty} \phi_k = 0$ in this case.

We shall treat the parameters (including H and α) describing the data generating mechanism as 'fixed' throughout, ignoring the dependence of any constants on these. Let $\{c_k\}$ denote a sequence with $c_0 = 1$ and

$$c_k = \begin{cases} \phi & \text{if } H = 1/\alpha \\ |H - 1/\alpha|^{-1} k^{H-1/\alpha} \pi_k & \text{otherwise.} \end{cases} \tag{2.3}$$

By Karamata's theorem (Bingham, Goldie, and Teugels, 1987, Thm. 1.5.11), $\sum_{l=0}^{k} \phi_k \sim c_k$ as $k \to \infty$. Set

$$d_k := k^{1/\alpha} c_k \varrho_k \qquad\qquad e_k := k d_k^{-1}, \tag{2.4}$$

and note that the sequences $\{c_k\}$, $\{d_k\}$ and $\{e_k\}$ are regularly varying with indices $H - 1/\alpha$, H and $1 - H$ respectively. Theorems 5.1–5.3 in Kasahara and Maejima (1988) yield

PROPOSITION 2.1. *Under Assumption 1,*

$$X_n(r) := \frac{1}{d_n} x_{\lfloor nr \rfloor} \rightsquigarrow_{\mathrm{fdd}} X(r), \qquad r \in [0, 1] \tag{2.5}$$

where X is the linear fractional stable motion (LFSM)

$$X(r) := \int_0^r (r - s)^{H-1/\alpha} \, \mathrm{d}Z_\alpha(s) + \int_{-\infty}^0 [(r - s)^{H-1/\alpha} - (-s)^{H-1/\alpha}] \, \mathrm{d}Z_\alpha(s) \tag{2.6}$$

with the convention that $X = Z_\alpha$ when $H = 1/\alpha$; Z_α is an α-stable Lévy motion on $\mathbb{R}$, with $Z_\alpha(0) = 0$.

REMARK 2.3. For a detailed discussion of the LFSM, see Samorodnitsky and Taqqu (1994). When $\alpha = 2$, Z_α is a Brownian motion with variance 2; if additionally $H \neq 1/\alpha$, X is thus a fractional Brownian motion.

REMARK 2.4. Excepting such cases as the following:

(i) $\alpha \in (1,2]$, $H > 1/\alpha$ (Astrauskas, 1983, Thm. 2);
(ii) $\alpha = 2$, $H = 1/\alpha$ and $\mathbb{E}\epsilon_0^2 < \infty$ (Hannan, 1979); and
(iii) $\alpha = 2$, $H < 1/\alpha$ and $\mathbb{E}|\epsilon_0|^q < \infty$ for some $q > 2$ (Davidson and de Jong, 2000, Thm. 3);

it may not be possible to strengthen the convergence in (2.5) to weak convergence on $\ell_\infty[0,1]$. Weak convergence may hold, however, with respect to a weaker topology, and we shall be principally concerned with whether this topology is sufficiently strong that

$$\inf_{r\in[0,1]} X_n(r) \rightsquigarrow \inf_{r\in[0,1]} X(r) \qquad \sup_{r\in[0,1]} X_n(r) \rightsquigarrow \sup_{r\in[0,1]} X(r), \tag{2.7}$$

such as would follow from weak convergence in the Skorokhod M_1 topology (see Skorokhod, 1956, 2.2.10). When $H = 1/\alpha$, sufficient conditions for this kind of convergence – which entail further restrictions on $\{\phi_k\}$ than are imposed here – are given in Avram and Taqqu (1992, Thm. 2) and Tyran-Kamińska (2010, Thm. 1 and Cor. 1). However, when $H < 1/\alpha$ and $\alpha \in (0,2)$, the sample paths of X are unbounded, and thus (2.7) cannot possibly hold (see Samorodnitsky and Taqqu, 1994, Example 10.2.5). In any case, (2.7) is *not* necessary for the main results of this paper; it merely permits Theorem 3.1 below to take a slightly strengthened form.

Remark 2.5. In consequence of Theorem 3(i) in Jeganathan (2004), the convergence in (2.5) occurs jointly with

$$\mathcal{L}_n^f(a) \coloneqq \frac{1}{e_n}\sum_{t=1}^n f(x_t - d_n a) \rightsquigarrow_{\mathrm{fdd}} \mathcal{L}(a)\int_{\mathbb{R}} f, \qquad a \in \mathbb{R}$$

for every $f \in \mathrm{BI}$. Here $\{\mathcal{L}(a)\}_{a\in\mathbb{R}}$ denotes the occupation density (*local time*) of X, a process which, almost surely, has continuous paths and satisfies

$$\int_{\mathbb{R}} f(x)\mathcal{L}(x)\,\mathrm{d}x = \int_0^1 f(X(r))\,\mathrm{d}r \tag{2.8}$$

for all Borel measurable and locally integrable f. (For the existence of $\mathcal{L}$, see Theorem 0 in Jeganathan, 2004; the path continuity may be deduced from Theorem 3.1 below.)

3. A uniform law for the convergence to local time.

Our main result concerns the convergence

$$\mathcal{L}_n^f(a, h_n) \coloneqq \frac{1}{e_n h_n}\sum_{t=1}^n f\left(\frac{x_t - d_n a}{h_n}\right) \rightsquigarrow \mathcal{L}(a)\int_{\mathbb{R}} f \tag{3.1}$$

where $\mathcal{L}_n^f(a, h_n)$ is regarded as a process indexed by $(f, a) \in \mathscr{F} \times \mathbb{R}$. ($\mathscr{F} \times \mathbb{R}$ is endowed with the product topology, $\mathscr{F}$ having the L^1 topology, and $\mathbb{R}$ the usual Euclidean topology.) $\{h_n\}$ is a measurable bandwidth sequence that may be functionally dependent on $\{x_t\}$, or indeed upon any other elements of the probability space; it is required only to satisfy

ASSUMPTION 2. *$h_n \in \mathscr{H}_n := [\underline{h}_n, \overline{h}_n]$ with probability approaching 1 (w.p.a.1), where $\overline{h}_n = o(d_n)$ and $\underline{h}_n^{-1} = o(e_n \log^{-2} n)$.*

Define

$$\mathrm{BI}_\beta := \left\{ f \in \mathrm{BI} \mid \int_{\mathbb{R}} |f(x)||x|^\beta \, \mathrm{d}x < \infty \right\} \tag{3.2}$$

and let BIL_β denote the subset of Lipschitz continuous functions in BI_β. In order to state conditions on $\mathscr{F} \subset \mathrm{BI}$ that are sufficient for (3.1) to hold, we first recall some definitions familiar from the theory of empirical processes. A function $F : \mathbb{R} \to \mathbb{R}_+$ is termed an *envelope* for $\mathscr{F}$, if $\sup_{f \in \mathscr{F}} |f(x)| \leq F(x)$ for every $x \in \mathbb{R}$. Given a pair of functions $l, u \in L^1$, define the bracket

$$[l, u] := \{f \in L^1 \mid l(x) \leq f(x) \leq u(x), \forall x \in \mathbb{R}\};$$

we say that $[l, u]$ is an *ϵ-bracket* if $\|u - l\|_1 < \epsilon$, and a *continuous* bracket if l and u are continuous. Let $N_{[]}^*(\epsilon, \mathscr{F})$ denote the minimum number of continuous ϵ-brackets required to cover $\mathscr{F}$.

ASSUMPTION 3.

(i) *$\mathscr{F} \subset \mathrm{BI}$ has envelope $F \in \mathrm{BIL}_\beta$, for some $\beta > 0$; and*
(ii) *for each $\epsilon > 0$, $N_{[]}^*(\epsilon, \mathscr{F}) < \infty$.*

We may now state our main result, the proof of which appears in Section 6.

THEOREM 3.1. *Suppose Assumptions 1–3 hold. Then*

(i) *(3.1) holds in $\ell_{\mathrm{ucc}}(\mathscr{F} \times \mathbb{R})$;*

and if additionally (2.7) holds, then

(ii) *(3.1) holds in $\ell_\infty(\mathscr{F} \times \mathbb{R})$.*

REMARK 3.1. The case where $h_n = 1$, $\mathscr{F} = \{f\}$ and $\{x_t\}$ is a random walk – which here entails $H = 1/\alpha$ and $\phi_i = 0$ for all $i \geq 1$ – has been studied extensively: see in particular Borodin (1981, 1982), Perkins (1982) and Borodin and Ibragimov (1995, Ch. V). In those works, it is proved (under these more restrictive assumptions on $\{x_t\}$) that

$$\frac{1}{e_n}\sum_{t=1}^{\lfloor nr \rfloor} f(x_t - d_n a) \rightsquigarrow \mathcal{L}(a;r)\int_{\mathbb{R}} f$$

on $\ell_\infty(\mathbb{R}\times[0,1])$, where $\mathcal{L}(a;r)$ denotes the local time of X restricted to $[0,r]$. Theorem 3.1 could be very easily extended in this direction; we have refrained from doing so only to keep the paper to a reasonable length. The principal contribution of Theorem 3.1 is thus to extend this convergence in a direction more suitable for statistical applications, by allowing $\{v_t\}$ to be serially correlated and the bandwidth sequence $\{h_n\}$ to be data-dependent.

REMARK 3.2. After the manuscript of this paper had been completed, we obtained a copy of an unpublished manuscript by Liu, Chan, and Wang (2014) in which, under rather different assumptions from those given here, a result similar to Theorem 3.1 is proved (for a fixed f and a deterministic sequence $\{h_n\}$). Regarding the differences between our main result and their Theorem 2.1, we may note particularly their requirement that there exist a sequence of processes $\{X_n^*\}$ with $X_n^* =_d X$, and a $\delta > 0$ such that

$$\sup_{r\in[0,1]} |X_n(r) - X_n^*(r)| = o_{\text{a.s.}}(n^{-\delta}), \tag{3.3}$$

a condition which excludes a large portion of the processes considered in this paper, in view of Remark 2.4 above. (The availability of (3.3) permits these authors to prove their result by an argument radically different from that developed here.) On the other hand, our results do *not* subsume theirs, since these authors do not require v_t to be a linear process.

Although Assumption 3 requires that $\mathscr{F}$ have a smooth envelope and smooth brackets, it is perfectly consistent with $\mathscr{F}$ containing discontinuous functions. Indeed, Assumption 3 is consistent with such cases as the following, as verified in Section A of the Supplement. (We expect that boundedness and $\int |f(x)||x|^\beta \,\mathrm{d}x < \infty$ could also be relaxed through the use of a suitable truncation argument, such as is employed in the proof of Theorem V.4.1 in Borodin and Ibragimov, 1995.)

EXAMPLE 3.1 (single function). $\mathscr{F} = \{f\}$ where $f \in \mathrm{BI}_\beta$, and is majorised by another function $F \in \mathrm{BIL}_\beta$, in the sense that $|f(x)| \le F(x)$ for all $x \in \mathbb{R}$. This obtains trivially if f is itself in BIL_β (simply take $F(x) \coloneqq |f(x)|$), but is also consistent with $f \in \mathrm{BI}_\beta$ having finitely many discontinuities (at the points $\{a_k\}_{k=1}^{K}$, where $a_k < a_{k+1}$), and being Lipschitz continuous on $(-\infty, a_1) \cup [a_K, \infty)$; all that is really necessary here is for f to have one-sided Lipschitz approximants. Importantly, this includes the case where $f(x) = \mathbf{1}\{x \in I\}$ for any bounded interval I.

EXAMPLE 3.2 (parametric family). $\mathscr{F} = \{g(x,\theta) \mid \theta \in \Theta\} \subset \mathrm{BIL}_\beta$, where Θ is compact, and there exists a $\tau \in (0,1]$ and a $\dot{g} \in \mathrm{BIL}_\beta$ such that

$$|g(x,\theta) - g(x,\theta')| \leq \dot{g}(x)\|\theta - \theta'\|^\tau$$

for all $\theta, \theta' \in \Theta$.

EXAMPLE 3.3 (smooth functions). $\mathscr{F} = \{f \in C^\tau(\mathbb{R}) \mid |f| \leq F\}$, where $F \in \mathrm{BIL}_\beta$ and

$$C_L^\tau(\mathbb{R}) \coloneqq \{f \in \mathrm{BI} \mid \exists C_f < L \text{ s.t. } |f(x) - f(x')| \leq C_f|x - x'|^\tau \ \forall x, x' \in \mathbb{R}\}$$

for some $\tau \in (0,1]$ and $L < \infty$.

4. Outline of proof and auxiliary results.

4.1. *Outline of proof.* The principal relationships between the results in this paper are summarised in Figure 1. The proof of Theorem 3.1, depicted in the top half of the figure, proceeds as follows. To reduce the difficulties arising by the randomness of $h = h_n$, we decompose

$$\mathcal{L}_n^f(a,h) = \mathcal{L}_n^\varphi(a)\int_{\mathbb{R}} f + \left[\mathcal{L}_n^f(a,h) - \mathcal{L}_n^\varphi(a)\int_{\mathbb{R}} f\right] \tag{4.1}$$

where

$$\varphi(x) \coloneqq (1 - |x|)\mathbf{1}\{|x| \leq 1\}. \tag{4.2}$$

denotes the triangular kernel function, and $\mathcal{L}_n^\varphi(a) \coloneqq \mathcal{L}_n^\varphi(a,1)$. (This choice of φ is made purely for convenience; any compactly supported Lipschitz function would serve our purposes equally well here.) It thus suffices to show that $\mathcal{L}_n^\varphi \rightsquigarrow \mathcal{L}$ in $\ell^\infty(\mathbb{R})$, and that the bracketed term on the right side of (4.1) is uniformly negligible over $(f,h) \in \mathscr{F} \times \mathscr{H}_n$.

In view of Remark 2.5 above, the finite-dimensional distributions of $\mathcal{L}_n^\varphi$ converge to those of $\mathcal{L}$. The asymptotic tightness of $\mathcal{L}_n^\varphi$ will follow from the bound on the spatial increments

$$\mathcal{L}_n^\varphi(a_1) - \mathcal{L}_n^\varphi(a_2) = \frac{1}{e_n}\sum_{t=1}^{n}[\varphi(x_t - d_n a_1) - \varphi(x_t - d_n a_2)] =: \frac{1}{e_n}\sum_{t=1}^{n} g_1(x_t),$$

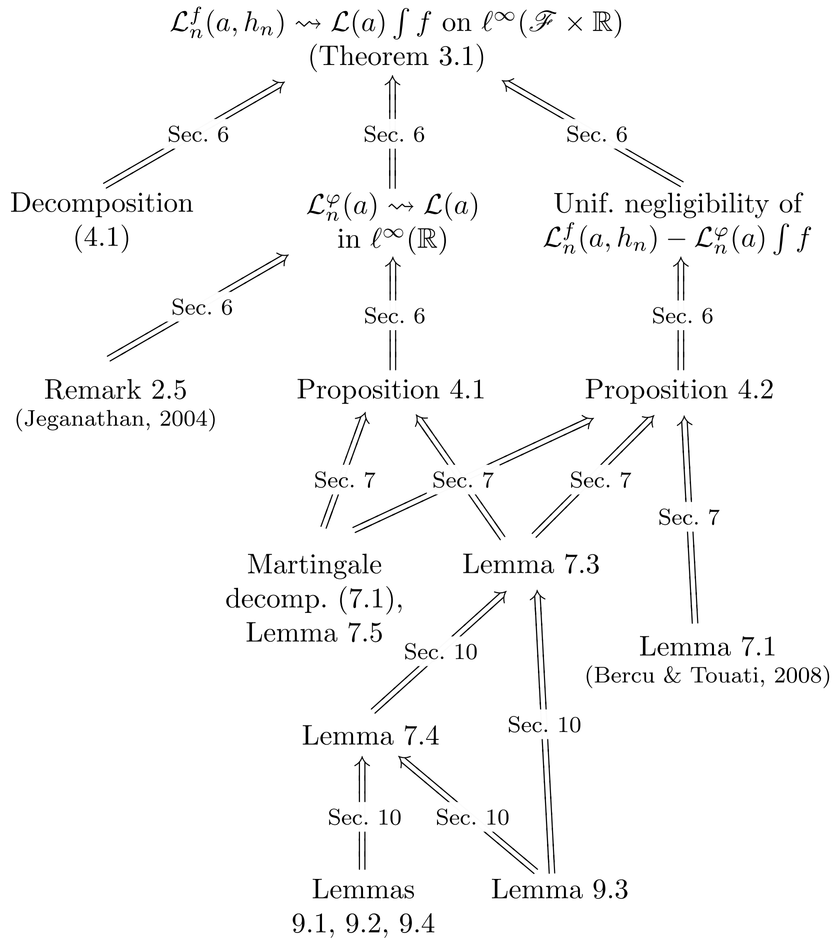

FIGURE 1. *Outline of proofs*

given in Proposition 4.1 below. The bracketed term on the right side of (4.1) may be written as

$$\frac{1}{e_n}\sum_{t=1}^{n}\left[\frac{1}{h}f\left(\frac{x_t-d_na}{h}\right)-\varphi(x_t-d_na)\int_{\mathbb{R}}f\right]=:\frac{1}{e_n}\sum_{t=1}^{n}g_2(x_t) \tag{4.3}$$

Control of (4.3) over progressively denser subsets of $\mathscr{F}\times\mathscr{H}_n$ is provided by Proposition 4.2 below; the conjunction of a bracketing argument and the continuity of the brackets suffices to extend this to the entirety of $\mathscr{F}\times\mathscr{H}_n$.

By construction, both $\int g_1 = 0$ and $\int g_2 = 0$. The proofs of Propositions 4.1 and 4.2 may therefore be approached in a unified way, through the analysis of sums of the form

$$\mathcal{S}_n g \coloneqq \sum_{t=1}^{n} g(x_t) \tag{4.4}$$

where g ranges over a class $\mathscr{G}$, all members of which have the property that $\int g = 0$. Such functions are termed *zero energy* functions (Wang and Phillips, 2011); we shall correspondingly term $\{\mathcal{S}_n g\}_{g\in\mathscr{G}}$ a *zero energy process*. Such processes are 'centred' in the sense that $e_n^{-1/2}\mathcal{S}_n g$ converges weakly to a mixed Gaussian variate (Jeganathan, 2008, Thm. 5); whereas $e_n^{-1}\mathcal{S}_n g \rightsquigarrow \mathcal{L}(0)\int g$ if $\int g \neq 0$.

(4.4) will be handled by decomposing $\mathcal{S}_n g$ as

$$\mathcal{S}_n g = \sum_{k=0}^{n-1} \mathcal{M}_{nk} g + \mathcal{N}_n g$$

where each $\mathcal{M}_{nk} g$ is a martingale: see (7.4) below. We provide order estimates for the sums of squares and conditional variances of the $\mathcal{M}_{nk} g$'s (Lemma 7.3); by an application of either Burkholder's inequality, or a tail bound due to Bercu and Touati (2008), these translate into estimates for the $\mathcal{M}_{nk} g$'s themselves. Propositions 4.1 and 4.2 then follow by standard arguments.

4.2. *Key auxiliary results.* To state these, we introduce the quantity

$$\|f\|_{[\beta]} \coloneqq \inf\{c \in \mathbb{R}_+ \mid |\hat{f}(\lambda)| \leq c|\lambda|^{\beta},\ \forall \lambda \in \mathbb{R}\} \tag{4.5}$$

for $f \in \mathrm{BI}$, $\beta \in (0,1]$, and $\hat{f}(\lambda) \coloneqq \int \mathrm{e}^{\mathrm{i}\lambda x} f(x)\,\mathrm{d}x$. It is easily verified that $\|f\|_{[\beta]}$ is indeed a norm on the space $\mathrm{BI}_{[\beta]} \coloneqq \{f \in \mathrm{BI} \mid \|f\|_{[\beta]} < \infty\}$ (modulo equality almost everywhere). Some useful properties of $\|f\|_{[\beta]}$ are collected in Lemma 9.1 below; in particular, it is shown that $\mathrm{BI}_{[\beta]}$ contains all $f \in \mathrm{BI}_\beta$ for which $\int f = 0$. Define

$$\overline{\beta}_H \coloneqq \frac{1-H}{2H} \wedge 1, \tag{4.6}$$

noting that $\overline{\beta}_H \in (0,1]$ for all $H \in (0,1)$, and let $\|\cdot\|_{\tau_{2/3}}$ denote the Orlicz norm associated to the convex and increasing function

$$\tau_{2/3}(x) \coloneqq \begin{cases} x(e-1) & \text{if } x \in [0,1], \\ \mathrm{e}^{x^{2/3}} - 1 & \text{if } x \in (1,\infty). \end{cases} \tag{4.7}$$

(See van der Vaart and Wellner, 1996, p. 95 for the definition of an Orlicz norm.) A bound on the spatial increments of $\mathcal{L}_n^{\varphi}$ is given by

PROPOSITION 4.1. *For every $\beta \in (0, \overline{\beta}_H)$, there exists $C_\beta < \infty$ such that*

$$\sup_{a_1,a_2\in\mathbb{R}} \|\mathcal{L}_n^\varphi(a_1) - \mathcal{L}_n^\varphi(a_2)\|_{\tau_{2/3}} \leq C_\beta |a_1 - a_2|^\beta.$$

The next result shall be applied to prove that the recentred sums (4.3) are uniformly negligible. Since the order estimate given below is of interest in its own right – see Duffy (2015), for an example of how it may be used to determine the uniform order of the first-order bias of a nonparametric regression estimator – we shall state it at a slightly higher level of generality than is needed for our purposes here. For $\mathscr{F} \subset \mathrm{BI}_{[\beta]}$, define

$$\delta_n(\beta, \mathscr{F}) := \|\mathscr{F}\|_\infty + e_n^{1/2}(\|\mathscr{F}\|_1 + \|\mathscr{F}\|_2) + e_n d_n^{-\beta}\|\mathscr{F}\|_{[\beta]} \tag{4.8}$$

where $\|\mathscr{F}\| := \sup_{f\in\mathscr{F}}\|f\|$.

PROPOSITION 4.2. *Suppose $\beta \in (0, \overline{\beta}_H)$ and $\mathscr{F}_n \subset \mathrm{BI}_{[\beta]}$ with $\#\mathscr{F}_n \lesssim n^C$. Then*

$$\max_{f\in\mathscr{F}_n} |\mathcal{S}_n f| \lesssim_p \delta_n(\beta, \mathscr{F}_n) \log n. \tag{4.9}$$

If also $\|\mathscr{F}_n\|_1 \lesssim 1$, $\|\mathscr{F}_n\|_{[\beta]} = o(d_n^\beta)$ and $\|\mathscr{F}_n\|_\infty = o(e_n \log^{-2} n)$, then

$$\max_{f\in\mathscr{F}_n} |\mathcal{S}_n f| = o_p(e_n).$$

REMARK 4.1. As is clear from the proof, if $\beta \in [\overline{\beta}_H, 1]$ then (4.9) holds in a modified form, with $e_n d_n^{-\beta}\|\mathscr{F}\|_{[\beta]}$ in (4.8) being replaced by

$$\left[\sum_{k=1}^{n-1} d_k^{-(1+\beta)} + e_n^{1/2}\sum_{k=1}^{n-1} k^{-1/2} d_k^{-(1+2\beta)/2}\right]\|\mathscr{F}\|_{[\beta]}.$$

The proofs of Propositions 4.1 and 4.2 are given in Section 7 below.

5. A preliminary application to nonparametric regression. Suppose that we observe $\{(y_t, x_t)\}_{t=1}^n$ generated according to the nonlinear cointegrating regression model

$$y_t = m_0(x_t) + u_t$$

where $\{u_t\}$ is some weakly dependent disturbance process. As shown in Wang and Phillips (2009b), under suitable smoothness conditions the unknown function m_0 may be consistently estimated, at each fixed $x \in \mathbb{R}$, by the Nadaraya-Watson estimator

$$\begin{aligned}\hat{m}(x) &= \frac{\sum_{t=1}^n K_{h_n}(x_t - x) y_t}{\sum_{t=1}^n K_{h_n}(x_t - x)} \\ &= m_0(x) + \frac{\sum_{t=1}^n K_{h_n}(x_t - x)[(m_0(x_t) - m_0(x)) + u_t]}{\sum_{t=1}^n K_{h_n}(x_t - x)} \end{aligned} \tag{5.1}$$

where $K_h(u) \coloneqq h^{-1}K(h^{-1}u)$, and $K \in \mathrm{BI}$ is a positive, mean-zero kernel with $\int_{\mathbb{R}} K = 1$.

Now consider the problem of determining the rate at which $\hat{m}$ converges uniformly to m_0. As a first step, we would need to obtain the uniform rate of divergence of the denominator in (5.1); it is precisely this rate that the preceding results allow us to compute. By Theorem 3.1,

$$\frac{1}{e_n} \sum_{t=1}^n K_{h_n}(x_t - d_n a) \rightsquigarrow \mathcal{L}(a) \tag{5.2}$$

in $\ell_\infty(\mathbb{R})$, provided $\mathbb{P}\{h_n \in \mathscr{H}_n\} \to 1$, and K satisfies the requirements of Example 3.1 (which seems broad enough to cover any reasonable choice of K). Since $a \mapsto \mathcal{L}(a)$ is random – being dependent on the trajectory of the limiting process X – we now face the problem of identifying a sequence of sets on which the left side of (5.2) can be uniformly bounded away from zero. A natural candidate is

$$A_n^\epsilon \coloneqq \{x \in \mathbb{R} \mid \mathcal{L}_n(d_n^{-1}x) \geq \epsilon\},$$

where $\epsilon > 0$. $\mathcal{L}_n$ is trivially bounded away from zero on this set, whence

$$\sup_{x \in A_n^\epsilon} \left[\sum_{t=1}^n K_{h_n}(x_t - x)\right]^{-1} \lesssim_p e_n^{-1}. \tag{5.3}$$

More significantly, for any given $\delta > 0$, we may choose $\epsilon > 0$ such that

$$\limsup_{n \to \infty} \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^n \mathbf{1}\{x_t \notin A_n^\epsilon\} \geq \delta\right\} \leq \delta; \tag{5.4}$$

see Section 8 for the proof. That is, $\epsilon > 0$ may be chosen such that A_n^ϵ contains as large a fraction of the observed trajectory $\{x_t\}_{t=1}^n$ as is desired, in the limit as $n \to \infty$. (Were we to allow $\epsilon = \epsilon_n \to 0$, we could permit $\delta = \delta_n \to 0$ here, but the order of (5.3) would necessarily be increased.)

Note that the sample-dependence of A_n^ϵ is necessary for it fulfil two roles here, by being both 'small' enough for (5.3) to hold, but also 'large' enough to be consistent with (5.4). If A_n^ϵ were replaced by a sequence of deterministic intervals (or sets, more generally), then the maintenance of (5.3) would necessarily come at the cost of violating (5.4). For example, when x_t is a random walk with finite variance ($\alpha = 2$), the 'widest' sequence of intervals $[-a_n, a_n]$ for which (5.3) holds is one for which $a_n = o(n^{1/2})$: but in consequence, the *fraction* of any trajectory $\{x_t\}_{t=1}^n$ falling within such an interval will converge to 0, as $n \to \infty$ (see Remark 2.8 in Duffy, 2015).

In this respect, the availability of Theorem 3.1 allows us to improve upon the analysis provided in an earlier paper by Chan and Wang (2014) – and, in the random walk case, that of Gao, Kanaya, Li, and Tjøstheim (2015) – who obtain uniform convergence rates for $\hat{m}_n$ on precisely such intervals (see Duffy, 2015 for further details). We expect that it would also play a similarly important role in the derivation of uniform convergence rates for series regression estimators in this setting, by ensuring the eigenvalues of the design matrix diverge at an appropriate rate, when attempting to estimate m_0 on a sequence of domains that contains most of the observed $\{x_t\}_{t=1}^n$.

6. Proof of Theorem 3.1. We shall prove only part (i) of Theorem 3.1 here; the relatively minor modifications required for the proof of part (ii) are detailed in Section B of the Supplement. Let $M < \infty$ be given; it suffices to prove that (3.1) holds in $\ell_\infty[-M, M]$. To simplify the exposition, we shall require that $h_n \in \mathscr{H}_n$ always; the proof in the general case (where this occurs w.p.a.1) requires no new ideas. The proof involves three steps:

(i) show that $\mathcal{L}_n^\varphi(a) \rightsquigarrow \mathcal{L}(a)$, using Proposition 4.1;
(ii) deduce $\mathcal{L}_n^f(a, h_n) \rightsquigarrow \mathcal{L}(a)\int_\mathbb{R} f$ for $f \in \mathrm{BIL}_\beta$, using a recentring, Proposition 4.2 and the Lipschitz continuity of f;
(iii) extend this to all $f \in \mathscr{F} \subset \mathrm{BI}$, where $\mathscr{F}$ satisfies Assumption 3, via a bracketing argument.

(i). Let φ be the triangular kernel function, as defined in (4.2) above, and set $\beta_0 \coloneqq \overline{\beta}_H/2$. Recall that $\mathcal{L}_n^f(a) \coloneqq \mathcal{L}_n^f(a, 1)$. By Proposition 4.1 and Theorem 2.2.4 in van der Vaart and Wellner (1996),

$$\left\| \sup_{\{a,a' \in M \mid |a-a'| \leq \delta\}} |\mathcal{L}_n^\varphi(a') - \mathcal{L}_n^\varphi(a)| \right\|_1 \lesssim \int_0^\delta \log^{3/2}(M\epsilon^{-1/\beta_0})\,\mathrm{d}\epsilon + \delta \log^{3/2}(M\delta^{-2/\beta_0}) \lesssim C_M \delta^{1/2}$$

whence $\mathcal{L}_n^{\varphi}$ is tight in $\ell_\infty[-M,M]$. Thus, in view of Remark 2.5,

$$\mathcal{L}_n^{\varphi}(a) \rightsquigarrow \mathcal{L}(a) \tag{6.1}$$

in $\ell_\infty[-M,M]$ (see van der Vaart and Wellner, 1996, Example 2.2.12).

(ii). Now let $f \in \mathrm{BIL}_\beta$; we may without loss of generality take f to be bounded by unity, with a Lipschitz constant of unity. For the subsequent argument, it will be more convenient to work with the inverse bandwidth $b := h^{-1}$. Define

$$\mathscr{B}_n := \{h^{-1} \mid h \in \mathscr{H}_n\} = [\underline{b}_n, \bar{b}_n] := [\bar{h}_n^{-1}, \underline{h}_n^{-1}].$$

and let $f_{(a,b)}(x) := bf[b(x-d_n a)]$, for $(a,b) \in \mathbb{R}\times\mathbb{R}_+$. Take $C_n := [-n^\gamma, n^\gamma] \times \mathscr{B}_n$, let $\mathscr{C}_n \subset C_n$ be a lattice of mesh $n^{-\delta}$, and let $p_n(a,b)$ denote the projection of $(a,b) \in C_n$ onto a nearest neighbour in $\mathscr{C}_n$ (with some tie-breaking rule). The following is a straightforward consequence of the Lipschitz continuity of f (see Section C of the Supplement for the proof).

Lemma 6.1. *For every $\gamma \geq 1$, there exists $\delta > 0$ such that*

$$\sup_{(a,b)\in C_n} \frac{1}{e_n} \sum_{t=1}^{n} |f_{(a,b)}(x_t) - f_{p_n(a,b)}(x_t)| = o_p(1).$$

By taking $\gamma \geq 1$, we may ensure that $C_n \supset [-M,M] \times \mathscr{B}_n$, for all n sufficiently large. Thus, for $\varphi_{(a)} := \varphi_{(a,1)}$ and $\mu_f := \int_{\mathbb{R}} f$,

$$\begin{aligned}
&\sup_{(a,b)\in[-M,M]\times\mathscr{B}_n} |\mathcal{L}_n^f(a, b^{-1}) - \mu_f \mathcal{L}_n^{\varphi}(a)| \qquad (6.2)\\
&\qquad \leq \sup_{(a,b)\in C_n} \frac{1}{e_n} \left| \sum_{t=1}^{n} [f_{(a,b)}(x_t) - \mu_f \varphi_{(a)}(x_t)] \right| \\
&\qquad \leq \sup_{(a,b)\in \mathscr{C}_n} \frac{1}{e_n} \left| \sum_{t=1}^{n} [f_{(a,b)}(x_t) - \mu_f \varphi_{(a)}(x_t)] \right| + o_p(1) \\
&\qquad = \sup_{g\in\mathscr{G}_n} \frac{1}{e_n} \left| \sum_{t=1}^{n} g(x_t) \right| + o_p(1), \qquad (6.3)
\end{aligned}$$

by Lemma 6.1, and we have defined $\mathscr{G}_n := \{f_{(a,b)} - \mu_f \varphi_{(a)} \mid (a,b) \in \mathscr{C}_n\}$. It is readily verified that $\|g\|_1 = 1$, $\#\mathscr{G}_n = \#\mathscr{C}_n \lesssim n^{1+\gamma+2\delta}$, and using Lemma 9.1(ii)

$$\sup_{g\in\mathscr{G}_n} \|g\|_{[\beta]} \lesssim \underline{b}_n^{-\beta} = o(d_n^\beta) \qquad \sup_{g\in\mathscr{G}_n} \|g\|_\infty \leq \bar{b}_n \lesssim e_n \log^{-2} n.$$

Thus $\mathscr{G}_n$ satisfies the requirements of Proposition 4.2, whence (6.3) is $o_p(1)$. Hence, in view of (6.1),

$$\mathcal{L}_n^f(a, h_n) \rightsquigarrow \mathcal{L}(a) \int_{\mathbb{R}} f \tag{6.4}$$

in $\ell_\infty[-M, M]$, for every $f \in \mathrm{BIL}_\beta$.

(iii). Finally, for $f \in \mathrm{BI}$ define the centred process

$$\nu_n(f, a) \coloneqq \mathcal{L}_n^f(a, h_n) - \mathcal{L}_n^\varphi(a) \int_{\mathbb{R}} f.$$

For a given $\epsilon > 0$, let $\{l_k, u_k\}_{k=1}^K$ denote a collection of *continuous* L^1 brackets that cover $\mathscr{F}$, with $\|u_k - l_k\|_1 < \epsilon$; the existence of these is guaranteed by Assumption 3. We first note (see Section C of the Supplement for the proof)

Lemma 6.2. *Under Assumption 3, the brackets $\{l_k, u_k\}_{k=1}^K$ can be chosen so as to lie in* BIL_β.

For each $f \in \mathscr{F}$, there exists a $k \in \{1, \ldots, K\}$ such that $l_k \leq f \leq u_k$, $\int_{\mathbb{R}}(u_k - f) < \epsilon$, and

$$\begin{aligned}\nu_n(f, a) &\leq \frac{1}{e_n} \sum_{t=1}^n \left[\frac{1}{h_n} u_k\left(\frac{x_t - d_n x}{h_n}\right) - \varphi(x_t) \int_{\mathbb{R}} f \right] \\ &\leq \nu_n(u_k, a) + \mathcal{L}_n^\varphi(a) \int_{\mathbb{R}} (u_k - f).\end{aligned}$$

Taking suprema,

$$\begin{aligned}\sup_{(f,a) \in \mathscr{F} \times [-M,M]} \nu_n(f, a) &\leq \max_{1 \leq k \leq K} \sup_{a \in [-M,M]} \nu_n(u_k, a) + \epsilon \sup_{a \in [-M,M]} \mathcal{L}_n^\varphi(a) \\ &= \epsilon \sup_{a \in [-M,M]} \mathcal{L}_n^\varphi(a) + o_p(1)\end{aligned}$$

with the second equality following by (6.4), since we may take $u_k \in \mathrm{BIL}_\beta$ by Lemma 6.2. Applying a strictly analogous argument to the lower bracketing functions, l_k, we deduce that

$$\sup_{(f,a) \in \mathscr{F} \times [-M,M]} |\nu_n(f, a)| \leq \epsilon \sup_{a \in [-M,M]} |\mathcal{L}_n^\varphi(a)| + o_p(1) = o_p(1) \tag{6.5}$$

whence (3.1) holds in $\ell_\infty[-M, M]$, in view of (6.1).

7. Controlling the zero energy process. The proofs of Propositions 4.1 and 4.2 rely on a telescoping martingale decomposition similar to that used to prove maximal inequalities for mixingales (for a textbook exposition, see e.g. Davidson, 1994, Sec. 16.2–16.3), which reduces $\mathcal{S}_n f$ to a sum of martingale components. In order to pass from control over each of these components to an order estimate for $\mathcal{S}_n f$ itself, we shall need the following results, the first of which is a straightforward consequence of Theorem 2.1 in Bercu and Touati (2008), and the second of which is well known. For a martingale $M \coloneqq \{M_t\}_{t=0}^n$ with associated filtration $\mathcal{G} \coloneqq \{\mathcal{G}_t\}_{t=0}^n$, define

$$[M] \coloneqq \sum_{t=1}^{n}(M_t - M_{t-1})^2 \qquad \langle M\rangle \coloneqq \sum_{t=1}^{n}\mathbb{E}[(M_t - M_{t-1})^2 \mid \mathcal{G}_{t-1}]. \tag{7.1}$$

We say that M is *initialised at zero* if $M_0 = 0$. Let $\|\cdot\|_{\tau_1}$ denote the Orlicz norm associated to $\tau_1(x) \coloneqq \mathrm{e}^x - 1$.

LEMMA 7.1. *Let $\{\Theta_n\}$ denote a sequence of index sets, and $\{K_n\}$ a real sequence such that $\#\Theta_n + K_n \lesssim n^C$. Suppose that for each $n \in \mathbb{N}$, $k \in \{1,\ldots,K_n\}$ and $\theta \in \Theta_n$, $M_{nk}(\theta)$ is a martingale, initialised at zero, for which*

$$\omega_{nk}^2 \coloneqq \max_{\theta\in\Theta_n}\{\|[M_{nk}(\theta)]\|_{\tau_1} \vee \|\langle M_{nk}(\theta)\rangle\|_{\tau_1}\} < \infty. \tag{7.2}$$

Then

$$\max_{\theta\in\Theta_n}\left|\sum_{k=1}^{K_n} M_{nk}(\theta)\right| \lesssim_p \left(\sum_{k=1}^{K_n}\omega_{nk}\right)\log n.$$

LEMMA 7.2. *Let Z be a random variable. Then*

(i) *$\|Z\|_p \lesssim p!^{1/p}\sigma$ for all $p \in \mathbb{N}$, if and only if $\|Z\|_{\tau_1} \lesssim \sigma$;*
(ii) *$\|Z\|_{2p} \lesssim (3p)!^{1/2p}\sigma$ for all $p \in \mathbb{N}$, if and only if $\|Z\|_{\tau_{2/3}} \lesssim \sigma$.*

The proofs of Lemmas 7.1 and 7.2 appear in Section D of the Supplement.

7.1. *The martingale decomposition.* For a fixed $f \in \mathrm{BI}_{[\beta]}$, it follows from Lemma 9.3(ii) below and the reverse martingale convergence theorem (Hall and Heyde, 1980, Thm 2.6) that

$$\|\mathbb{E}_t f(x_{t+k})\|_\infty \lesssim d_k^{-(1+\beta)} \to 0 \qquad \mathbb{E}_{t-k} f(x_t) \xrightarrow{p} \mathbb{E} f(x_t) \neq 0$$

for each $t \geq 0$ as $k \to \infty$; here $\mathbb{E}_t f(x_{t+k}) \coloneqq \mathbb{E}[f(x_{t+k}) \mid \mathcal{F}^t_{-\infty}]$, for $\mathcal{F}^t_s \coloneqq \sigma(\{\epsilon_r\}^t_{r=s})$. Because $\{f(x_t)\}$ is asymptotically unpredictable only in the 'forwards' direction, we truncate the 'usual' decomposition at $t = 0$, writing

$$f(x_t) = \sum_{k=1}^{t} [\mathbb{E}_{t-k+1} f(x_t) - \mathbb{E}_{t-k} f(x_t)] + \mathbb{E}_0 f(x_t).$$

Performing this for each $1 \leq t \leq n$ gives

$$\begin{aligned}
\sum_{t=1}^{n} f(x_t) = {} & \\
& \mathbb{E}_0 f(x_1) \;\; + [f(x_1) - \mathbb{E}_0 f(x_1)] \\
& + \mathbb{E}_0 f(x_2) \;\; + [f(x_2) - \mathbb{E}_1 f(x_2)] \quad + [\mathbb{E}_1 f(x_2) - \mathbb{E}_0 f(x_2)] \\
& + \cdots \\
& + \mathbb{E}_0 f(x_n) \;\; + [f(x_n) - \mathbb{E}_{n-1} f(x_n)] \;\; + [\mathbb{E}_{n-1} f(x_n) - \mathbb{E}_{n-2} f(x_n)] \\
& \qquad\qquad + \cdots + [\mathbb{E}_1 f(x_n) - \mathbb{E}_0 f(x_n)].
\end{aligned}$$

Defining

$$\xi_{kt} f \coloneqq \mathbb{E}_t f(x_{t+k}) - \mathbb{E}_{t-1} f(x_{t+k}) \tag{7.3}$$

and collecting terms appearing in the same 'column' of the preceding display, we thus obtain

$$\begin{aligned}
\mathcal{S}_n f = \sum_{t=1}^{n} f(x_t) &= \sum_{t=1}^{n} \mathbb{E}_0 f(x_t) + \sum_{k=0}^{n-1} \sum_{t=k+1}^{n} [\mathbb{E}_{t-k} f(x_t) - \mathbb{E}_{t-k-1} f(x_t)] \\
&= \sum_{t=1}^{n} \mathbb{E}_0 f(x_t) + \sum_{k=0}^{n-1} \sum_{t=1}^{n-k} [\mathbb{E}_t f(x_{t+k}) - \mathbb{E}_{t-1} f(x_{t+k})] \\
&= \sum_{t=1}^{n} \mathbb{E}_0 f(x_t) + \sum_{k=0}^{n-1} \sum_{t=1}^{n-k} \xi_{kt} f \\
&= \mathcal{N}_n f + \sum_{k=0}^{n-1} \mathcal{M}_{nk} f \qquad (7.4)
\end{aligned}$$

where

$$\mathcal{N}_n f \coloneqq \sum_{t=1}^{n} \mathbb{E}_0 f(x_t) \qquad\qquad \mathcal{M}_{nk} f \coloneqq \sum_{t=1}^{n-k} \xi_{kt} f.$$

A bound for $\|\mathcal{N}_n f\|_\infty$ is provided by Lemma 9.3(ii) below. $\{\xi_{kt} f, \mathcal{F}_{-\infty}^t\}_{t=1}^{n-k}$, by construction, forms a martingale difference sequence for each k, and so control over each of the martingale 'pieces' $\mathcal{M}_{nk} f$ will follow from control over

$$\mathcal{U}_{nk} f \coloneqq [\mathcal{M}_{nk} f] = \sum_{t=1}^{n-k} \xi_{kt}^2 f \qquad \mathcal{V}_{nk} f \coloneqq \langle \mathcal{M}_{nk} f \rangle = \sum_{t=1}^{n-k} \mathbb{E}_{t-1} \xi_{kt}^2 f,$$

in combination with either Burkholder's inequality (Hall and Heyde, 1980, Thm 2.10) or Lemma 7.1 above, as appropriate.

7.2. *Proofs of Propositions 4.1 and 4.2.* Define

$$\varsigma_n(\beta, f) \coloneqq \|f\|_\infty + \|f\|_1 + \|f\|_{[\beta]} \sum_{t=1}^{n} d_t^{-(1+\beta)}$$

and

$$\sigma_{nk}^2(\beta, f) \coloneqq \begin{cases} \|f\|_\infty^2 + \|f\|_2^2 e_n & \text{if } k \in \{0, \dots, k_0\} \\ e_n \left[k^{-1} d_k^{-(1+2\beta)} \|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k} \|f\|_1^2 \right] & \text{if } k \in \{k_0 + 1, \dots, n-1\}. \end{cases}$$

The following provides the requisite control over the components of (7.4).

Lemma 7.3. *For any $\beta \in [0, 1]$,*

$$\|\mathcal{N}_n f\|_\infty \lesssim \varsigma_n(\beta, f), \tag{7.5}$$

and for all $0 \le k \le n-1$,

$$\|\mathcal{U}_{nk} f\|_{\tau_1} \vee \|\mathcal{V}_{nk} f\|_{\tau_1} \lesssim \sigma_{nk}^2(\beta, f). \tag{7.6}$$

The proof of (7.6), in turn, relies upon the following.

Lemma 7.4. *For every $k \in \{0, \dots, n-1\}$, $t \in \{1, \dots, n-k\}$ and $\beta \in (0, 1]$*

$$\|\xi_{kt}^2 f\|_\infty + \sum_{s=1}^{n-k-t} \|\mathbb{E}_t \xi_{k,t+s}^2 f\|_\infty \lesssim \sigma_{nk}^2(\beta, f).$$

The proofs of these results are deferred to Sections 9 and 10. We shall also need the following, for which we recall the definition of $\delta_n(\beta, \mathscr{G})$ given in (4.8) above.

LEMMA 7.5. *If $\beta \in (0, \overline{\beta}_H)$ and $\mathscr{G} \subset \mathrm{BI}_{[\beta]}$, then there exists a $C_\beta < \infty$ such that*

$$\sup_{f \in \mathscr{G}} \varsigma_n(\beta, f) + \sum_{k=0}^{n-1} \sup_{f \in \mathscr{G}} \sigma_{nk}(\beta, f) \leq C_\beta \delta_n(\beta, \mathscr{G}).$$

The proof appears in Section D of the Supplement. We now turn to the

PROOF OF PROPOSITION 4.1. Let $g \in \mathrm{BI}_{[\beta]}$. Burkholder's inequality, and Lemmas 7.2(i) and 7.3 and give

$$\|\mathcal{M}_{nk} g\|_{2p} \leq b_{2p}^{1/2p} \|\mathcal{U}_{nk} g\|_p^{1/2} \lesssim (b_{2p} \cdot p!)^{1/2p} \sigma_{nk}(\beta, g) \lesssim (3p)!^{1/2p} \sigma_{nk}(\beta, g)$$

for every $p \in \mathbb{N}$, where b_{2p} depends on p in the manner prescribed by Burkholder's inequality. Hence $\|\mathcal{M}_{nk} g\|_{\tau_{2/3}} \lesssim \sigma_{nk}(\beta, g)$ by Lemma 7.2(ii). Thence by (7.4), Lemma 7.3 and Lemma 7.5 (taking $\mathscr{G} = \{g\}$)

$$\|\mathcal{S}_n g\|_{\tau_{2/3}} \leq \|\mathcal{N}_n g\|_\infty + \sum_{k=0}^{n-1} \|\mathcal{M}_{nk} g\|_{\tau_{2/3}} \leq C \delta_n(\beta, g) \tag{7.7}$$

for some $C < \infty$ depending on β.

For $a_1, a_2 \in \mathbb{R}$, set $\Delta \coloneqq |a_1 - a_2|$ and define

$$\varphi_{[a_1, a_2]}(x) \coloneqq \varphi(x - d_n a_1) - \varphi(x - d_n a_2).$$

Let $\beta \in (0, \overline{\beta}_H)$. Since φ is bounded and Lipschitz,

$$\|\varphi_{[a_1, a_2]}\|_\infty \leq (d_n \Delta) \wedge 1 \leq d_n^\beta \Delta^\beta;$$

and further, since φ is bounded and compactly supported,

$$\|\varphi_{[a_1, a_2]}\|_p \leq 2 \|\varphi_{[a_1, a_2]}\|_\infty^\beta \|\varphi^{1-\beta}\|_p \lesssim d_n^\beta \Delta^\beta,$$

for $p \in \{1, 2\}$. Finally, by Lemma 9.1(iii),

$$\|\varphi_{[a_1, a_2]}\|_{[\beta]} \lesssim d_n^\beta \Delta^\beta.$$

Thence by (7.7) and the definition of $\delta_n(\beta, \mathscr{F})$,

$$\begin{aligned}
\|\mathcal{L}_n^\varphi(a_1) - \mathcal{L}_n^\varphi(a_2)\|_{\tau_{2/3}} &= e_n^{-1} \|\mathcal{S}_n \varphi_{[a_1, a_2]}\|_{\tau_{2/3}} \\
&\leq C(e_n^{-1/2} \cdot d_n^\beta \Delta^\beta + d_n^{-\beta} \cdot d_n^\beta \Delta^\beta) \\
&\lesssim C \Delta^\beta,
\end{aligned}$$

for some C depending on β; here we have used the fact that since $\beta < \overline{\beta}_H \leq \frac{1-H}{2H}$, $\{e_n^{-1/2} d_n^\beta\}$ is regularly varying with index $H(\beta - \frac{1-H}{2H}) < 0$. $\square$

Proof of Proposition 4.2. In view of Lemmas 7.3 and 7.5, we have

$$\max_{f\in\mathscr{F}_n}|\mathcal{N}_n f| \leq \max_{f\in\mathscr{F}_n}\varsigma_n(\beta,f) \lesssim_p \delta_n(\beta,\mathscr{F}_n),$$

and by an application of Lemma 7.1,

$$\max_{f\in\mathscr{F}_n}\left|\sum_{k=0}^{n-1}\mathcal{M}_{nk}f\right| \lesssim_p \delta_n(\beta,\mathscr{F}_n)\log n.$$

Thus (4.9) follows from (7.4).

For the second part of the result, note that under the stated conditions on $\mathscr{F}_n$,

$$\|\mathscr{F}_n\|_2 \leq \|\mathscr{F}_n\|_\infty^{1/2}\|\mathscr{F}_n\|_1^{1/2} = o[e_n^{1/2}\log^{-1} n]$$

whence

$$e_n^{-1}\delta_n(\beta,\mathscr{F}_n) = o_p(\log^{-1} n) + d_n^{-\beta}o_p(d_n^\beta) = o_p(1)$$

whereupon the result follows by (4.9). □

8. Proof of (5.4). Let $\mu_n(a) \coloneqq \frac{1}{n}\sum_{t=1}^n \mathbf{1}\{d_n^{-1}x_t \leq a\}$ and $\mu(a) \coloneqq \int_{-\infty}^a \mathcal{L}(x)\,\mathrm{d}x$. It is shown in Section E of the Supplement that

$$\mu_n \rightsquigarrow \mu \tag{8.1}$$

in $\ell_\infty(\mathbb{R})$, jointly with the convergence in Theorem 3.1.

Let $T(x) \coloneqq \mathbf{1}\{x < \epsilon\}$ and $\mathcal{L}_n(a) \coloneqq \mathcal{L}_n^K(a,h_n)$. We first note that

$$\frac{1}{n}\sum_{t=1}^n \mathbf{1}\{x_t \notin A_n^\epsilon\} = \frac{1}{n}\sum_{t=1}^n \mathbf{1}\{\mathcal{L}_n(d_n^{-1}x_t) < \epsilon\} = \int_{\mathbb{R}} T(\mathcal{L}_n(a))\,\mathrm{d}\mu_n(a)$$

by definition of A_n^ϵ and μ_n. We shall now suppose that $\mathcal{L}_n \overset{\text{a.s.}}{\to} \mathcal{L}_n$ in $\ell_{\text{ucc}}(\mathbb{R})$, and $\mu_n \overset{\text{a.s.}}{\to} \mu$ in $\ell_\infty(\mathbb{R})$, as may be justified (in view of Theorem 3.1 and (8.1)) by Theorem 1.10.3 in van der Vaart and Wellner (1996); let $\Omega_0 \subset \Omega$ denote a set, having $\mathbb{P}\Omega_0 = 1$, on which this convergence occurs. Define

$$\overline{T}(x) = \begin{cases} 1 & \text{if } x \leq \epsilon \\ \epsilon^{-1}(2\epsilon - x) & \text{if } x \in (\epsilon, 2\epsilon) \\ 0 & \text{if } x \geq 2\epsilon. \end{cases}$$

Then, fixing an $\omega \in \Omega_0$,

$$\int_{\mathbb{R}} T(\mathcal{L}_n^\omega(a))\,\mathrm{d}\mu_n^\omega(a) \leq \int_{\mathbb{R}} \overline{T}(\mathcal{L}_n^\omega(a))\,\mathrm{d}\mu_n^\omega(a) = \int_{\mathbb{R}} F_n(a)\,\mathrm{d}\mu_n^\omega(a),$$

where $F_n(a) := (\overline{T} \circ \mathcal{L}_n^\omega)(a)$. Now let $[c,d]$ be chosen such that $\mu^\omega(d) - \mu^\omega(c) < \epsilon$. Since $\overline{T}$ is uniformly continuous, $F_n(a) \to F(a) := (\overline{T} \circ \mathcal{L}^\omega)(a)$ uniformly over $a \in [c,d]$, whence

$$\begin{aligned}\int_{\mathbb{R}} F_n(a)\,\mathrm{d}\mu_n^\omega(a) &\leq \int_{[c,d]^c} \mathrm{d}\mu_n(a) + \int_{[c,d]} F(a)\,\mathrm{d}\mu_n^\omega(a) + \sup_{a\in[c,d]} |F_n(a) - F(a)| \\ &\to \epsilon + \int_{[c,d]} F(a)\,\mathrm{d}\mu^\omega(a) \\ &\leq \epsilon + \int_{\mathbb{R}} F(a)\,\mathrm{d}\mu^\omega(a)\end{aligned}$$

where the convergence follows by the Portmanteau theorem (van der Vaart and Wellner, 1996, Thm. 1.3.4), since F is continuous.

Thus

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\left\{\frac{1}{n}\sum_{t=1}^{n} \mathbf{1}\{x_t \notin A_n^\epsilon\} \geq \delta\right\} &\leq \limsup_{n\to\infty} \mathbb{P}\left\{\int_{\mathbb{R}} \overline{T}(\mathcal{L}_n(a))\,\mathrm{d}\mu_n(a) \geq \delta\right\} \\ &\leq \mathbb{P}\left\{\epsilon + \int_{\mathbb{R}} \overline{T}(\mathcal{L}(a))\,\mathrm{d}\mu(a) \geq \delta\right\} \\ &\leq \mathbb{P}\left\{\epsilon + \int_{\mathbb{R}} \mathbf{1}\{\mathcal{L}(a) \leq 2\epsilon\}\mathcal{L}(a)\,\mathrm{d}a \geq \delta\right\}. \end{aligned} \tag{8.2}$$

where, noting that $\mathcal{L}$ is the density of μ, the final inequality follows from

$$\int_{\mathbb{R}} \overline{T}(\mathcal{L}(a))\,\mathrm{d}\mu(a) \leq \int_{\mathbb{R}} \mathbf{1}\{\mathcal{L}(a) \leq 2\epsilon\}\,\mathrm{d}\mu(a) = \int_{\mathbb{R}} \mathbf{1}\{\mathcal{L}(a) \leq 2\epsilon\}\mathcal{L}(a)\,\mathrm{d}a.$$

Finally,

$$\epsilon + \int_{\mathbb{R}} \mathbf{1}\{\mathcal{L}(a) \leq 2\epsilon\}\mathcal{L}(a)\,\mathrm{d}a \overset{\text{a.s.}}{\to} \int_{\mathbb{R}} \mathbf{1}\{\mathcal{L}(a) = 0\}\mathcal{L}(a)\,\mathrm{d}a = 0$$

as $\epsilon \to 0$, by dominated convergence, and so $\epsilon > 0$ may be chosen such that the right side of (8.2) is less than δ.

9. Results preliminary to the proofs of Lemmas 7.3 and 7.4. Our arguments shall rely heavily on the use of the inverse Fourier transform to analyse objects of the form $\mathbb{E}_t f(x_{t+k})$, similarly to Borodin and Ibragimov (1995), Jeganathan (2004, 2008) and Wang and Phillips (2009b, 2011). Provided that $f \in \mathrm{BI}$ and Y has an integrable characteristic function ψ_Y, the 'usual' inversion formula

$$\mathbb{E}f(y_0 + Y) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda y_0}\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda Y}\,\mathrm{d}\lambda \tag{9.1}$$

for $y_0 \in \mathbb{R}$, is still valid, even when $\hat{f}(\lambda) = \int f(x) \mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x$ is not integrable; these conditions will always be met whenever the inversion formula is required below. The following provides some useful bounds for $\hat{f}$.

LEMMA 9.1. *For every $f \in \mathrm{BI}$ and $\beta \in (0,1]$,*

(i) $|\hat{f}(\lambda)| \le (|\lambda|^{\beta} \|f\|_{[\beta]}) \wedge \|f\|_1$;

(ii) *if $\int f = 0$, then*

$$\|f\|_{[\beta]} \le 2^{1-\beta} \inf_{y\in\mathbb{R}} \int_{\mathbb{R}} |f(x-y)||x|^{\beta}\,\mathrm{d}x$$

and so $\mathrm{BI}_{[\beta]} \supseteq \{f \in \mathrm{BI}_{\beta} \mid \int f = 0\}$;

(iii) *if $f(x) \coloneqq g(x-a_1) - g(x-a_2)$ for some $a_1, a_2 \in \mathbb{R}$, then*

$$\|f\|_{[\beta]} \le 2^{1-\beta}|a_1 - a_2|^{\beta}\|g\|_1.$$

Let $\mathcal{F}_s^t \coloneqq \sigma(\{\epsilon_r\}_{r=s}^t)$, noting that $\mathcal{F}_{s_1}^{s_2} \perp\!\!\!\perp \mathcal{F}_{s_3}^{s_4}$ for $s_1 \le s_2 < s_3 \le s_4$. For $0 < s < t$, we shall have frequent recourse to the following decomposition,

$$\begin{aligned} x_t = \sum_{k=1}^{t} v_t = \sum_{k=1}^{t}\sum_{l=0}^{\infty} \phi_l \epsilon_{k-l} \\ =: x^*_{s-1,t} + \sum_{i=0}^{t-s} \epsilon_{t-i} \sum_{j=0}^{i} \phi_j =: x^*_{s-1,t} + x'_{s,t,t}, \end{aligned} \tag{9.2}$$

where $x^*_{s-1,t} \perp\!\!\!\perp x'_{s,t,t}$ and $x^*_{s-1,t}$ is $\mathcal{F}_{-\infty}^{s-1}$-measurable.[3] Defining $a_i \coloneqq \sum_{j=0}^{i} \phi_j$, we may further decompose $x'_{s,t,t}$ as

$$x'_{s,t,t} = \sum_{i=s}^{t} a_{t-i}\epsilon_i = \sum_{i=s}^{r} a_{t-i}\epsilon_i + \sum_{i=r+1}^{t} a_{t-i}\epsilon_i =: x'_{s,r,t} + x'_{r+1,t,t}, \tag{9.3}$$

where $x'_{s,r,t}$ is $\mathcal{F}_s^r$-measurable, and $x'_{r+1,t,t}$ is $\mathcal{F}_{r+1}^t$-measurable. The following property of the coefficients $\{a_i\}$ is particularly important: there exist $0 < \underline{a} \le \overline{a} < \infty$, and a $k_0 \in \mathbb{N}$ such that

$$\underline{a} \le \inf_{k_0+1 \le k} \inf_{\lfloor k/2 \rfloor \le l \le k} c_k^{-1}|a_l| \le \sup_{k_0+1\le k} \sup_{\lfloor k/2\rfloor \le l \le k} c_k^{-1}|a_l| \le \overline{a}. \tag{9.4}$$

This is an easy consequence of Karamata's theorem. Throughout the remainder of the paper, k_0 refers to the object of (9.4); it is also implicitly maintained $k_0 \ge 8p_0$ for p_0 as in Assumption 1(i).

[3] $x^*_{s-1,t}$ is weighted sum of $\{\epsilon_t\}_{t=-\infty}^{s-1}$: since these weights are not important for our purposes, we have refrained from giving an explicit formula for these here.

Having decomposed x_t into a sum of independent components, we shall proceed to control such objects as the right side of (9.1) with the aid of Lemma 9.1 and the following, which provides bounds on integrals involving the characteristic functions of some of those components of x_t. Recall that Assumption 1(i) is equivalent to the statement that

$$\log\psi(\lambda) = -|\lambda|^{\alpha}G(\lambda)\left[1 + \mathrm{i}\beta\,\mathrm{sgn}(\lambda)\tan\left(\frac{\pi\alpha}{2}\right)\right] \tag{9.5}$$

for all λ in a neighbourhood of the origin, where G is even and slowly varying at zero (see Ibragimov and Linnik, 1971, Thm. 2.6.5). Here, as throughout the remainder of this paper, a slowly varying (or regularly varying) function is understood to take only strictly positive values, and have the property that $G(\lambda) = G(|\lambda|)$ for every $\lambda \in \mathbb{R}$.

Lemma 9.2. *Let $p \in [0, 5]$, $q \in (0, 2]$ and $z_1, z_2 \in \mathbb{R}_+$. Then*

(i) *there exists a $\gamma_1 > 0$ such that, for every $t \geq 0$ and $k \geq k_0 + 1$,*

$$\int_{\mathbb{R}} (z_1|\lambda|^p \wedge z_2)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|\,\mathrm{d}\lambda \lesssim z_1 d_k^{-(1+p)} + z_2\mathrm{e}^{-\gamma_1 k}$$

and if $F(u) \asymp G^{p/\alpha}(u)$ as $u \to 0$,

$$\int_{\mathbb{R}} (z_1|a_k|^p|\lambda|^{p+q}F(a_k\lambda) \wedge z_2)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|\,\mathrm{d}\lambda \lesssim z_1 k^{-p/\alpha} d_k^{-(1+q)} + z_2\mathrm{e}^{-\gamma_1 k};$$

(ii) *for every $t \geq 1$, $k \geq k_0 + 1$ and $s \in \{k_0 + 1, \ldots, t\}$,*

$$\int_{\mathbb{R}} |\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t-s+1,t-1,t+k}}|\,\mathrm{d}\lambda \lesssim \frac{c_s}{c_{k+s}} d_s^{-1}.$$

The preceding summarises and refines some of the calculations presented on pp. 15–21 of Jeganathan (2008). It further implies

Lemma 9.3. *Let $f \in \mathrm{BI}$. Then*

(i) *for every $t \geq 0$ and $k \geq k_0 + 1$*

$$\mathbb{E}_t|f(x_{t+k})| \lesssim d_k^{-1}\|f\|_1;$$

(ii) *if in addition $f \in \mathrm{BI}_{[\beta]}$, then for every $t \geq 0$ and $k \geq k_0 + 1$,*

$$|\mathbb{E}_t f(x_{t+k})| \lesssim \mathrm{e}^{-\gamma_1 k}\|f\|_1 + d_k^{-(1+\beta)}\|f\|_{[\beta]}.$$

For the next result, define

$$\vartheta(z_1, z_2) \coloneqq \mathbb{E}\big[\mathrm{e}^{-\mathrm{i}z_1\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_1\epsilon_0}\big]\big[\mathrm{e}^{-\mathrm{i}z_2\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_2\epsilon_0}\big].$$

LEMMA 9.4. *Uniformly over $z_1, z_2 \in \mathbb{R}$,*

$$|\vartheta(z_1, z_2)| \lesssim [|z_1|^{\alpha}\tilde{G}(z_1) \wedge 1]^{1/2}[|z_2|^{\alpha}\tilde{G}(z_2) \wedge 1]^{1/2}$$

where $\tilde{G}(u) \asymp G(u)$ as $u \to 0$.

Proofs of (9.1), (9.4) and the preceding lemmas are given in Section F of the Supplement.

10. Proofs of Lemmas 7.3 and 7.4.

PROOF OF LEMMA 7.3. By Lemma 9.3(ii),

$$\begin{aligned}|\mathcal{N}_n f| &\le \sum_{t=1}^{k_0}|\mathbb{E}_0 f(x_t)| + \sum_{t=k_0+1}^{n}|\mathbb{E}_0 f(x_t)| \\ &\lesssim \|f\|_{\infty} + \sum_{t=k_0+1}^{n}\Big[\mathrm{e}^{-\gamma_1 t}\|f\|_1 + d_t^{-(1+\beta)}\|f\|_{[\beta]}\Big]\end{aligned}$$

whence (7.5). Regarding (7.6), it follows from repeated application of the law of iterated expectations that

$$\begin{aligned}\mathbb{E}|\mathcal{V}_{nk} f|^p \le p! \cdot \sum_{t_1=1}^{n-k}\cdots\sum_{t_{p-1}=t_{p-2}}^{n-k}\mathbb{E}\Big[\mathbb{E}_{t_1-1}(\xi^2_{kt_1}f)\cdots\mathbb{E}_{t_{p-1}-1}(\xi^2_{kt_{p-1}}f)\Big] \\ \cdot\left(\|\xi^2_{kt_{p-1}}f\|_{\infty} + \sum_{s=1}^{n-k-t_{p-1}}\|\mathbb{E}_{t_{p-1}-1}\xi^2_{k,t_{p-1}+s}f\|_{\infty}\right);\end{aligned} \tag{10.1}$$

more details of the calculations leading to (10.1) are given in Section G of the Supplement. By Lemma 7.4, the final term on the right is bounded by $C\sigma^2_{nk}(\beta, f)$. Proceeding inductively, we thus obtain

$$\mathbb{E}|\mathcal{V}_{nk} f|^p \lesssim p! \cdot C^p\sigma^{2p}_{nk}(\beta, f),$$

whence the required bound follows by Lemma 7.2(i). An analogous argument yields the same bound for $\mathcal{U}_{nk} f$. □

PROOF OF LEMMA 7.4. We shall obtain the required bound for $\mathbb{E}_t\xi^2_{k,t+s}f$ by providing a bound for $\mathbb{E}_{t-s}\xi^2_{kt}f$ (for $s \in \{1,\ldots,t\}$) that depends only on k and s (and *not* t), separately considering the cases where

(i) $k \in \{k_0+1,\ldots,n-t\}$; and
(ii) $k \in \{0,\ldots,k_0\}$.

(i). Recall the decomposition given in (9.2) and (9.3) above, applied here to reduce x_{t+k} to a sum of independent pieces,

$$\begin{aligned} x_{t+k} &= x^*_{0,t+k} + x'_{1,t-1,t+k} + x'_{t,t,t+k} + x'_{t+1,t+k,t+k} \\ &= x^*_{0,t+k} + x'_{1,t-1,t+k} + a_k\epsilon_t + x'_{t+1,t+k,t+k} \end{aligned}$$

with the convention that $x'_{1,t-1,t+k} = 0$ if $t = 1$, so that by Fourier inversion,

$$\begin{aligned} \xi_{kt}f &= \mathbb{E}_t f(x_{t+k}) - \mathbb{E}_{t-1}f(x_{t+k}) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda x^*_{0,t+k}}\mathrm{e}^{-\mathrm{i}\lambda x'_{1,t-1,t+k}} \\ &\qquad\cdot\left[\mathrm{e}^{-\mathrm{i}\lambda a_k\epsilon_t} - \mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda a_k\epsilon_t}\right]\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}\,\mathrm{d}\lambda. \end{aligned} \tag{10.2}$$

Thence

$$\begin{aligned} \xi^2_{kt}f = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} &\hat{f}(\lambda_1)\hat{f}(\lambda_2)\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x^*_{0,t+k}}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{1,t-1,t+k}} \\ &\cdot\left[\mathrm{e}^{-\mathrm{i}\lambda_1 a_k\epsilon_t} - \mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_1 a_k\epsilon_t}\right]\left[\mathrm{e}^{-\mathrm{i}\lambda_2 a_k\epsilon_t} - \mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_2 a_k\epsilon_t}\right] \\ &\cdot\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_1 x'_{t+1,t+k,t+k}}\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_2 x'_{t+1,t+k,t+k}}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2. \end{aligned} \tag{10.3}$$

Now suppose $s \in \{k+1,\ldots,t\}$. Taking conditional expectations on both sides of (10.3) gives

$$\begin{aligned} \mathbb{E}_{t-s}\xi^2_{kt}f = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} &\hat{f}(\lambda_1)\hat{f}(\lambda_2)\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x^*_{0,t+k}}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{1,t-s,t+k}} \\ &\cdot\mathbb{E}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{t-s+1,t-1,t+k}}\cdot\vartheta(\lambda_1 a_k,\lambda_2 a_k) \\ &\cdot\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_1 x'_{t+1,t+k,t+k}}\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_2 x'_{t+1,t+k,t+k}}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2, \end{aligned}$$

where we have defined

$$\vartheta(z_1,z_2) \coloneqq \mathbb{E}\left[\mathrm{e}^{-\mathrm{i}z_1\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_1\epsilon_0}\right]\left[\mathrm{e}^{-\mathrm{i}z_1\epsilon_0} - \mathbb{E}\mathrm{e}^{-\mathrm{i}z_2\epsilon_0}\right]$$

for $z_1, z_2 \in \mathbb{R}$, and made the further decomposition

$$x'_{1,t-1,t+k} = x'_{1,t-s,t+k} + x'_{t-s+1,t-1,t+k}$$

with the convention that $x'_{1,t-s,t+k} = 0$ if $s = t$. Thence, using (9.4), Lemma 9.4, and $|ab| \lesssim |a|^2 + |b|^2$, we obtain

$$
\begin{aligned}
\mathbb{E}_{t-s}\xi_{kt}^2 f \lesssim \iint_{\mathbb{R}^2} & |\hat{f}(\lambda_1)\hat{f}(\lambda_2)| \\
& \cdot [|a_k\lambda_1|^\alpha \tilde{G}(a_k\lambda_1) \wedge 1]^{1/2}[|a_k\lambda_2|^\alpha \tilde{G}(a_k\lambda_2) \wedge 1]^{1/2} \\
& \cdot |\mathbb{E}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{t-s+1,t-1,t+k}}| \\
& \cdot |\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_1 x'_{t+1,t+k,t+k}}||\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_2 x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2
\end{aligned}
\tag{10.4}
$$

$$
\begin{aligned}
\lesssim \int_{\mathbb{R}} & |\hat{f}(\lambda_1)|^2(|a_k|^\alpha|\lambda_1|^\alpha \tilde{G}(a_k\lambda_1) \wedge 1)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda_1 x'_{t+1,t+k,t+k}}| \\
& \int_{\mathbb{R}} |\mathbb{E}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{t-s+1,t-1,t+k}}| \,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1
\end{aligned}
\tag{10.5}
$$

where we have appealed to symmetry (in λ_1 and λ_2) to reduce the final bound to a single term. By a change of variables and Lemma 9.2(ii),

$$
\begin{aligned}
\int_{\mathbb{R}} & |\mathbb{E}\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)x'_{t-s+1,t-1,t+k}}| \,\mathrm{d}\lambda_2 \\
& = \int_{\mathbb{R}} |\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t-s+1,t-1,t+k}}| \,\mathrm{d}\lambda \lesssim \frac{c_s}{c_{k+s}} d_s^{-1},
\end{aligned}
\tag{10.6}
$$

while Lemma 9.1(i) and then Lemma 9.2(i) give

$$
\begin{aligned}
\int_{\mathbb{R}} & |\hat{f}(\lambda)|^2(|a_k|^\alpha|\lambda|^\alpha \tilde{G}(a_k\lambda) \wedge 1)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \\
& \leq \int_{\mathbb{R}} [(|a_k|^\alpha|\lambda|^{\alpha+2\beta}\tilde{G}(a_k\lambda)\|f\|_{[\beta]}^2) \wedge \|f\|_1^2]|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \\
& \lesssim k^{-1} d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2.
\end{aligned}
\tag{10.7}
$$

Together, (10.5)–(10.7) yield

$$
\mathbb{E}_{t-s}\xi_{kt}^2 f \lesssim \frac{c_s}{c_{k+s}} d_s^{-1}(k^{-1}d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2).
\tag{10.8}
$$

When $s \in \{1, \ldots, k\}$, (10.4) continues to hold, whence

$$
\begin{aligned}
\mathbb{E}_{t-s}\xi_{kt}^2 f & \lesssim \left(\int_{\mathbb{R}} |\hat{f}(\lambda)|(|\lambda|^{\alpha/2}\tilde{G}^{1/2}(a_k\lambda) \wedge 1)|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}| \,\mathrm{d}\lambda \right)^2 \\
& \lesssim \left(\int_{\mathbb{R}} [(|a_k|^{\alpha/2}|\lambda|^{(\alpha/2+\beta)}\tilde{G}^{1/2}(a_k\lambda)\|f\|_{[\beta]}) \wedge \|f\|_1] \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \cdot |\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|\,\mathrm{d}\lambda\Big)^2 \\
&\lesssim \left(k^{-1/2} d_k^{-(1+\beta)} \|f\|_{[\beta]} + \mathrm{e}^{-\gamma_1 k}\|f\|_1\right)^2 \\
&\lesssim d_s^{-1}(k^{-1}d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2) \qquad (10.9)
\end{aligned}
$$

by Lemmas 9.1(i) and 9.2(i); in obtaining the final result, we have used the fact that $s \leq k$ to replace a d_k^{-1} by d_s^{-1}. Since $\{c_k\}$ is regularly varying and $k \geq k_0+1$, it follows from Potter's inequality (Bingham, Goldie, and Teugels, 1987, Thm. 1.5.6(iii)) that

$$
\sum_{s=1}^{k} d_s^{-1} + \sum_{s=k+1}^{n} \frac{c_s}{c_{k+s}} d_s^{-1} \lesssim \sum_{s=1}^{n} d_s^{-1} \lesssim n d_n^{-1} = e_n,
$$

with the final bound following by Karamata's theorem. As noted above, since the bounds (10.8) and (10.9) do not depend on t, they apply also to $\mathbb{E}_t \xi_{k,t+s}^2 f$. Hence, in view of the preceding,

$$
\begin{aligned}
\sum_{s=1}^{n-k-t} \mathbb{E}_t \xi_{k,t+s}^2 f &\lesssim (k^{-1}d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2) \\
&\qquad\qquad \cdot \left[\sum_{s=1}^{k} d_s^{-1} + \sum_{s=k+1}^{n-k-t} \frac{c_s}{c_{k+s}} d_s^{-1}\right] \\
&\lesssim e_n(k^{-1}d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2).
\end{aligned}
$$

Turning now to $\|\xi_{kt}^2 f\|_\infty$, note that (10.2) still holds, with the convention that $x_{1,t-1,t+k} = 0$ if $t = 1$. Thus, again by Lemmas 9.1(i) and 9.2(i),

$$
\begin{aligned}
\|\xi_{kt}^2 f\|_\infty &\lesssim \left(\int_{\mathbb{R}} |\hat{f}(\lambda)||\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|\,\mathrm{d}\lambda\right)^2 \\
&\lesssim \left(\int_{\mathbb{R}} [|\lambda|^\beta \|f\|_{[\beta]} \wedge \|f\|_1]|\mathbb{E}\mathrm{e}^{-\mathrm{i}\lambda x'_{t+1,t+k,t+k}}|\,\mathrm{d}\lambda\right)^2 \\
&\lesssim \left(\|f\|_{[\beta]} d_k^{-(1+\beta)} + \|f\|_1 \mathrm{e}^{-\gamma_1 k}\right)^2 \\
&\lesssim d_k^{-2(1+\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2 \\
&\lesssim e_n(k^{-1}d_k^{-(1+2\beta)}\|f\|_{[\beta]}^2 + \mathrm{e}^{-\gamma_1 k}\|f\|_1^2);
\end{aligned}
$$

where the final bound follows because $k \leq n$, and so $d_k^{-1} \lesssim k^{-1} n d_n^{-1} = k^{-1} e_n$.

(ii). When $s \in \{1, \ldots, k_0\}$, the crude bound $\mathbb{E}_{t-s}\xi_{kt}^2 f \lesssim \|f\|_\infty^2$ suffices, since k_0 is fixed and finite. On the other hand, if $s \in \{k_0+1, \ldots, t\}$, we have by Jensen's inequality and Lemma 9.3(i) that

$$\mathbb{E}_{t-s}\xi_{kt}^2 f \leq \mathbb{E}_{t-s}(\mathbb{E}_t f(x_{t+k}) - \mathbb{E}_{t-1} f(x_{t+k}))^2 \lesssim \mathbb{E}_{t-s} f^2(x_{t+k}) \lesssim d_s^{-1}\|f\|_2^2.$$

Thence, by Karamata's theorem,

$$\begin{aligned}\sum_{s=1}^{n-k-t} \mathbb{E}_t \xi_{k,t+s}^2 f &\leq \sum_{s=1}^{k_0} \mathbb{E}_t \xi_{k,t+s}^2 f + \sum_{s=k_0+1}^{n-k-t} \mathbb{E}_t \xi_{k,t+s}^2 f \\ &\lesssim \|f\|_\infty^2 + \|f\|_2^2 \sum_{s=k_0+1}^{n-k-t} d_s^{-1} \\ &\lesssim \|f\|_\infty^2 + \|f\|_2^2 e_n.\end{aligned}$$

Regarding $\|\xi_{kt}^2 f\|_\infty$, the bound $\|\xi_{kt}^2 f\|_\infty \lesssim \|f\|_\infty^2$ obtains trivially. □

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References.

Astrauskas, A. (1983): "Limit theorems for sums of linearly generated random variables," *Lithuanian Mathematical Journal*, 23(2), 127–34.

Avram, F., and M. S. Taqqu (1992): "Weak convergence of sums of moving averages in the α-stable domain of attraction," *Annals of Probability*, 20(1), 483–503.

Bercu, B., and A. Touati (2008): "Exponential inequalities for self-normalized martingales with applications," *Annals of Applied Probability*, 18(5), 1848–69.

Bingham, N. H., C. M. Goldie, and J. L. Teugels (1987): *Regular Variation.* C.U.P., Cambridge (UK).

Borodin, A. N. (1981): "On the asymptotic behavior of local times of recurrent random walks with finite variance," *Theory of Probability and its Applications*, 26(4), 758–72.

——— (1982): "On the asymptotic behavior of local times of recurrent random walks with infinite variance," *Theory of Probability and its Applications*, 29(2), 318–33.

Borodin, A. N., and I. A. Ibragimov (1995): "Limit theorems for functionals of random walks," *Proceedings of the Steklov Institute of Mathematics*, 195(2), 1–259.

Chan, N., and Q. Wang (2014): "Uniform convergence for nonparametric estimators with nonstationary data," *Econometric Theory*, 30(5), 1110–33.

Davidson, J. (1994): *Stochastic Limit Theory: an introduction for econometricians.* OUP, Oxford (UK).

Davidson, J., and R. M. de Jong (2000): "The functional central limit theorem and weak convergence to stochastic integrals II," *Econometric Theory*, 16(5), 643–66.

Duffy, J. A. (2015): "Uniform convergence rates over maximal domains in structural nonparametric cointegrating regression," Unpublished, University of Oxford.

Gao, J., S. Kanaya, D. Li, and D. Tjøstheim (2015): "Uniform consistency for nonparametric estimators in null recurrent time series," forthcoming in *Econometric Theory*, dx.doi.org/10.1017/S0266466614000577.

Hall, P., and C. C. Heyde (1980): *Martingale Limit Theory and Its Application.* Academic Press, New York (USA).

Hannan, E. J. (1979): "The central limit theorem for time series regression," *Stochastic Processes and their Applications*, 9(3), 281–89.

Ibragimov, I. A., and Y. V. Linnik (1971): *Independent and stationary sequences of random variables.* Wolters–Noordhoff, Groningen (Netherlands).

Jeganathan, P. (2004): "Convergence of functionals of sums of r.v.s to local times of fractional stable motions," *Annals of Probability*, 32, 1771–95.

——— (2008): "Limit theorems for functionals of sums that converge to fractional Brownian and stable motions," Cowles Foundation Discussion Paper No. 1649, Yale University.

Karlsen, H. A., T. Myklebust, and D. Tjøstheim (2007): "Nonparametric estimation in a nonlinear cointegration type model," *Annals of Statistics*, 35(1), 252–99.

Karlsen, H. A., and D. Tjøstheim (2001): "Nonparametric estimation in null recurrent time series," *Annals of Statistics*, 29(2), 372–416.

Kasahara, Y., and M. Maejima (1988): "Weighted sums of I.I.D. random variables attracted to integrals of stable processes," *Probability Theory and Related Fields*, 78(1), 75–96.

Kasparis, I., E. Andreou, and P. C. B. Phillips (2012): "Nonparametric predictive regression," Cowles Foundation Discussion Paper No. 1878, Yale University.

Kasparis, I., and P. C. B. Phillips (2012): "Dynamic misspecification in nonparametric cointegrating regression," *Journal of Econometrics*, 168(2), 270–84.

Liu, W., N. Chan, and Q. Wang (2014): "Uniform approximation to local time with applications in non-linear cointegrating regression," Unpublished, University of Sydney.

Park, J. Y., and P. C. B. Phillips (2001): "Nonlinear regressions with integrated time series," *Econometrica*, 69(1), 117–161.

Perkins, E. (1982): "Weak invariance principles for local time," *Probability Theory and Related Fields*, 60(4), 437–51.

Samorodnitsky, G., and M. S. Taqqu (1994): *Stable Non-Gaussian Random Processes: stochastic models with infinite variance.* CRC Press, Boca Raton (USA).

Skorokhod, A. V. (1956): "Limit theorems for stochastic processes," *Theory of Probability and its Applications*, 1(3), 261–90.

Tyran-Kamińska, M. (2010): "Functional limit theorems for linear processes in the domain of attraction of stable laws," *Statistics and Probability Letters*, 80(11), 975–81.

van der Vaart, A. W. (1998): *Asymptotic Statistics.* C.U.P., Cambridge (UK).

van der Vaart, A. W., and J. A. Wellner (1996): *Weak Convergence and Empirical Processes: with applications to statistics.* Springer, New York (USA).

Wang, Q., and P. C. B. Phillips (2009a): "Asymptotic theory for local time density estimation and nonparametric cointegrating regression," *Econometric Theory*, 25(3), 710–38.

——— (2009b): "Structural nonparametric cointegrating regression," *Econometrica*, 77(6), 1901–48.

——— (2011): "Asymptotic theory for zero energy functionals with nonparametric regression applications," *Econometric Theory*, 27(2), 235–259.

——— (2012): "A specification test for nonlinear nonstationary models," *Annals of Statistics*, 40(2), 727–58.

——— (2015): "Nonlinear cointegrating regression with endogeneity and long memory,"

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