Strategic Bidding in Product-Mix, Sequential, and Simultaneous Auctions

Simon Finster*

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Abstract

We study equilibria in Product-Mix, sequential, and simultaneous auctions, which are used to sell differentiated, indivisible goods. A flexible bidder with unit demand, interested in buying any of the goods, competes against several inflexible bidders, each interested in only one specific good. For first-price and second-price payments, we obtain theoretical results on equilibrium bidding, and compare efficiency, revenue, and bidder surplus numerically. Differences in outcomes between Product-Mix and sequential auctions are small for a range of value distributions. The simultaneous auction performs worst in all dimensions, and differences in performance vary substantially with the degree of competition the flexible bidder faces.

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JEL Classification: C72, D44, D47, D61, D82

1 Introduction

If buyers are interested in two different goods, but want to buy at most one of them, they face a difficult strategic situation in standard simultaneous auction: they do not know which auction to participate in. Holding auctions sequentially mitigates this difficulty, and the Product-Mix auction design resolves it altogether. Following the financial crisis in 2007, the Product-Mix auction (PMA) was designed for the Bank of England to allocate loans to commercial banks against different collateral (Klemperer (2008, 2010, 2018)) and has been in use ever since. In a single-round procedure, bidders can express trade-offs between different varieties of goods. The auction implements the efficient allocation and if bidders act as price takers it determines a competitive equilibrium.

The PMA allows a bidder with the aforementioned preferences to submit bids on both goods, under the constraint to win at most one. It therefore greatly simplifies the bidder's bidding decision, and it is expected to result in a more efficient allocation, if the number of bidder is sufficiently large. However, this auction has not been studied in game-theoretic models: if the number of bidders in

*Nuffield College and Department of Economics, University of Oxford, simon.finster@nuffield.ox.ac.uk.
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the auction is small, bidders are have a strong incentive to bid strategically in order to influence auction prices and maximise their surplus. My contribution is, to the best of our knowledge, the first comparison of equilibrium bidding in PMAs with standard simultaneous and sequential auctions. We consider first-price and second-price payment rules, where the standard PMA with uniform prices falls under the second-price rule. The first-price PMA we discuss is an instance of the menu auction by Bernheim and Whinston (1986), equipped with the Product-Mix bidding language.

The PMA has further applications in finance and various other industries, e.g. in online advertisement. Advertisers bid for web space to display their ads on publishers' websites. If two websites have different audiences, an advertiser with say only one ad they want to display has a trade-off between publishing on either (but not both) of the two websites. Another important application are electricity markets. Capacity-constrained suppliers, e.g. operators of pumped-storage hydro power facilities, face a trade-off between bidding for adjacent time slots. Suppose they can supply either in the morning or the afternoon, but not both. The Product-Mix format would allow them to express a trade-off between the morning-product and the afternoon-product, without taking on the risk of having to supply both. Consumer good auctions provide many other examples for possible applications of the PMA.[1]

In our analysis, simultaneous auctions perform worst in terms of efficiency, revenue, and bidder surplus. The performance of PMAs and sequential auctions is similar, with a slight advantage for sequential auctions in terms of efficiency and bidder surplus. When bidders are restricted to bid for at most one unit, the standard PMA is outcome-equivalent to the general VCG mechanism[2,3] and thus sets the benchmark for efficiency. The first-price PMA achieves the highest revenue, closely followed by the sequential auction. The flexible bidder prefers the PMA under the second-price rule, and she prefers a sequential auction under the first-price rule. Among first-price auctions, efficiency is highest in the PMA, close to a sequential format. We obtain most of our results assuming bidders' values are uniformly distributed; but varying the distributions does not significantly change the results.

In a simple version of our model, the auctioneer sells two differentiated goods, one unit of each. All bidders have constant marginal values for one or two goods and unit demand. There is one flexible bidder who values both goods, up to one unit overall; and there are two groups of inflexible bidders who are interested in only either one of the goods, respectively. Inflexible bidders are modelled as a competitive fringe: in equilibrium, it is optimal for them to bid their true value. This allows us to focus on the flexible bidder's behaviour. She has to balance the trade-off between winning one good or the other and not too often winning both goods at the same time. Under the first-price rule, there is an additional incentive to shade bids. In Product-Mix auctions, the former

[1] Vintage cars are an obvious example. I have also run several Product-Mix auctions among students at Oxford, to sell licences for rental of community-owned exercise equipment: we have rowing machines of different quality in storage and each student (most likely) wants not more than one.

[2] Ausubel and Milgrom (2005)

[3] A proof for this can be found e.g. in Leonard (1991). When bidders are restricted to bid for one unit, the PMA is equivalent to his formulation of an assignment problem of individuals to positions. Demange (1982) proves a closely related result in a two-sided market setup.

trade-off is eliminated because the bidding format allows to specify bids (so called "paired" bids[4]) that prevent the flexible bidder from winning both goods at once. Similarly, in sequential auctions the flexible bidder can choose when to expose herself to potentially winning more than one unit.

We find that differences between auction formats are more extreme for high values of the flexible bidder.[5] We also vary the inflexible bidders' value distribution in a way that is straightforwardly interpreted as varying the degree of competition the flexible bidder faces. We find that outcomes in the first-price PMA and the first-price sequential auction formats do not differ by much, irrespective of the degree of competition. However, in the comparison of the first-price PMA/sequential auction versus the first-price simultaneous auction, differences in bidder surplus increase considerably with increased competition, while differences in efficiency and revenue decline with increased competition.

Our paper is among the first to study strategic bidding in PMAs, and it is the first paper to compare relevant auction formats for the sale of heterogeneous goods with respect to equilibrium bidding, efficiency, revenue, and bidder surplus. Holmberg et al. (2018) extend Klemperer and Meyer (1989)'s supply function equilibrium to two goods. Their characterisation of linear equilibria under demand uncertainty can be used to describe equilibrium bidding in two-variety PMAs with a number of symmetric bidders holding private information. However, the authors provide no information on how these equilibria compare to other mechanisms used in practice.

Our setting builds on Krishna and Rosenthal (1996)'s model with "global" and "local" bidders. In contrast to their work, we study the sale of heterogeneous goods and assume different preferences for the flexible (global) bidder: if she wins both goods she can only use one of them, whereas in Krishna and Rosenthal (1996) the flexible bidder has increasing returns.[6] Albano et al. (2001) study a similar framework and draw comparisons with the VCG mechanism, but they also restrict to super-additive valuations.[7] Vickrey (1961) discussed simultaneous and "successive" auctions with unit-demand bidders, however, with identical objects.

Simultaneous auctions have been also studied by Gerding et al. (2008), in a setup similar to ours. For identical goods and identical second-price auctions, they show that a flexible bidder participates in each auction with a strictly positive bid. The setup is extended to non-identical auctions, but identical goods. Our model differs in that we assume identical distributions for the inflexible bidders' values across all auctions, but we allow for the goods to be non-identical to the flexible bidder. In addition, we provide a result on the uniqueness of equilibria with two goods, and also consider first-price payments. Simultaneous first-price auctions under full information have been studied by Palfrey (1980), and a related optimal bidding problem is discussed in Rothkopf (1977). Both papers consider a setup with constraints on the sum of all amounts bid for.

A related approach to modelling multi-unit auctions with large and small bidders is taken by

[4]In the original design of the PMA, these bids are called paired bids (Klemperer (2010)); in the computer science literature, they are better known as XOR-bids.

[5]We only compare mechanisms with the same payment rule, as comparisons between payment rules are inherently flawed by the setup of our model: inflexible bidders are comparatively stronger under the first-price rule.

[6]In their paper, the authors establish equilibria for simultaneous second-price auctions and, in an example, compare equilibrium bidding and revenue to sequential second-price and combinatorial auctions.

[7]Albano et al. (2001) provide efficiency and revenue comparisons between a variant of the 1994 FCC simultaneous ascending auction, simultaneous and sequential second-price auctions, using the VCG mechanism as a benchmark.

Baisa and Burkett (2017). In their model, the small bidders each demand only an infinitesimal amount. They bid truthfully in the uniform-price auction, but shade their bids under discriminatory payments. Their model is less distorting than ours under the first-price rule; however, we found it not to be tractable for Product-Mix auctions.

Our paper is structured as follows. We present a simple version of the model in section 2 and provide results on equilibrium behaviour. In section 3, we describe the general model. We present our numerical analysis in section 4, assuming different distributions for the inflexible bidders' values. Section 5 concludes.

2 A simple two-good model

The flexible bidder's decision problem is best illustrated in a simple model with just two differentiated goods. We describe a model where two indivisible goods A and B are for sale, one unit of each, but all our results hold if the goods are divisible as well. Three types of bidders compete for the goods: a flexible bidder with unit demand, bidder F (which we refer to as "she") values one unit of good A at v_A or one unit of good B at v_B and has no value for more than one unit in total. The two goods are differentiated for bidder F with $v_B \geq v_A > 0$.[8] All bidders are risk neutral. Formally, we have

$$u^F(p_A, p_B) = \begin{cases} v_A - p_A & \text{if A is won} \\ v_B - p_B & \text{if B is won} \\ v_B - p_B - p_A & \text{if A and B are won} \\ 0 & \text{otherwise} \end{cases}$$

where p_A and p_B denote the prices of good A and B. There is a group G_A of $n_A \geq 2$ inflexible bidders, each of which is interested only in good A, and there is another group G_B of $n_B \geq 2$ inflexible bidders, each of which is interested only in good B. It is without loss of generality to assume that $n_A = n_B = n$; the size of each competitive fringe does not matter. Bidders of group G_K value one unit of good $K = A, B$ at $k = a, b, k \geq 0$, that is

$$u^K(p_A, p_B) = \begin{cases} k - p_K & \text{if good } K \text{ is won} \\ 0 & \text{otherwise} \end{cases}$$

The auctioneer ("he") has one unit of good A and one unit of good B for sale. Both goods are identical to him at a zero reserve price. Each bidder knows their value privately, but the distribution of the inflexible bidders' values is common knowledge. The inflexible bidders also know that they have at least one competitor with an identical value. The inflexible bidders' values a and b are random variables drawn from an absolutely continuous probability distribution G with probability density function g and support $[0, \bar{v}]$ where $\bar{v} \in \mathbb{R} \cup \{\infty\}$. We make the following assumption:

[8] We will sometimes note illustrative results for the corner case $v_A = 0$, or $v_A = v_B = 0$.

Assumption 1. The reverse hazard rate $\frac{g(x)}{G(x)}$ is weakly decreasing on $[0, \bar{v}]$.

For most of our results, we assume $0 < v_A \leq v_B < \bar{v}$.[9] We now describe the auction formats we are interested in.

Product-Mix auctions. In describing the PMA for our setup, we define (i) the bidding language, (ii) the allocation rule, and (iii) the payment rule.

(i) The bidding language allows single and paired bids.[10] The flexible bidder submits a bid (w_A, w_B) to state demand for 1 unit of good A *or* good B, but not both, at prices of up to w_A or w_B respectively. The inflexible bidders of group G_K submit a single bid each for good $K = A, B$. Call the bid value of this bid y_K.

(ii) The auctioneer is given bids $(w_A, w_B), (y_A, 0), (0, y_B)$. In the standard PMA, he then determines the efficient allocation given the reported values, i.e. the flexible bidder obtains good A if and only if $w_A - y_A > w_B - y_B$ and $w_A \geq y_A$, and she wins good B if and only if $w_B - y_B \geq w_A - y_A$ and $w_B \geq y_B$.[11] In the first-price PMA, the same allocation rule applies. Together with the pricing rule, this is equivalent to choosing the revenue-maximising allocation, as in the menu auction by Bernheim and Whinston (1986).

(iii) In the standard PMA, the auctioneer determines the lowest competitive equilibrium prices. In our model, prices will always be set by the inflexible bidders' bids y_A and y_B.[12] In the first-price PMA, each winning bidder pays their bid price for the unit won.

Simultaneous auction. Bidders can participate in either one or both of two simultaneous single-unit auctions. The auction price is set by the highest losing bid, respectively, under the second-price rule. A winner pays her bid under the first-price rule.[13]

Sequential auction. Two single-unit auctions for good A and B, respectively, are held sequentially. After the first auction, the auctioneer announces the winning allocation and the clearing price. Bidders then choose their bid in the second auction. They are allowed to participate in either or both auctions. Payment rules are as in the simultaneous auction. We distinguish the case where good A is sold first and the case where good B is sold first.

Remark: In our setup the standard PMA (second-price rule) is equivalent to the VCG mechanism. Bidders pay precisely their social externality from participating in the auction. This also holds more generally for the standard PMA when bidders are restricted to bid for only one unit (possibly with a paired bid).

[9] We note some corner solutions for $v_B \geq \bar{v}$, but this is not the focus of this study.

[10] Paired bids are an instance of XOR bids (see, e.g., Sandholm (2002)) where each component specifies the quantity and price of exactly one good. Let p_A and p_B denote the auction prices. A paired bid (w_A, w_B) for good A and good B respectively expresses the following preference: if $w_A - p_A > w_B - p_B$, then the bidder would like to receive up to one unit of good A. If the inequality sign is switched, she would like to receive up to one unit of good B. If the inequality sign is replaced by an equality, the bidder is indifferent between receiving good A and B, assuming indivisibility of goods.

[11] Ties are broken in favour of good B in case of indifference between the components of the paired bid, and in favour of the flexible bidder between her and the inflexible bidders; but this is not important for our analysis.

[12] In general, prices are determined in a linear programme. A detailed description of the price setting concept would go beyond the scope of our study and can be found in Baldwin and Klemperer (2019).

[13] Ties are broken in favour of the flexible bidder, but again this is not important in our analysis.

Throughout the paper, we use subscripts to denote the auction format. The subscripts are "$P2$" (standard PMA), "$Sim2$" (simultaneous second-price), "$SeqK2$" (sequential second-price with good K sold first), "$P1$" (first-price PMA), "$Sim1$" (simultaneous first-price), and "$SeqK1$" (sequential first-price with good K sold first).

2.1 Equilibrium bidding

First, we formalise that inflexible bidders bid indeed truthfully as claimed above. All proofs are relegated to the appendix.

Lemma 1. *In the auction formats introduced above, at least two inflexible bidders in each group bid their true values in any equilibrium.*

Under the second-price rule, truthful bidding is a dominant strategy for the inflexible bidders in all auction formats. With first-price payments, a simple Bertrand-type argument can be made to establish the lemma. We show this for the first-price PMA in the appendix, and the arguments for the other first-price auctions are analogous. The flexible bidder's choice set can be restricted by eliminating weakly dominated strategies: she never bids above her true values. Knowing the inflexible bidders' equilibrium strategy, we can illustrate the allocation to the flexible bidder, as an outcome of each mechanism's allocation rule, graphically in a-b-space in figures 1(a) - 1(d). The dark shaded region marked with "A" corresponds to bidder F winning good A, the light shaded region to winning good B, and the striped region to winning both goods. Note that w_A and w_B, assuming $w_A \leq w_B$,[14] are not drawn at the optimal values, although proportions approximately resemble equilibrium bidding.[15]

Product-Mix auctions.

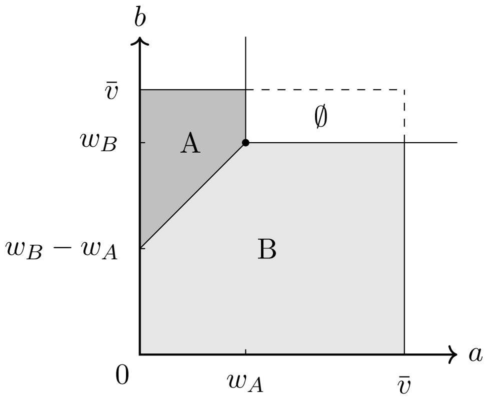
The Product-Mix auction caters to the unit demand preference of the flexible bidder. The format alleviates the exposure to win more than one unit, maintaining the facility to win either of the goods. Given the flexible bidder's preference, it is without loss of generality to assume that she makes one flat paired bid.[16] Denote this bid (w_A, w_B). In the same way, each inflexible bidder of group G_A or G_B makes a bid $(y_A, 0) = (a, 0)$ and $(0, y_B) = (0, b)$, respectively. As stated above, the flexible bidder obtains good A if and only if $w_A - a > w_B - b$ and $w_A \geq a$, i.e. north of the 45 degree line in figure 1(a). She wins good B if and only if $w_B - b \geq w_A - a$ and $w_B \geq b$. Ties such that $w_A = a$, $w_B = b$, or $w_A + b = w_B + a$ occur with zero probability in equilibrium.

Proposition 1. *In the standard PMA, there exists a unique equilibrium in which the flexible bidder bids truthfully.*

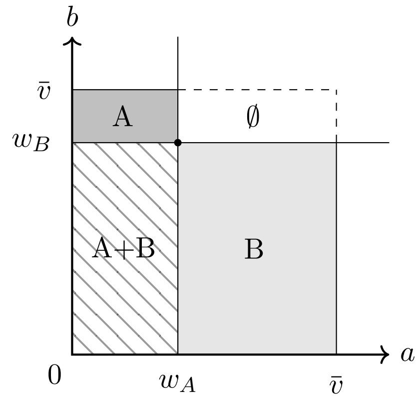
[14] The figure corresponding to $w_A > w_B$ is symmetric.

[15] The only precisely optimal value drawn is w_A in the sequential auction with good B sold first.

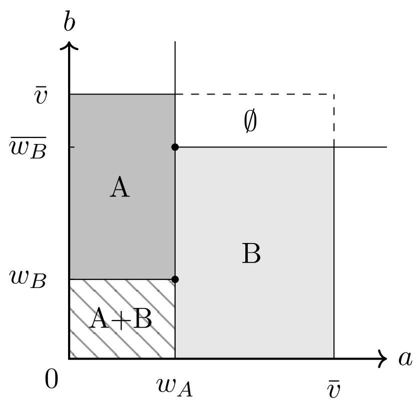
[16] This is argued in many related models, see for example Baisa and Burkett (2017). Note that, when the good is divisible, this only holds if the flexible bidder knows that her bid (or a part of it) will never win *and* set the auction price at the same time. For example, if she knew that the inflexible bidders might demand only half a unit of some good, it might be her bid which ends up setting the price. In this case, the flexible bidder would have an incentive to reduce demand on the second half of the unit she is bidding for.



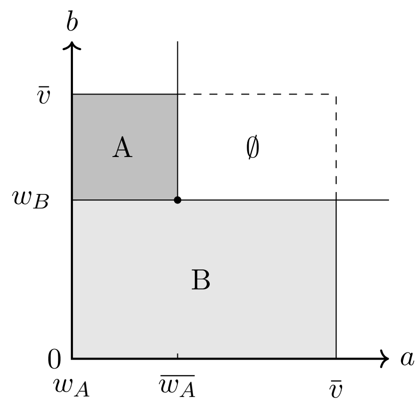
(a) PMA



(b) Simultaneous



(c) Sequential, good A sold first



(d) Sequential, good B sold first

Figure 1: Allocations to the flexible bidder

The standard PMA achieves full efficiency in our model. Indeed, it is equivalent to VCG-mechanism. It is straightforward to calculate bidder surplus, revenue, and efficiency, which can be found in the appendix and will be needed for our numerical analysis later.

The Product-Mix auction can also be implemented with first-price payments, i.e. each winner pays their respective bid for the goods won. Naturally, the flexible bidder then has an incentive to shade her bids. Again, she makes a flat paired bid (w_A, w_B), given the inflexible bidders' flat, truthful bid. Clearly, any decreasing bid would leave her with an opportunity to improve her payoff ex-post. We denote by $P_A(w_A, w_B)$ and $P_B(w_A, w_B)$ denote the probability of bidder F winning good A and good B, respectively. The allocation rule remains unchanged from the standard PMA; or, looking at it differently, the auctioneer now chooses the revenue-maximising allocation. To study equilibrium play, we now write out the flexible bidder's expected payoff function. We simply have

$$\Pi^F_{P1|v_A,v_B}(w_A, w_B) = P_A(w_A, w_B)\,[v_A - w_A] + P_B(w_A, w_B)\,[v_B - w_B] \tag{1}$$

To determine the probabilities of winning good A or B, respectively, we note the following lemma.

Lemma 2. *In any BNE, bids must be such that $w_A \leq w_B$.*

The argument is simple: $w_A > w_B$ cannot be part of an equilibrium because simply switching the bid amounts between good A and B would guarantee the bidder a strictly better payoff. Using figure 1(a), it is not hard to see that

$$P_A(w_A, w_B) = \int_0^{w_A} \int_{w_B - w_A + a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) \text{ and} \tag{2}$$

$$P_B(w_A, w_B) = \int_0^{w_A} \int_0^{w_B - w_A + a} \mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}} \int_0^{w_B} \mathrm{d}G(b)\,\mathrm{d}G(a) \tag{3}$$

So the flexible bidder's maximisation problem is

$$\max_{w_A, w_B} \Pi^F_{P1|v_A,v_B}(w_A, w_B)$$
$$\text{s.t. } 0 \leq w_K \leq v_K,\ K = A, B, \text{ and (2) and (3) hold}$$

Proposition 2. *In the first-price Product-Mix auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid $(w_A{}^*, w_B{}^*)$, characterised by equations (17) and (18) (in the appendix).*

The fact that bidder F makes strictly positive bids on both goods in equilibrium may seem intuitive, but in fact is not obvious considering carefully the probabilities of winning. Notice that by increasing her bid on one of the goods, while increasing the probability of winning on that respective good, but also the bid price, she decreases the probability of winning the respective other good. A priori, it is not clear in what direction the trade-off between these three effects goes. This is illustrated in figure 13 in the appendix. We demonstrate that any point on the boundary of $[0, v_A] \times [0, v_B]$ cannot be optimal and therefore the global optimum must be interior and coincide with a stationary point; i.e., first-order conditions fully characterise the global optimum of the flexible bidder's maximisation problem.

Sequential auctions.
A standard alternative to sell good A and B, without creating exposure to win both goods at once, is to sell them sequentially. The flexible bidder learns the outcome of the first auction before she decides how to bid in the second auction. Let ${w_K}^*$ denote the equilibrium bid after winning the first auction, and let $\overline{w_K}^*$ denote the equilibrium bid after losing the first auction, for $K = A, B$, where good $J \neq K$ is sold prior to good K. The flexible bidder's optimal strategy is found by straightforward backwards-induction. Because of the asymmetry in the flexible bidder's values the order in which good A and B are sold matters; we first present the case where good A is sold first.

Proposition 3. *In the sequential second-price auction, where good A is sold prior to good B, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder always submits a strictly positive bid.

All characterisations of optimal bids can be found in the appendix. Intuitively, the second stage adjusts the flexible bidder's true first-stage value for good A; it is the difference in the expected second-stage payoffs conditional on winning or losing the first auction. If she won the first auction, she will make a strictly positive bid in the auction for good B again, but only so high that her potential additional payment is entirely hedged: she bids exactly the difference in value between good B and good A. This is different, of course, when good B is sold first; there is no incentive to bid again in the second auction if the first auction and the higher-value good is won.

Proposition 4. *In the sequential second-price auction, where good B is sold prior to good A, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder bids always submits a strictly positive, truthful bid after losing the first auction, and a zero bid after losing the first auction.

Suppose now that good A and B are sold in two sequential first-price auctions. We first discuss the case where good A is prior to good B. Fix v_B and let $\widehat{v_A} := G(\overline{w_B}^*(\widehat{v_A}, v_B))(v_B - \overline{w_B}^*(\widehat{v_A}, v_B)) - G({w_B}^*(\widehat{v_A}, v_B))(v_B - {w_B}^*(\widehat{v_A}, v_B) - v_A)$.

Proposition 5. *In the sequential first-price auction, where good A is sold prior to good B, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, iff $v_A < \widehat{v_A}$ for given v_B, and he submits a zero bid iff $v_A \geq \widehat{v_A}$ for given v_B, and

(ii) in the second auction, the flexible bidder bids always submits a strictly positive bid.

Intuitively, if the expected additional gain from winning the second auction after the first auction is won already, is too small relative to v_A, then it is not worth bidding in the first auction to begin with. Instead, a higher expected gain can be achieved by forgoing the bid in the first auction, and making an optimal standard first-price auction bid in the subsequent auction. The proof proceeds by backwards-induction. After having won the first auction, bidder F occurs a sunk cost of w_A. To determine her bid in the second auction, she updates her value for good B and makes a correspondingly optimal bid. Again, the incremental gain from winning good B is $v_B - v_A$, but has to be adjusted for optimal shading due to the first-price rule. Now consider the sequential auction where good B is sold prior to good A.

Proposition 6. *In the sequential first-price auction, where good B is sold prior to good A, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder always submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder bids always submits a strictly positive bid after losing the first auction, and a zero bid after winning the first auction.

Naturally, after winning the higher value good in the first auction, there is nothing to be gained from participating in the second auction. If the first auction is lost, the flexible bidder can bid again in the second auction. The difference in expected payoff between the case where she wins or loses the first auction determines her true value.

Simultaneous auctions.
Suppose now the auctioneer sells good A and good B in two separate auctions which take place simultaneously. Bidder F can participate either in both auctions or bid only in either the A-auction or the B-auction. Participating in both auctions comes with the risk of potentially winning in both auction, and having to pay for a good that brings no additional value. In reality, this exposure may prevent the flexible bidder from bidding optimally. Her maximisation problem is

$$\max_{w_A, w_B} \Pi^F_{Sim2|v_A,v_B}(w_A, w_B) = G(w_A)\,(1 - G(w_B))\,v_A + G(w_B)v_B - \int_0^{w_A} a\,\mathrm{d}G(a) - \int_0^{w_B} b\,\mathrm{d}G(b)$$
$$\text{s.t. } 0 \leq w_K \leq v_K,\ K = A, B$$

Proposition 7. *In the simultaneous second-price auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid $({w_A}^*, {w_B}^*)$, characterised by equations (26) and (27) (in the appendix).*

In the appendix, we show that the two equations fully characterise the global optimum of bidder F's profit maximisation problem. This global optimum is always interior, i.e. despite the risk of having to pay for both goods, the flexible bidder participates in both auctions, if she bids optimally. This result also holds under first-price payments. Proceeding similarly, we write bidder F's maximisation problem as

$$\max_{w_A, w_B} \Pi^F_{Sim1|v_A,v_B}(w_A, w_B) = G(w_A)\left(1 - G(w_B)\right) v_A - G(w_A) w_A + G(w_B)\left(v_B - w_B\right)$$
$$\text{s.t. } 0 \leq w_K \leq v_K, \; K = A, B$$

Proposition 8. *In the simultaneous first-price auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid $({w_A}^*, {w_B}^*)$, characterised by equations (34) and (35) (in the appendix).*

In addition to the standard trade-off in a single good first-price auction, increasing the probability to win the respective other good decreases the expected payoff on the first good, because of the looming possibility to have to overpay for the higher-value good. Through the equilibrium characterisation we also obtain the following result, which will be useful for numerical analysis.

Proposition 9. *In the simultaneous auction (first-price or second-price), if $G(x)$ is convex $\forall x \in [0, \bar{v}]$, or if $G(x)$ is concave $\forall x \in [0, \bar{v}]$, the equilibrium is unique.*

2.2 Example with uniform distributions of a and b

We illustrate some further properties of the equilibria described above in an example where a and b are uniformly distributed on $[0, 1]$. Equilibrium bids follow from the characterisation in the general case, given in the proof of the corresponding proposition. For the first-price PMA, we additionally establish a uniqueness result.

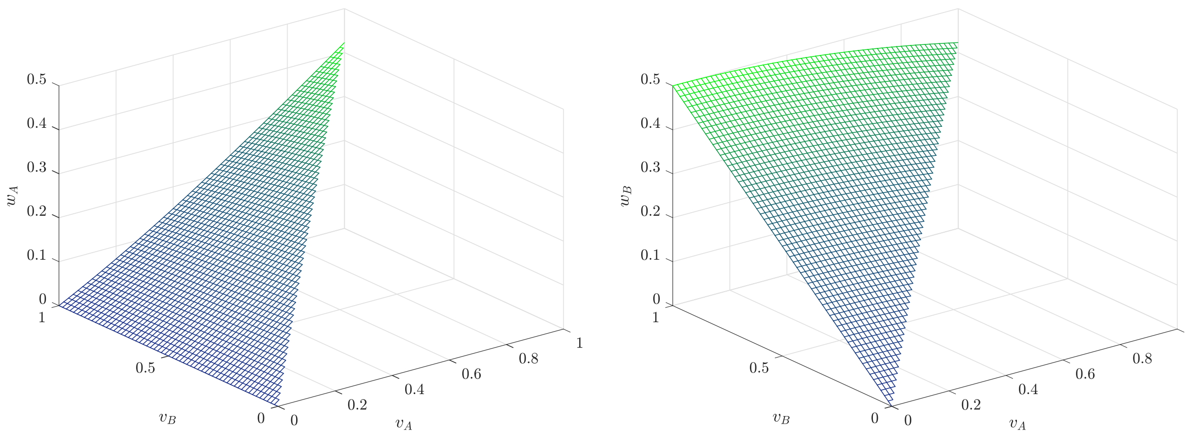
Product-Mix auctions. In the standard PMA, the flexible bidder bids truthfully as discussed above. Under the first-price rule, the equilibrium is characterised by the following equations:

$$-\frac{3}{2}{w_A}^{*2} + {w_A}^*\left(-2 + 3{w_B}^* + v_A - v_B\right) + v_A(1 - {w_B}^*) = 0 \tag{4}$$
$$\frac{3}{2}{w_A}^{*2} + v_B - 2{w_B}^* - v_A {w_A}^* = 0 \tag{5}$$

Proposition 10. *In the first-price PMA, when a and b are uniformly distributed on $[0, 1]$ there exists a* unique *equilibrium.*

Uniqueness is established by noticing that equations (4) and (5) describe a third-degree polynomial (function of w_A). Analysing its roots and limit behaviour, the intermediate value theorem guarantees that exactly one of these roots is between 0 and 1. The two equations above allow us to perform simple comparative statics. The implicit function theorem yields results for $\frac{\partial w_A}{\partial v_A}$, $\frac{\partial w_A}{\partial v_B}$, $\frac{\partial w_B}{\partial v_A}$, and $\frac{\partial w_B}{\partial v_B}$ (see appendix). We can evaluate the derivatives at $(v_A, v_B, {w_A}^*, {w_B}^*)$, and obtain, e.g., $\frac{\partial w_A}{\partial v_A}(0,0,0,0) = \frac{1}{2}$, $\frac{\partial w_A}{\partial v_B}(0,0,0,0) = \frac{\partial w_B}{\partial v_A}(0,0,0,0) = 0$, and $\frac{\partial w_B}{\partial v_B}(0,0,0,0) = \frac{1}{2}$. We also have $\frac{\partial w_A}{\partial v_A}(0,1,0,0.5) = \frac{1}{3}$, $\frac{\partial w_A}{\partial v_B}(0,1,0,0.5) = \frac{\partial w_B}{\partial v_A}(0,1,0,0.5) = 0$, and $\frac{\partial w_B}{\partial v_B}(0,1,0,0.5) = \frac{1}{2}$.

We plot the solution of equation (4) and (5) in Figure 2(a) and 2(b), which illustrate the derivatives stated above. It is evident that bidder F's equilibrium bids are effectively competing with one another. For given v_K (v_J), bidder F's bid ${w_K}^*$ is weakly decreasing (weakly increasing) in v_J (v_K), $K \neq J = A, B$, and this can be shown by numerically evaluating the partial derivatives of ${w_A}^*$ and ${w_B}^*$.



(a) ${w_A}^*(v_A, v_B)$ (b) ${w_B}^*(v_A, v_B)$

Figure 2: Equilibrium bids for first-price PMA with uniform a and b

Sequential auctions. In the sequential auction, we obtain closed form equilibrium characterisations for uniformly distributed a and b. Bidder F always makes a strictly positive bid in the first auction, given by ${w_A}^* = \frac{1}{2}{v_A}^2 + v_A(1 - v_B)$. Second-stage bids are given by ${w_B}^* = v_B - v_A$ and ${\overline{w_B}}^* = v_B$. If good B is sold prior to good A, we have the first-stage bid ${w_B}^* = v_B - \frac{{v_A}^2}{2}$, and second-stage bids ${w_A}^* = 0$ and ${\overline{w_A}}^* = v_A$.

For the sequential first-price auction where good A is sold prior to good B, first note the second-stage bids ${w_B}^* = \frac{1}{2}(v_B - v_A)$ and ${\overline{w_B}}^* = \frac{v_B}{2}$. Then, we determine $\widehat{v_A} = 2v_B - 4$, which is always negative. So bidder F makes a strictly positive bid in the first auction, given by ${w_A}^* = \frac{1}{4}\left(\frac{{v_A}^2}{2} + v_A(2 - v_B)\right)$. When good B is sold prior to good A, bidder F bids ${w_B}^* = \frac{1}{2}\left(v_B - \frac{{v_A}^2}{4}\right)$ in the first auction, and ${w_A}^* = \frac{v_A}{2}$ and ${\overline{w_A}}^* = 0$ in the second auction.

Both in the first-price and second-price auction, comparative statics are of equilibrium bids are obvious; intuitively, they are analogous to those in the first-price PMA. For given v_K (v_J), equilibrium bids ${w_K}^*$ are weakly decreasing (weakly increasing) in v_J (v_K), $K \neq J = A, B$.

Simultaneous auctions. In the simultaneous auction when a and b are uniformly distributed on $[0, 1]$ we also obtain closed form solutions. Under the second-price rule we have ${w_A}^* = \frac{v_A(1-v_B)}{1-{v_A}^2}$ and ${w_B}^* = \frac{v_B-{v_A}^2}{1-{v_A}^2}$. In the simultaneous first-price auction, bidder F bids ${w_A}^* = \frac{v_A(2-v_B)}{4-{v_A}^2}$ and ${w_B}^* = \frac{2v_B-{v_A}^2}{4-{v_A}2}$. Comparative statics are analogous to the sequential and Product-Mix auctions (details are in appendix B).

3 General model

There are M indivisible goods sold in an auction, one unit of each good. $M+1$ types of bidders compete for the goods. There is one flexible bidder F ("she") who values one unit of good j at v_j, $j \in \mathcal{J} := \{1, ..., M\}$, but at most one unit in total. We assume $v_M \geq v_{M-1} \geq ... \geq v_1 > 0$. Let $\mathbf{v} = (v_1, ..., v_M)$, let $q_j \in \{0,1\}$ denote the quantity of good j allocated to the flexible bidder at price p_j, and let $\mathbf{q}$ and $\mathbf{p}$ denote the corresponding M-vectors. Furthermore, let j^{max} denote the good with the highest index she is allocated, i.e. $j^{max} := \max\{j \in \mathcal{J} : q_j = 1\}$. $\langle x, y\rangle$ denotes the dot product of two vectors x and y. The flexible bidder's utility is then

$$u^F(\mathbf{p}, \mathbf{q}) = v_{j^{max}} - \langle \mathbf{p}, \mathbf{q}\rangle$$

There is a group G_j of $n_j \geq 2$ inflexible bidders, each of which is interested only in good j; there are M groups in total. Each such group can be seen as a competitive fringe for a specific good. It is without loss of generality to assume that $n_j = n \ \forall j$: the size of each competitive fringe does not matter. Bidders of group G_j value one unit of good $j \in \mathcal{J}$ at $x_j \geq 0$. Letting $\mathbf{q}$ denote the vector of quantities allocated, we have

$$u^j(\mathbf{p}, \mathbf{q}) = x_j q_j - \langle \mathbf{p}, \mathbf{q}\rangle$$

All M goods are identical to the auctioneer at a zero reserve price. Each bidder knows their value privately, but the distribution of the inflexible bidders' values is common knowledge. The inflexible bidders also know that they have at least one competitor with an identical value. All x_j are random variables drawn from an absolutely continuous probability distribution G with probability density function g and support $[0, \bar{v}]$, where $\bar{v} \in \mathbb{R} \cup \{\infty\}$. Assumption 1 still holds. For most of our results, we assume $v_M \leq \bar{v}$. The description of the auction formats is easily extended to the case with M goods.

Product-Mix auctions. We define (i) the bidding language, (ii) the allocation rule, and (iii) the payment rule.

(i) The bidding language allows single and paired bids.[17] The flexible bidder submits a bid $\mathbf{w} = (w_j)_{j=1,...,M}$ to state demand for 1 unit of good 1 *or* good 2 *or* ... *or* good M, but not more than one in total, at prices of up to w_j for good j, respectively. The inflexible bidders of group G_j submit a single bid each for good j.

(ii) The auctioneer is given a bid list $\mathbf{w}, (y_1, 0, ..., 0), ..., (0, ..., y_M)$. In the standard PMA, he then determines the efficient allocation given the reported values, i.e. the flexible bidder obtains

[17]Paired bids are an instance of XOR bids (see, e.g., Sandholm (2002)) where each component specifies the quantity and price of exactly one good. Let $\mathbf{p}$ denote the auction prices. A paired bid $\mathbf{w} = (w_j)_{j=1,...,M}$ expresses the following preference: if $b_k - p_k > b_j - p_j$ for all $j \neq k$, then the bidder would like to receive up to one unit of good k. If the inequality sign is replaced by an equality for some good j, the bidder is indifferent between receiving good k and j, assuming indivisibility of goods.

good k if and only if $w_k - y_k > w_j - y_j$ for all $j \neq k$, and $w_k - y_k \geq 0$.[18] In the first-price PMA, the same allocation rule applies. Together with the pricing rule, this is equivalent to choosing the revenue-maximising allocation, as in the menu auction by Bernheim and Whinston (1986).

(iii) In the standard PMA, the auctioneer determines the lowest competitive equilibrium prices. In our model, prices will always be set by the inflexible bidders' bids $y_j, j \in \mathcal{J}$.[19] In the first-price PMA, each winning bidder pays their bid price for the unit won.

Simultaneous auctions. Bidders can participate in any number of M simultaneous single-unit auctions. In particular, a bidder could choose to bid only in one auction, or in all of them. The auction price is set by the highest losing bid, respectively, under the second-price rule. A winner pays her bid under the first-price rule.

Sequential auctions. M single-unit auctions for one of the goods, respectively, are held sequentially. Bidders can participate in any number of the M auctions. After each auction, the auctioneer announces the winning allocation and the clearing price. Bidders then choose their bid in the subsequent auction. Payment rules are as in the simultaneous auction. There are $M!$ ways of ordering the sale. We are only interested in two orderings: the case where the goods are ordered according to the flexible bidder's values from lowest to highest, and reverse.

Remark: Again, the standard PMA (second-price rule) is equivalent to the VCG mechanism (see section 2).

3.1 Equilibrium bidding

As in the case with two goods, we make use of the following preliminary lemma. The proof from the two good case trivially extends to this model, and again, the flexible bidder never chooses a weakly dominated strategy, i.e. bids not more than her value on any good.

Lemma 3. *In the auction formats introduced above, at least two inflexible bidders in each group bid their true values in any equilibrium.*

Product-Mix auctions. Again, it is without loss of generality to assume that the flexible bidder makes one flat paired bid. Denote this bid $\mathbf{w} = (w_1, ..., w_M)$. In the same way, each bidder of group G_j makes a single bid $(0, ...0, x_j, 0, ...0), j \in \mathcal{J}$. Then, the efficient allocation is such that bidder F wins good k if and only if $w_k \geq x_k$ and $w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j$.[20] Given the equilibrium behaviour of all inflexible bidders, it is easy to argue the flexible bidder's optimal strategy, which straightforwardly extends the two-good model.

Proposition 11. *In the PMA with the second-price rule, there exists an equilibrium in which the flexible bidder bids truthfully. If $v_M \geq \bar{v}$, there exists a continuum of equilibria in which $w_M \geq \bar{v}$ and $w_j = \max\{w_M + v_j - v_M, 0\}$ $\forall j \in \mathcal{J}, j \neq M$.*

[18] Ties are broken in favour of higher value goods in case of indifference between the components of the paired bid, and in favour of the flexible bidder between her and the inflexible bidders; but this is not important for our analysis.

[19] In general, prices are determined in a linear programme. A detailed description of the price setting concept would go beyond the scope of our study and can be found in Baldwin and Klemperer (2019).

[20] The events $w_j = x_j$, $w_k = x_k$, and $w_j + x_k = w_k + x_j$ for $k \neq j \in \mathcal{J}$ occur with zero probability.

The flexible bidder's payoff function is not straightforward to construct with M goods. We skip this step and discuss the first-price PMA instead. The payoff function in the first-price PMA can easily be translated to the standard PMA.

In the first-price PMA, by straightforward extension of the two-good case, it is without loss of generality to assume that she makes a flat paired bid $\mathbf{w} = (w_1, ..., w_M)$. The inflexible bidders bid their value in equilibrium. Let $P_k(\mathbf{w})$ denote the probability of bidder F winning good k. She wins good k if and only if $w_k \geq x_k$ and $w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j$. So the probability of winning good k is given by

$$P_k(\mathbf{w}) = Prob(w_k \geq x_k \text{ and } w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j) \tag{6}$$

As a first step, we write bidder F's expected payoff function simply as

$$\Pi^F_{P1|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} P_k(\mathbf{w}) \left[v_k - w_k\right] \tag{7}$$

To find a closed form expression for $P_k(\mathbf{w})$, we first attempt to find some structure in the flexible bidder's equilibrium bids. Given her values $v_M \geq v_{M-1} \geq ... \geq v_1 > 0$, we can show that $w_{k'} > w_k$ for $k' < k$ cannot be part of an equilibrium, because switching the bid prices between good k and k' would guarantee a strictly higher profit.

Lemma 4. *In equilibrium, the flexible bidder's bids must be such that* $w_M \geq w_{M-1} \geq ... \geq w_1$.

Using Lemma 4 and the fact that all inflexible bidders bid truthfully, we can now write down an expression for $P_k(\mathbf{w})$. Let k denote the good that the flexible bidder wins. Let $\{l_1, l_2, ..., l_P\} =: \mathcal{F} \subseteq \mathcal{J}$ denote any possible subset of goods (including the empty set), which the flexible bidder obviously loses to the inflexible bidders, that is good $l \in \mathcal{F} \Leftrightarrow w_l < x_l$. Let σ denote the permutations of goods that the flexible bidder does *not obviously* lose to the inflexible bidders, i.e.

$$\sigma = \begin{pmatrix} j_1 & j_2 & ... & j_{M-P-1} \\ \sigma(j_1) & \sigma(j_2) & ... & \sigma(j_{M-P-1}) \end{pmatrix}$$

where $\{j_1, ..., j_{M-P-1}\} = \mathcal{J} \setminus \mathcal{F}$. A permutation σ may be interpreted as follows: given a realisation $\mathbf{x}$ and a bid $\mathbf{w}$, the order of σ corresponds to the "order of efficiency" among the goods the flexible bidder does not obviously lose to the inflexible bidders. That is, the permutation σ is such that $w_{\sigma(j_1)} - x_{\sigma(j_1)} \geq w_{\sigma(j_2)} - x_{\sigma(j_2)} \geq ... \geq w_{\sigma(j_{M-P-1})} - x_{\sigma(j_{M-P-1})}$. We also define $\sigma(j_0) := k$.

Further, let $\bar{x}_{\sigma(j)}$ denote the upper bound of the integral corresponding to integration over $x_{\sigma(j)}$, and let $\underline{x}_{\sigma(j)}$ denote the lower bound of the integral corresponding to integration over $x_{\sigma(j)}$.

Then, we have

$$
\begin{aligned}
P_k(\mathbf{w}) = &\sum_{\mathcal{F}\subseteq\mathcal{J}\setminus\{k\}} \sum_{\substack{\sigma(j):\\ j\in\mathcal{J}\setminus\{\mathcal{F}\cup\{k\}\}}} \int_{w_{l_1}}^{\bar{v}} \dots \int_{w_{l_P}}^{\bar{v}} \\
&\int_{\max\{\underline{x}_{\sigma(j_{M-P-2})}-\bar{x}_{\sigma(j_{M-P-2})}+x_{\sigma(j_{M-P-1})},0\}}^{w_{\sigma(j_{M-P-1})}} \int_{\max\{\underline{x}_{\sigma(j_{M-P-3})}-\bar{x}_{\sigma(j_{M-P-3})}+x_{\sigma(j_{M-P-2})},0\}}^{w_{\sigma(j_{M-P-2})}-w_{\sigma(j_{M-P-1})}+x_{\sigma(j_{M-P-1})}} \cdots \\
&\int_{\max\{\underline{x}_{\sigma(j_0)}-\bar{x}_{\sigma(j_0)}+x_{\sigma(j_1)},0\}}^{w_{\sigma(j_1)}-w_{\sigma(j_2)}+x_{\sigma(j_2)}} \int_0^{w_{\sigma(j_0)}-w_{\sigma(j_1)}+x_{\sigma(j_1)}} \\
&\mathrm{d}G(x_{\sigma(j_0)})\,\mathrm{d}G(x_{\sigma(j_1)}) \dots \mathrm{d}G(x_{\sigma(j_{M-P-2})})\,\mathrm{d}G(x_{\sigma(j_{M-P-1})})\,\mathrm{d}G(x_{l_P}) \dots \mathrm{d}G(x_{l_1})
\end{aligned} \tag{8}
$$

We are adding $\sum_{l=0}^{M-1}\binom{M-1}{l}(M-1-l)!$ integral terms.

Using equations (7) and the expression for P_k, bidder F's maximisation problem is

$$\max_{\mathbf{w}} \quad \Pi^F_{P1|\mathbf{v}}(\mathbf{w}) \quad \text{s.t. } 0 \leq \mathbf{w} \leq \mathbf{v}, \text{ and (7) and (8) hold}$$

The choice set $\mathcal{C}_M := [0, v_1] \times \dots \times [0, v_M]$ is an M-dimensional parallelepiped. We show that the function $\Pi^F(\mathbf{w})$ cannot attain its maximum on the boundary of $\mathcal{C}_M$. This immediately implies that the global optimum must be interior and coincide with a stationary point. Hence, the first-order conditions fully characterise the solution to bidder F's optimal bidding problem.

Proposition 12. *In the first-price PMA, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $(w_k{}^*)_{k=1,\dots,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla}\Pi^F_{P1|\mathbf{v}}(\mathbf{w}^*) = 0$.

The intuition behind this proposition is similar to the two-good case: the decrease in probability of winning a more favourable good is of a different (lower) order than the increase in probability of winning some good in events where otherwise the bidder would have won nothing. For small bids, the order effect outweighs that a higher value could be potentially achieved from winning more favourable goods, i.e. it is better to win more often overall, but potentially a lower-value good.

Simultaneous auctions. In a model with M simultaneous second-price auctions, we show that the flexible bidder always bids a strictly positive bid in each of the M auctions. Note that Gerding et al. (2008) have developed a similar result, when the goods are all identical to the flexible bidder.[21] However, we are interested in a model of differentiated goods, i.e. we maintain different values for different goods for the flexible bidder. Of course, dominated strategies we will never played, so bidder F's reduced choice set is again $\mathcal{C}_M := [0, v_1] \times \dots \times [0, v_M]$. Because each inflexible bidder of

[21] For certain distributions of the inflexible bidders' values, they provide additional results on equilibrium characterisation.

group G_k bids their value x_k in equilibrium, the flexible bidder's expected payoff function is

$$\Pi^F_{Sim2|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0^{w_k} x g(x) \,\mathrm{d}x \right]$$

and she solves the maximisation problem

$$\max_{\mathbf{w}} \quad \Pi^F(\mathbf{w}) \quad \text{s.t. } 0 \le \mathbf{w} \le \mathbf{v}$$

The first-order condition for w_j of the corresponding unconstrained problem is

$$w_j = -\sum_{k=1}^{j-1} v_k G(w_k) \prod_{l=k+1, l \neq j}^{M} (1 - G(w_l)) + v_j \prod_{l=j+1}^{M} (1 - G(w_l)) \tag{9}$$

Proposition 13. *In the simultaneous second-price auctions, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $(w_k{}^*)_{k=1,\dots,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla}\Pi^F_{Sim2|\mathbf{v}}(\mathbf{w}^*) = 0$.

Example 1. Note that symmetric values of the flexible bidder do not necessarily imply symmetric bids in equilibrium. To demonstrate this, let $M = 2$ and $v_1 = v_2 = v < 1$. We assume a probability distribution with support $[0, 1]$ and a uniform spike around $x \in [0, 1]$, where $x < v$. Let $h := \frac{1-\epsilon+2\epsilon^2}{2\epsilon}$ and $\epsilon < x$. Formally, the probability density function is

$$g(t) = \begin{cases} \epsilon & \text{if} \quad t < x - \epsilon \\ h & \text{if} \quad t \in [x - \epsilon, x + \epsilon) \\ \epsilon & \text{if} \quad t \ge x + \epsilon \end{cases}$$

Now let $\epsilon = 0.1$ (hence $h = 4.6$), $v = 0.7$, and $x = 0.5$. In the appendix, we show that the global maximum is at $(w_A{}^*, w_B{}^*) \approx (0.02, 0.70)$. Depending on how much probability mass concentrates around the spike, we may also obtain a symmetric equilibrium. For example, let $\epsilon = 0.4$ (hence $h = 1.15$), $v = 0.7$, and $x = 0.5$. The global maximum here is $(w_A{}^*, w_B{}^*) \approx (0.42, 0.42)$.

Our results under the second-price rule also hold under first-price payments. Although obvious that truthful bidding cannot be optimal in any one auction, it is not clear that the flexible bidder still makes a strictly positive bid in each auction. The flexible bidder's expected payoff function is

$$\Pi^F_{Sim1|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - G(w_k) w_k \right]$$

with the first-order conditions

$$w_j = -\sum_{k=1}^{j-1} v_k G(w_k) \prod_{l=k+1, l \neq j}^{M} (1 - G(w_l)) + v_j \prod_{l=j+1}^{M} (1 - G(w_l)) - \frac{G(w_j)}{g(w_j)} \tag{10}$$

In addition to the standard trade-off between payment and probability of winning in a single good first-price auction, increasing the probability to win one good decreases the expected payoff on any higher-value good.

Proposition 14. *In the simultaneous first-price auctions, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $(w_k{}^*)_{k=1,...,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla}\Pi^F_{Sim1|\mathbf{v}}(\mathbf{w}^*) = 0$*.*

Sequential auctions. When M auctions are held sequentially, the history of past auction outcomes at each stage is relevant for bidder F's bidding decision. There are $M!$ different orders in which the goods can be sold. We focus on the two most natural orders, which are of course ordering the goods by value from high to low, and from low to high. Let us assume first that goods are sold in the order $1, 2, ..., M$. In each auction, the flexible bidder knows which past auctions she has won, and at what price. Prices paid in the past are sunk costs, however, and do not matter in the backwards-induction. It only matters to the bidder by how much she could potentially improve her current payoff. Denote by H_j the history known in the auction of good $j+1$, i.e. auction outcomes of auctions $1, ..., j$. Let v^{max}_{j-1} denote the value of the highest-value object bidder F obtained in the history H_{j-1}.

Proposition 15. *In the sequential second-price auction, in which goods* $1, ..., M$ *are sold ordered from the lowest-value to the highest-value good, there exists an equilibrium, in which the flexible bidder's bid in the auction for good* j *is characterised by equation (45).*

When the order of sale is reversed, i.e. goods are sold in the order $M, ..., 1$, bidder F's strategy simplifies. She submits a strictly positive bid in each auction until she wins for the first time; thereafter she submits zero bids in all subsequent auctions. Details can be found in the appendix.

A similar result as above can be obtained for M sequential first-price auctions.

Proposition 16. *In the sequential first-price auction, in which goods* $1, ..., M$ *are sold ordered from the lowest-value to the highest-value good, there exists an equilibrium, in which the flexible bidder's bid in the auction for good* j *is characterised by equation (46).*

The case with the reverse sales order is again similar to the second-price auction and omitted.

3.2 Identical values

Equilibrium bidding can be illustrated further if we consider identical values for the flexible bidder, i.e. $v_j = v \ \forall j \in \mathcal{J}$. In the Product-Mix auction and in the simultaneous auctions, we restrict our analysis to symmetric equilibria.[22] In sequential auctions, equilibria are inherently asymmetric. We characterise equilibrium bidding, the flexible bidder's payoff function and the auctioneer's revenue for first-price auctions in appendix C.7. Revenue in second-price auctions is

[22] Symmetric equilibria may not exist for more extreme distributions of the inflexible bidders' values; see example 1.

identical for all auction formats, because auction prices are always given by the inflexible bidders' bids. We use the equilibrium characterisation in the example below.

Example 2. More insightful are some numerical computations for different numbers of goods M. In this example, let the flexible bidder's value $v = 0.9$ and let the inflexible bidders' values be uniformly distributed on $[0, 1]$. The flexible bidder's equilibrium bids and profits are shown in figures 3 - 6. Note that each dot of the sequential auction, corresponding to M goods, represents the equilibrium bid in the first of M auctions. Because the equilibrium is subgame-perfect, the bids up to good M also depict the sequence of equilibrium bids when M goods are sold. Under second-

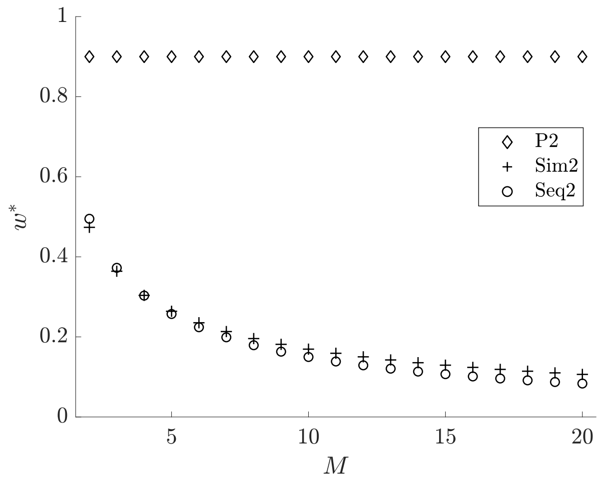


Figure 3: Equilibrium bids in second-price auctions

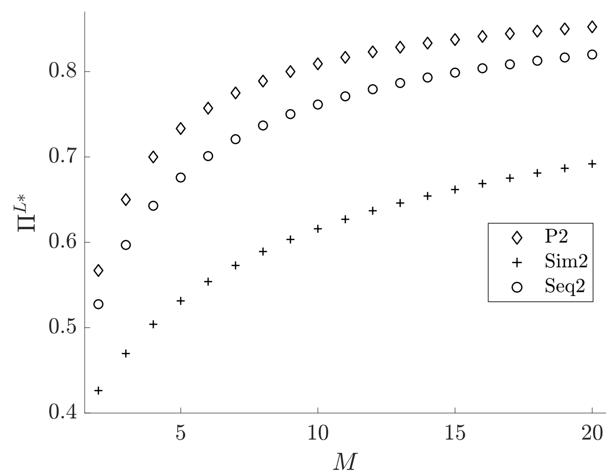


Figure 4: Bidder surplus in second-price auctions

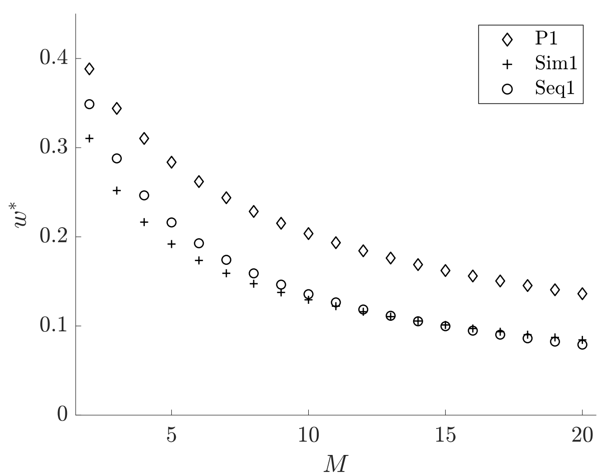


Figure 5: Equilibrium bids in first-price auctions

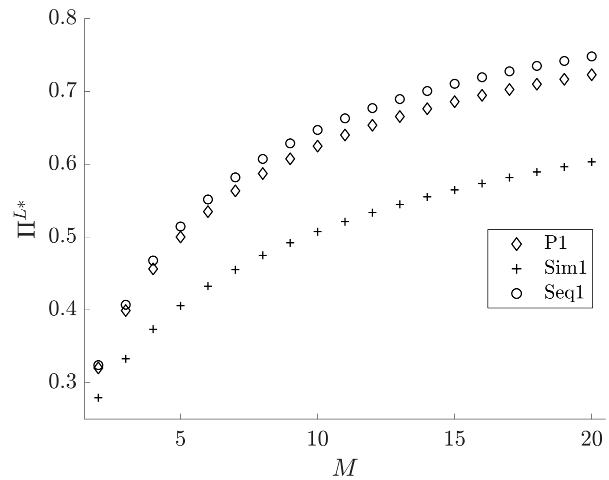


Figure 6: Bidder surplus in first-price auctions

price payments, as the number of goods increases, the flexible bidder increasingly shades her bids in the first of the sequential auctions. As evident from figure 3, with 20 goods, the optimal bid in the first of 20 sequential auctions is even lower than her optimal bid in 20 simultaneous auctions.[23]

[23] We are assuming symmetric equilibria in simultaneous auctions. Asymmetric equilibria may exist for distributions of the inflexible bidders' values with more concentrated probability mass than in this uniform example.

This translates directly into the flexible bidder's payoffs: differences between the sequential and the simultaneous auction are increasing in the number of goods. In both auction formats, the flexible bidder is more likely to win at least one of the goods with more goods for sale. However, in the sequential auction, she can better avoid the risk of winning more than one good. Interestingly, the differences in payoffs between the PMA and the sequential auction are first increasing and then decreasing slightly in the number of goods.

Under first-price payments, the effects observed in the flexible bidder's optimal bidding are similar to those under second-price payments. The optimal bids in the sequential and simultaneous auction move closer together as the sequential auction offers a higher probability to win exactly one good. The optimal bids in the first-price PMA compared to sequential and simultaneous auctions are moving further apart: the more goods are for sale, the less bid shading is necessary in the PMA relative to the other formats. The flexible bidder increasingly benefits from stronger bid shading in the sequential auction as the number of goods increases, comparatively to the PMA. The simultaneous auction does worse in terms of payoffs as more goods are sold: the risk of winning more than one good paired with the first-price disadvantage outweighs the benefit of a higher probability of winning at least one good.

Note that for approximately $M \geq 10$, the differences in payoffs stagnate for all auction formats. This suggests some kind of saturation in how much the flexible bidder can exploit the benefits of a specific auction format, as the number of goods for sale grows larger.

4 Numerical results

In this section, we derive and compute results on bidder surplus, revenue, and efficiency for the two-good model. First, we focus on the example where a and b are uniformly distributed on $[0, 1]$, and we provide a detailed discussion. Then, we also present results for different distributions of a and b, varying the strength of the competitive fringes (symmetrically). Many of our results turn out to be robust to changes in the prior on a and b. Bidder surplus is denoted Π^F_Y, revenue R_Y, and efficiency/welfare W_Y, where Y is substituted by the abbreviation for the respective auction format introduced in section 2.

4.1 Uniform distributions of a and b

First, we compare the bidder surplus, revenue, and efficiency given bidder F's values v_A and v_B (we call these "interim" outcomes). We restrict our attention to $(v_A, v_B) \in [0, 1]^2$. For the comparison with sequential auctions, we introduce a straightforward metric which combines the respective outcomes from the sequential format with good A sold first, and the format with good B sold first: $\Pi^F_{Seq2|v_A,v_B} = \frac{1}{2}\Pi^F_{SeqA2|v_A,v_B} + \frac{1}{2}\Pi^F_{SeqB2|v_A,v_B}$, and analogously for the first-price sequential auction.[24]

[24] This metric is due to the fact that sometimes we may not know which of two differentiated objects will be higher value for the buyer, e.g. a red car or a blue car, because it may depend more on taste than an objective difference in quality. Being unaware of the true ordering in value, we simply assume that each ordering happens with equal

For most auction formats, a closed form solution can be derived when a and b are uniform on $[0,1]$ (closed form solutions are stated in the appendix). The first-price PMA is an exception because the equilibrium bids are characterised implicitly, and comparisons with this auction are done numerically (implemented in *Matlab*; to compare closed form solutions we used *Mathematica*).

Interim bidder surplus. First we note the obvious: inflexible bidders make zero surplus in equilibrium in all auction types. Thus, the flexible bidder's equilibrium surplus equals overall bidder surplus. We state our results for $0 < v_A < v_B < 1$. When $v_A = 0$, $v_B = 1$, or $v_A = v_B$ (or a combination of these conditions), the flexible bidder may be indifferent between some of the auction types.[25] We find that

$$\Pi^F_{P2|v_A,v_B} > \Pi^F_{Seq2|v_A,v_B} > \Pi^F_{Sim2|v_A,v_B} \text{ and } \Pi^F_{Seq1|v_A,v_B} > \Pi^F_{P1|v_A,v_B} > \Pi^F_{Sim1|v_A,v_B}$$

The magnitude of the flexible bidder's preference for one auction format relative to another is illustrated in figure 7. We plot the relative deviation of her equilibrium payoff in the PMA from her payoff in the sequential and simultaneous format under the corresponding payment rule as a contour plot, i.e. the 2D-projection of given levels of relative deviations as a function of v_A and v_B. For example, the left-most line in Figure 7(a) describes all combinations of v_A and v_B for which the standard PMA results in a 1% higher payoff for the flexible bidder than the sequential second-price auction. The contour lines are mirrored at the 45-degree line because equilibria are symmetric for valuations $v_A \geq v_B$.

Interim revenue. We state our results for $0 < v_A < v_B \leq 1$. When $v_A = 0$, or $v_A = v_B$ (or a combination of these conditions), revenue may be identical for some of the auction types.[26]

Under the second-price rule, prices are always determined by the inflexible bidders' bids. We have

$$R_{Y|v_A,v_B} = E_{a,b}\left[a + b \mid v_A, v_B\right] = 1 \text{ and } R_{P1|v_A,v_B} > R_{Seq1|v_A,v_B} > R_{Sim1|v_A,v_B}$$

with $Y \in \{P2, Sim2, Seq2\}$. We illustrate the magnitude of the relative difference in revenues between one auction format and another in figure 8.

Interim efficiency. To compare efficiency between the auction formats, we compute welfare as the sum of the flexible bidder's payoff and revenue (closed form solutions are listed in the appendix). We state our results for $0 < v_A < v_B < 1$, and by composition of the previous comparisons, indifference may hold for $v_A = 0$ or $v_A = v_B$. We find that

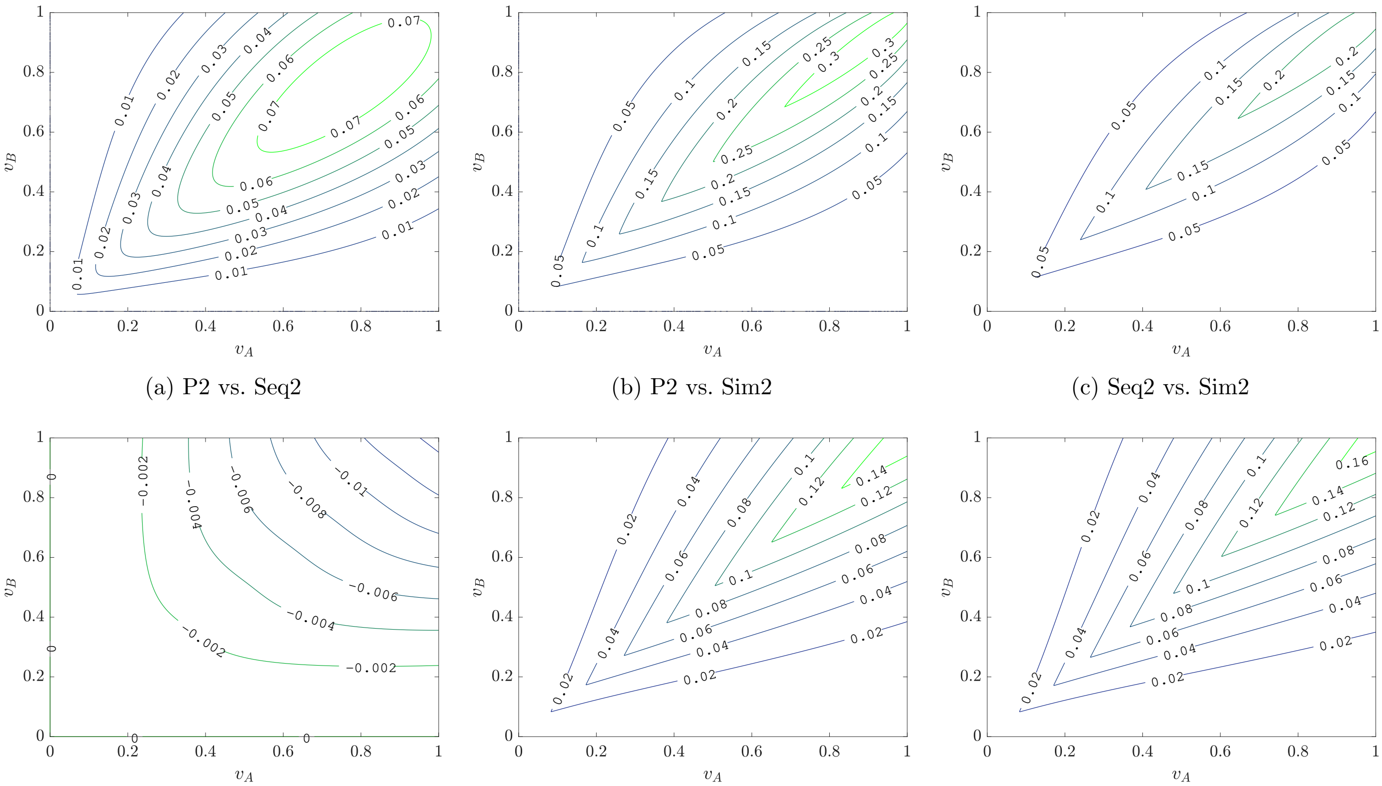
$$\mathcal{W}_{P2|v_A,v_B} > \mathcal{W}_{Seq2|v_A,v_B} > \mathcal{W}_{Sim2|v_A,v_B} \text{ and } \mathcal{W}_{P1|v_A,v_B} > \mathcal{W}_{Sim1|v_A,v_B}$$

The relation between (P1) and (Seq1) is ambiguous, depending on the values of v_A and v_B. The

possibility.

[25] When $v_A = v_B$, the sales order in sequential auctions is obviously irrelevant. When $v_B = 1$, the sales order in the sequential second-price auction is irrelevant. When $v_A = 0$, all auction types under the same pricing rule are strategically equivalent.

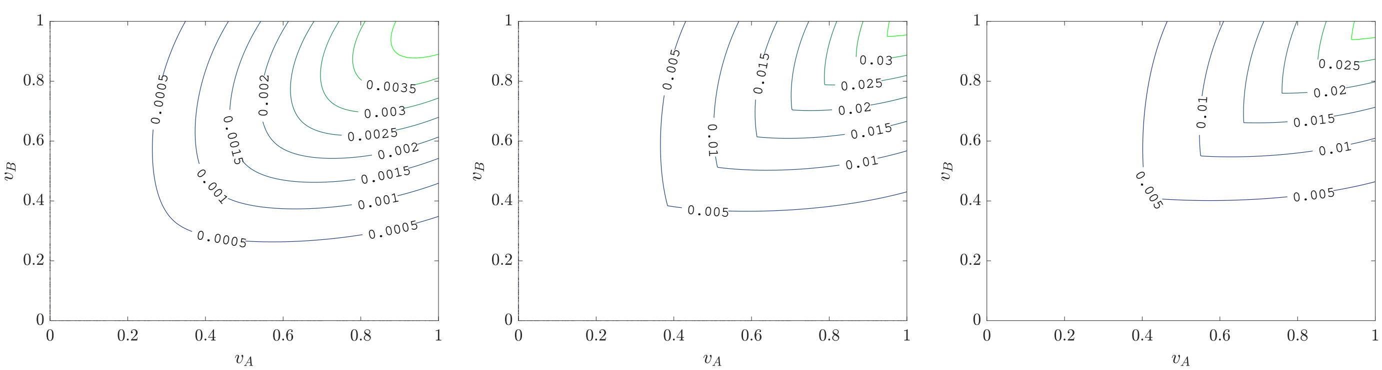
[26] Again, when $v_A = v_B$, the sales order in sequential auctions is irrelevant. When $v_A = 0$, all auction types under the same pricing rule are strategically equivalent, hence revenue is also the same.



(a) P2 vs. Seq2 (b) P2 vs. Sim2 (c) Seq2 vs. Sim2

(d) P1 vs. Seq1 (e) P1 vs. Sim1 (f) Seq1 vs. Sim1

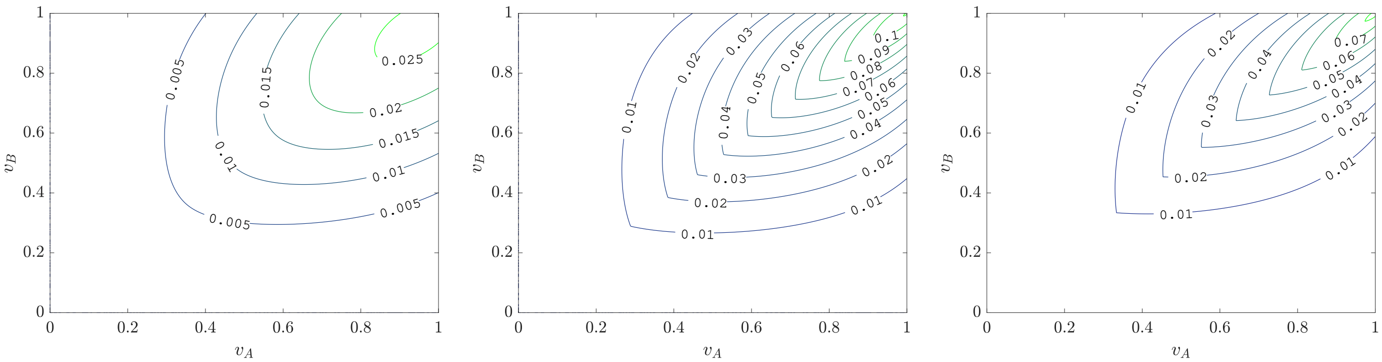
Figure 7: The flexible bidder's interim payoffs: "P2 vs. Seq2" shows the relative gain of mechanism $P2$ over mechanism $Seq2$, i.e. $(\Pi^F_{P2|v_A,v_B} - \Pi^F_{Seq2|v_A,v_B})/\Pi^F_{Seq2|v_A,v_B})$, etc.



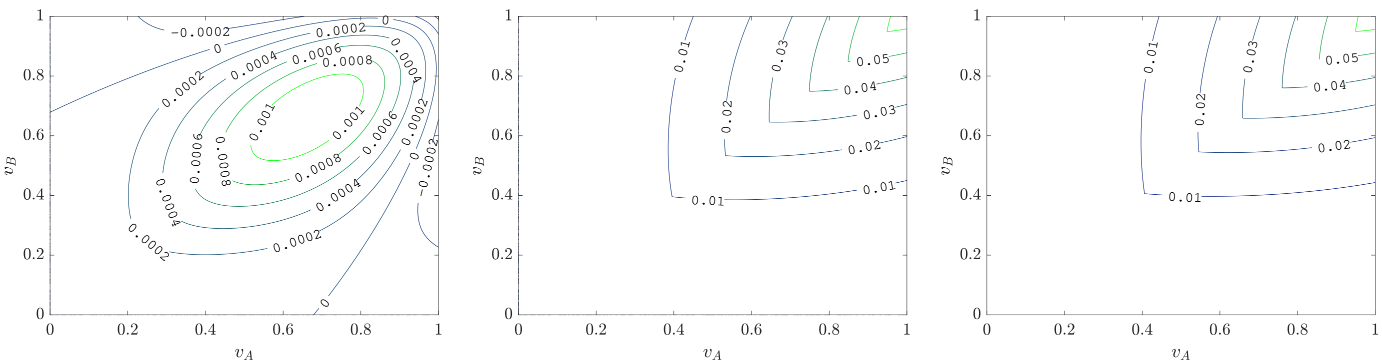
(a) P1 vs. Seq1 (b) P1 vs. Sim1 (c) Seq1 vs. Sim1

Figure 8: Interim revenue: "P1 vs. Seq1" shows the relative gain of mechanism $P1$ over mechanism $Seq1$, i.e. $(R_{P1|v_A,v_B} - R_{Seq1|v_A,v_B})/R_{Seq1|v_A,v_B})$, etc.

magnitudes of deviations are illustrated in figure 9.



(a) P2 vs. Seq2 (b) P2 vs. Sim2 (c) Seq2 vs. Sim2



(d) P1 vs. Seq1 (e) P1 vs. Sim1 (f) Seq1 vs. Sim1

Figure 9: Interim efficiency: “P2 vs. Seq2” shows the relative gain of mechanism $P2$ over mechanism $Seq2$, i.e. $(W_{P2|v_A,v_B} - W_{Seq2|v_A,v_B})/W_{Seq2|v_A,v_B})$, etc.

Ex-ante outcomes. We also derive average outcomes across all possible realisations of v_A and v_B, and we call these “ex-ante outcomes”. Table 1 shows the ex-ante bidder surplus, revenue, and efficiency rounded to 3 decimals. Note these are absolute values. They are computed assuming each value pair $(v_A, v_B) \in \mathbb{R}^2|0 < v_A < 1, v_A < v_B < 1$ occurring with equal probability.[27] Differences are very small; so in order to get a full and meaningful comparison, we distinguish the two sequential auction types in this table. The ordering of interim outcomes largely translates

Table 1: Ex-ante bidder surplus, revenue, and efficiency

$M =$	(P2)	(SeqA2)	(SeqB2)	(Sim2)	(P1)	(SeqA1)	(SeqB1)	(Sim1)
Π_M^L	0.283	0.272	0.275	0.258	0.149	0.149	0.151	0.142
R_M	1.000	1.000	1.000	1.000	1.069	1.067	1.068	1.062
W_M	1.283	1.272	1.275	1.258	1.218	1.215	1.219	1.204

into ex-ante outcomes, despite the disaggregation of the two sequential auctions. Under the second-

[27] If v_A and v_B were truly continuous, they would be drawn from a uniform distribution on $\{\mathbb{R}^2|0 < v_A < 1, v_A < v_B < 1\}$ with density function $g(v_A, v_B) = 2$. We compute the average for approx. 2M value pairs on the above domain.

price rule, the ordering Product-Mix, Sequential, Simultaneous holds, with the sequential auction with the higher-value good sold first performing better than the other sequential format. Under the first-price rule, the sequential format with the lower-value good sold first performs identical to Product-Mix in bidder surplus and revenue, but falls back in efficiency. In contrast, the sequential format with the higher-value good sold first outperforms the Product-Mix auction in bidder surplus and efficiency, but not in revenue. The first-price PMA yields the highest revenues across all auction formats, and, naturally, the standard PMA achieves the highest efficiency overall.

4.2 Discussion

In Table 2 we present several examples of equilibrium bids and interim outcomes for specific valuations v_A and v_B, which are helpful for understanding the intuition behind equilibrium outcomes. Note that the multi-column of the sequential auction formats displays the second-stage bid conditional on winning or losing the first stage, when applicable. Relating to payment rules and auction formats, we give some intuition for bidding behaviour and outcomes with the help of figures 7, 8, and 9.

First-price and second-price rule. First, we note a significant difference between first-price and second-price formats. The structure of our model means that first-price auctions are unusually bad for the flexible bidder because her competitors also bid truthfully under the first-price rule. This effect is even more pronounced when her values are small. To see this, consider the sale of a single unit. The flexible bidder always bids half of her value in a first-price auction,[28] whether her valuation is 0.1 or 1. In a second-price auction, she always bids truthfully. Now consider the sale of 2 units, one of each good. Then, in the standard PMA, she still bids truthfully for any values. In the first-price auction (e.g. the sequential), with small values, the flexible bidder simply repeats her strategy (approximately) from the single good case and bids (approximately) half her valuation, because the likelihood of winning one unit (or both units) is negligible. With high values, things are less bad for her: because she needs to win only one unit, she can still significantly shade her bids. This implies that, comparing payment rules, the difference in payoffs is large especially for small valuations.

Bidder surplus. Because of the structural difference between pricing rules, our focus is on comparing different auction formats under the same pricing rule. The fact that winning two units is unlikely with low valuations also has a straightforward implication for those comparisons. There is little difference between the flexible bidder's payoffs in different second-price or first-price auction formats for small valuations, and similarly for strongly asymmetric valuations, because the flexible bidder focuses her bidding mainly on the high value good.[29]

Sequential auctions. An important aspect in sequential auctions is the "scope for gambling" for the good sold in the first auction, i.e. bidding for it at a very low price with a low probability of success. The second auction offers the possibility to rebid in case of losing the first auction, or even

[28] Bidding half of her value maximises her expected payoff given the trade-off between maximising the probability of winning and minimising her payment.

[29] See figure 7(a)-(c), (e),(f).

to improve on one's payoff when the lower value good was sold first (i.e. the flexible bidder can make up for a "too low" bid in the first auction). Three related effects are worth noting.

(i) "Commitment effect." The "scope for gambling" is higher in second-price auctions than in first-price auctions, because paying her own bid diminishes the opportunity to make up for losing the first auction. In other words, under the first-price rule the flexible bidder is more nervous about entering the second auction. Hence it is optimal for her to commit more to her first-stage bid, that is to shade her bid less relative to her value.[30]

(ii) "Reversed bid effect." Another effect can be observed in the sequential auction where the higher value good is sold first: the ratio of bid price to value is always higher in the second auction. When both values are high, the flexible bidder may even choose to make a higher bid for the low value good than for the high value good.

(iii) "Preferred gamble." Keeping the pricing rule fixed, the sequential format is better for the flexible bidder when the higher value good is sold first, compared to when the lower value good is sold first. In the former case, she has a higher chance of winning her preferred good at a cheap price compared to the latter case, because "gambling" in the first auction may strike her a good bargain.

The sequential first-price auction does, on average, better for the flexible bidder than the first-price PMA. For low valuations the differences are negligible, and similarly for strongly asymmetric valuations: bidding behaviour from a single-unit sale is (approximately) replicated, and the "scope for gambling" is very low. This results in the contour lines in figure 7 (d). If values are high, in the sequential auction where the high value good is sold first, we have the "reverse bid effect". Reversing her bid order relative to her values is, intuitively, a bad bidding strategy in a simultaneous or Product-Mix format. In the sequential decision, however, this reverse bid ordering can make the flexible bidder better off ex-ante.[31,32] Note that the "commitment" effect is beneficial to the flexible bidder. Under the first-price rule, the sequential auction with the higher-value good sold first is, on average, indeed better for her. Stronger "gambling" behaviour (although optimal ex-ante) makes her worse off comparatively.

Simultaneous auctions. Under the second-price rule, it is also true that the flexible bidder's payoff is always weakly higher in the PMA than in the simultaneous auction. Any action the flexible bidder takes in the simultaneous auction is also available to her in the PMA (two separate bid instead of one paired bid would allowed), yet she chooses to bid differently and must therefore be better off. This holds for general distributions of a and b.

Under the first-price rule, our numerical results show that with uniform distributions, on average,

[30] This is illustrated for example in Table 2 for values $v_A = 0.5$ and $v_B = 1$: comparing the equilibrium bids between (SeqA2) and (SeqA1), ${w_A}^*_{SeqA2} = 0.125$ and ${w_A}^*_{SeqA1} = 0.156$, the commitment effect in the first auction is obvious.

[31] See Table 2 for $v_A = 0.9$ and $v_B = 1$: $\overline{w_A}^*_{SeqB1} = 0.450$ and ${w_B}^*_{SeqB1} = 0.399$, and payoffs are larger in the sequential auction than the first-price PMA.

[32] The relationship between the first-price PMA and the sequential auction with the lower value good sold first is ambiguous, in line with the flexible bidder's "preferred gamble" described above. For sufficiently dissimilar valuations the flexible bidder slightly prefers the first-price PMA, because the incentive to "gamble" for the higher value good introduces a loss in expectations compared to the flexibility between goods that the PMA offers. When values are similar and sufficiently high, however, the disadvantage from bidding for the lower value good first becomes less significant, and the flexible bidder can better utilise her market power from the information update.

the flexible bidder is better off in the PMA compared to the simultaneous auction (see section 4.1). Generally, however, there may be states where the flexible bidder is better off in the simultaneous auction.[33,34]

Revenue. Our model will naturally tend to generate higher prices for first-price than second-price payment rules. To see this, consider again the single unit case. Assume the flexible bidder has value v and competes against a number of inflexible bidders who always bid their value, which is uniform on $[0, 1]$. Then, in the first-price auction the flexible bidder bids $\frac{v}{2}$, and in the second-price auction she bids her value v. The auctioneer's revenue is higher under the first-price rule: $R_{FP} = \int_0^{v/2} \frac{v}{2}\,\mathrm{d}x + \int_{v/2}^1 x\,\mathrm{d}x = \frac{1}{2} + \frac{v^2}{8} > \frac{1}{2} = \int_0^1 x\,\mathrm{d}x = R_{SP}$. In the two-good case, the same effect - the inflexible bidders bid the same under the first-price and the second-price rule - means that first-price auctions outperform second-price auctions in terms of revenue. Intuitively, and obvious from the single-unit case, this effect is stronger for high valuations of the flexible bidder.

Due to the "commitment effect" in sequential first-price auctions described above, the flexible bidder's bid prices in the first-price PMA and sequential first-price auctions differ only slightly. Committing to a higher bid in the first auction is beneficial for the flexible bidder, whereas the "gambling" behaviour (although optimal ex-ante) makes her worse off comparatively. Obviously, this "gambling" is not only bad for the flexible bidder but also for the auctioneer: when values are high, the PMA's advantage becomes slightly more pronounced. The relationship between revenues in the sequential and the simultaneous auction is similar to "P1 vs. Sim" as described below (figure 8 (c)).

It is intuitive that prices in the first-price PMA are always higher than in the simultaneous first-price auction because expected losses from winning both goods are anticipated through higher bid shading. As explained above, differences for low and strongly asymmetric values of the flexible bidder are negligible, whereas high and more similar values correspond to higher anticipated losses and hence more extreme differences between the two auction formats (illustrated in figure 8(d)). The difference in bid shading between the first-price PMA and the simultaneous first-price auction is *greater on the low value good*, and thus the minimum of the two values predominantly determines the magnitude of the deviation in revenue ("Leontief-like" contour lines).[35]

Efficiency. It is clear that the standard PMA is efficient when each bidder's total demand does not exceed one unit. All other auction formats we consider do not achieve full efficiency due to the exposure to win more than one unit, or the first-price payment rule. Keeping the auction format fixed, the welfare increment in second-price auctions relative to first-price auctions, gained from a

[33]Such states must be such that she wins good A in the PMA because the distance between her bid and the inflexible bidders' bid is smaller on good A, but wins both goods in the simultaneous auction, and is still better off despite having to pay for both goods. For such scenario to arise, it would have to be true that $v_A - w_A^{*P1} < v_B - w_B^{*Sim1} - w_A^{*Sim1}$. Due to the complexity of the first-order conditions, we cannot show this to be true or false for general G. Note that under second-price payments this condition becomes $v_A - a < v_B - b - a$, which contradicts the flexible bidder having won good A in the PMA.

[34]Under the second-price rule, the inflexible bidders do not respond to the flexible bidder's behaviour. Under the first-price rule, if the inflexible bidders do not behave competitively among themselves, their behaviour has to take into account the flexible bidder's optimal shading, and vice versa. In this case, it is still true that the flexible bidder's strategy space expands in the PMA, but her bidding would have to account for the inflexible bidders' shading too.

[35]See for example Table 2 for $v_A = 0.5, v_B = 1$.

more efficient allocation to the flexible bidder, outweighs the welfare loss in terms of revenue due to her exercising some market power. Thus, second-price payments lead to higher welfare in each auction format considered.[36]

Due to the invariance of revenues in second-price formats in our model, differences in efficiency between those auction formats are entirely determined by the flexible bidder's profits. This naturally implies that figures 9(a) - (c) are just a reshaped version of the deviations in payoffs.[37] Comparing auction formats within the first-price rule, variations in the flexible bidder's payoffs are slightly stronger (most of the time), but revenues are much higher in absolute magnitudes. Hence, the *relative* difference in efficiency between auction formats is similar to the difference in revenues, when comparing simultaneous auctions to Product-Mix and sequential auctions. Comparing PMA and sequential auctions, the opposing effects of bidder surplus and revenue nearly cancel each other out.

4.3 Varying the competitive fringes

In this section, we vary the strength of the competitive fringes the flexible bidder is competing against. Again, we consider the model with two goods A and B for sale, and two competitive fringes. Each variation considered is identical for the two fringes. We present the analysis for first-price auctions only (but may extend this to second-price auctions in the future). In the previous section, we discussed the case where a and b are uniformly distributed on $[0, 1]$. Here, we examine distributions with probability density functions $g_k(x) = kx^{k-1}$ and $\widetilde{g}_k(x) = k(1-x)^{k-1}$, for $k \in \mathbb{N}$, and support $[0, 1]$. The corresponding distribution functions are $G_k(x) = x^k$ and $\widetilde{G}_k(x) = 1-(1-x)^k$.

Notice that these distributions have a straightforward interpretation in terms of increased or decreased competition. g_k corresponds to k independent groups, where each group is composed of at least two identical bidders in perfect competition. All k groups are interested in the same good. Values across groups are drawn independently from a uniform distribution on $[0, 1]$. The flexible bidder faces the maximum of k values, i.e. increased competition, represented by g_k. Similarly, $\widetilde{g}_k$ also corresponds to k independent groups, where each group is composed of at least two identical bidders in perfect competition. Again, values across groups are drawn independently from a uniform distribution on $[0, 1]$; however, the group with the lowest value among the k groups will be selected as the competition the flexible bidder faces.[38] The distributions are depicted in figure 10.

In each of the tables 3 - 5 in appendix 4.3, we compare two auction formats. We choose two different metrics: (i) the relative difference with (approximate) maximum absolute value on $[0, v_A] \times [0, v_B]$, denoted "max diff", and (ii) the relative difference of average outcomes across

[36]A trivial observation from table 2 is that absolute differences between the flexible bidder's payoffs under the first-price and the second-price rule are higher than absolute differences between revenues under the first-price and the second-price rule.

[37]Because we use relative deviations, the magnitudes of deviations in profits and revenues do not simply add up to magnitudes in efficiency deviations.

[38]One interpretation might be, that there are exactly two bidders in each group, and $2k - 1$ units of one respective good are for sale, i.e. decreased competition. Important here is the modelling: the flexible bidder competes only against the weakest of k groups.

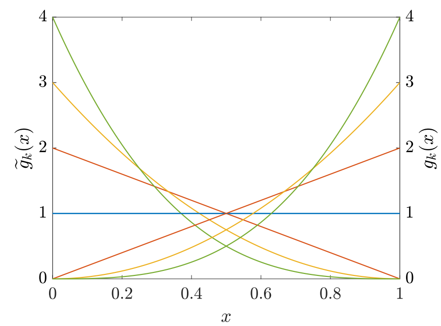


Figure 10: Probability density functions on $[0, 1]$

$[0, v_A] \times [0, v_B]$, denoted "avg diff". For example, in the second line in table 3 we read that, for the uniform distribution, efficiency in the PMA is maximally 6.5% higher than in the simultaneous auction, at values $(v_A, v_B) = (1, 1)$ of the flexible bidder. In line 9, we read that, on average, for the uniform distribution, efficiency in the PMA is 1.2% higher than in the simultaneous auction.

We calculated outcomes for 1326 values pairs $(v_A, v_B) \in \{(x, y)|0 \leq x, y \leq 1, x \leq y\}$.[39] In each comparison of auction formats, for each outcome dimension and for every distribution of a and b, the value pair with the maximum absolute value relative difference is selected among those pairs, and the average across those pairs is computed.[40]

In table 3, we compare the first-price PMA with the first-price simultaneous auction. Note that maximum differences in efficiency and revenue decrease as competition for the flexible bidder increases, but the maximum difference in bidder surplus increases first slightly, and then rather dramatically from 15.5% to 43.4%. Average values change in the same direction, but not with the same magnitude. In fact, differences in the average are nearly invariant to increased or decreased competition for efficiency and revenue. The difference in averages rises from 4.3% to 7.2% for bidder surplus, but by far not as much as the maximum difference. With high values and against strong competition, the flexible bidder has an incentive to bid much higher in the simultaneous auction; therefore, the exposure to win two units instead of one is much more pronounced compared to the PMA.

Comparing the first-price PMA with the first-price sequential auction with good A sold first in table 4, we find virtually no difference in efficiency and revenue, for maximum differences and differences in average. The flexible bidder does between 1.4% and 2.3% worse in the PMA, but this is also largely invariant to the strength of her competition. Differences in average bidder surplus disappear entirely. In table 5, we compare differences in outcomes between the first-price PMA and the first-price sequential auction with good B sold first. Varying the flexible bidder's

[39] For distributions of a and b other than the uniform distribution, we cannot obtain closed form solutions. Moreover, we only know for simultaneous and sequential auctions that the equilibrium is unique. In the first-price PMA, multiple equilibria, and multiple stationary points may exists. Our propositions guarantee interior equilibria; therefore, we simply search among all stationary points.

[40] We chose approx. 2M value pairs for calculating averages for the uniform distribution in section 4.1. With different distributions, the model is computationally much more demanding; therefore we need to dispense with much fewer value pairs. However, varying the number of pairs convinced us to achieve more than sufficient accuracy with 1326 pairs.

competition does not impact the differences by much. Simply comparing absolute values across different distributions, we naturally find that efficiency and revenue are improving significantly, whereas bidder surplus drops by a factor of 10 going from low to high competition (see tables 6-8).

To summarise, varying the degree of competition impacts the differences between simultaneous and PMA/sequential auctions. Maximum differences in efficiency and revenue decrease as competition increases. The maximum difference in bidder surplus increases with increasing competition. Average difference do not respond as much to varying competition, and comparisons between other auction formats are also largely invariant to the degree of competition.

5 Conclusion

This paper compares equilibrium bidding in Product-Mix, sequential, and simultaneous auctions. Considering first-price and second-price payment rules, we characterise unique equilibria in most auction formats under some regularity assumptions on value distributions. For the example of uniform distributions, equilibria are unique in all auction formats. Our results are derived for a model of indivisible goods, but all results go through for the divisible good case.

For a broad class of value distributions and for any number of goods, we demonstrate that the flexible bidder faces an exposure problem in simultaneous auctions. Because it is ex-ante optimal to participate in every auction with a strictly positive bid, she runs the risk of winning both goods, when she can use only one (assuming no possibility for resale). Our papers compares alternative formats that alleviate this exposure problem. Our answer is inconclusive: the Product-Mix and the sequential auction perform similarly in terms of efficiency, revenue, and bidder surplus; but subtle differences can be observed.

For uniform value distributions, the performance comparisons depend on the variation in the flexible bidder's values. Relative differences across auction formats, keeping the payment rule fixed, are most extreme for high values of the flexible bidder. The reason is that for small values the flexible bidder's probability of winning one (or both) goods is very small; her optimal strategy is to approximately repeat her optimal bid from the single-unit case twice. Averaging over the flexible bidder's values, we find that the first-price PMA performs best compared to other first-price auctions, in terms of revenue and efficiency. The flexible bidder, on the other hand, prefers the sequential format (on average) to the first-price PMA. Among second-price auctions, the sequential auction with the higher-value good sold first is the runner-up to the standard PMA, from the flexible bidder's and an efficiency perspective.

For first-price mechanisms, we vary the inflexible bidders' distribution to evaluate the robustness of our findings. We also obtain comparative statics with respect to the competition the flexible bidder is faced with. We select distributions that may be interpreted as varying the strength of the two competitive fringes. Differences between the first-price Product-Mix and first-price sequential formats are very subtle, and nearly invariant to changing the strength of competition. As competition becomes more fierce, the flexible bidder does increasingly worse in the simultaneous auction compared to an auction format eliminating the exposure to win both goods. In contrast, differences in efficiency and revenue between the PMA and the simultaneous auction become less pronounced with increased competition.

We plan to generate further insights into bidding behaviour in the three considered auction formats through a series of lab experiments. Beyond that, it remains an open problem to characterise equilibrium bidding under the first-price rule if inflexible bidders did not have identical competitors, or if two or more flexible bidders participated.

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Appendix

A Proofs for the 2-good case

Proof of Lemma 1. The proof hinges on the fact that there are (at least) two bidders bidding for good A only and two bidders bidding for good B only, as these two bidder engage in a Bertrand-type competition (the argument here is made for the case of $n = 2$, but is analoguous for $n > 2$).[41] First, it is obvious that no equilibrium bid made by any bidder can be higher than their valuation under discriminatory pricing. It is also easy to see that in any equilibrium two bidders of group G_k have to bid the same amount. We call those bidders 1_k and 2_k. If their bid wins, the unit of good k is randomly allocated among them with equal probability. Suppose they make different bids in equilibrium, say 1_k bids higher than 2_k. Then, in the case where 2_k wins, bidder 1_k could improve his payoff by bidding at least as much as bidder 1_k. In the case where 2_k loses, 1_k's payoff would remain unchanged by this deviation, i.e. bidder 2_k could deviate to improve his expected payoff. Now suppose that the two bidders with an identical valuation $v_k \in \{a, b\}$ bid $s_k < v_k$. Then, again, 1_k for example could profitably deviate by bidding $s_k + \epsilon \leq v_k$. He would win the unit of good k *always*, not only half of the time, at an ϵ-higher price in the cases where s_k is a winning bid, which is a strict payoff improvement. In the cases where s_k isn't winning, his payoff would either remain zero (if $s_k + \epsilon$ remains a losing bid or is winning at $s_k + \epsilon = v_k$) or increase strictly (if $s_k + \epsilon < v_k$ becomes a winning bid). It follows that $s_k = v_k$ is the only possible candidate for a BNE. □

A.1 Product-Mix auctions

Proof of Proposition 1. The flexible bidder's preference for her ex-post allocation coincides with the efficient allocation: she wants to win good A if and only if

$$v_A > a \qquad \text{and} \tag{11}$$

$$v_A - a > v_B - b \tag{12}$$

and good B if and only if

$$v_B > b \qquad \text{and} \tag{13}$$

$$v_B - b > v_A - a. \tag{14}$$

□

[41] Note that we are looking for pure strategy Bayes-Nash equilibria only.

We also establish bidder F's expected payoff function. We have

$$\begin{aligned}
\Pi^F_{P2|v_A,v_B}(w_A,w_B) &= P\,(\text{F wins A})\,\mathrm{E}[v_A - a \mid \text{F wins A}] \\
&\quad + P\,(\text{F wins B})\,\mathrm{E}[v_B - b \mid \text{F wins B}] \\
&\quad + \int_0^{w_A}\int_{w_B-w_A+a}^{\bar{v}} v_A - a\,\mathrm{d}G(b)\,\mathrm{d}G(a) \\
&= \int_0^{w_A}\int_0^{w_B-w_A+a} v_B - b\,\mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}}\int_0^{w_B} v_B - b\,\mathrm{d}G(b)\,\mathrm{d}G(a)
\end{aligned}$$

Proof of Lemma 2. First, note that because a and b are *iid* distributed, we have $P_A(w_B,w_A) = P_B(w_A,w_B)$. Now suppose $w_A > w_B$. Then we can easily show that bidding $\widetilde{w_A} = w_B$ and $\widetilde{w_B} = w_A$ is a profitable deviation:

$$\begin{aligned}
\Pi^F(\widetilde{w_A},\widetilde{w_B}) =& P_A(\widetilde{w_A},\widetilde{w_B})\,[v_A - \widetilde{w_A}] + P_B(\widetilde{w_A},\widetilde{w_B})\,[v_B - \widetilde{w_B}] \\
=& P_A(w_B,w_A)\,[v_A - w_B] + P_B(w_B,w_A)\,[v_B - w_A] \\
=& P_A(w_A,w_B)\,[v_B - w_A] + P_B(w_A,w_B)\,[v_A - w_B] \\
>& P_A(w_A,w_B)\,[v_A - w_A] + P_B(w_A,w_B)\,[v_B - w_B]
\end{aligned}$$

The last inequality holds because $P_A(w_A,w_B) > P_B(w_A,w_B)$ for $w_A > w_B$. □

Proof of Proposition 2. Putting together equations (1), (2), and (3), we obtain

$$\begin{aligned}
\Pi^F_{P1|v_A,v_B}(w_A,w_B) &= \left[\int_0^{w_A}\int_{w_B-w_A+a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a)\right](v_A - w_A) \\
&\quad + \left[\int_0^{w_A}\int_0^{w_B-w_A+a} \mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}}\int_0^{w_B} \mathrm{d}G(b)\,\mathrm{d}G(a)\right](v_B - w_B)
\end{aligned}$$

The first-order conditions write

$$\begin{aligned}
\frac{\partial \Pi^F_{P1|v_A,v_B}}{\partial w_A} &= (v_A - w_A)\left[g(w_A)\,(1 - G(w_B)) + \int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a)\right] \\
&\quad - (v_B - w_B)\int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a) - \int_0^{w_A}\int_{w_B-w_A+a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) = 0
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\frac{\partial \Pi^F_{P1|v_A,v_B}}{\partial w_B} &= (v_B - w_B)\left[g(w_B)\,(1 - G(w_A)) + \int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a)\right] \\
&\quad - (v_A - w_A)\int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a) \\
&\quad - \left[G(w_B)\,(1 - G(w_A)) + \int_0^{w_A}\int_0^{w_B-w_A+a} \mathrm{d}G(b)\,\mathrm{d}G(a)\right] = 0
\end{aligned} \tag{16}$$

Simplifying yields

$$\begin{aligned}&(v_A - {w_A}^*)\, g({w_A}^*)\,(1 - G({w_B}^*)) - \int_0^{{w_A}^*} \int_{{w_B}^* - {w_A}^* + a}^{\overline{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ (v_A - {w_A}^* - (v_B - {w_B}^*)) \int_0^{{w_A}^*} g({w_B}^* - {w_A}^* + a)\,\mathrm{d}G(a) = 0\end{aligned} \tag{17}$$

$$\begin{aligned}&[(v_B - {w_B}^*)\, g({w_B}^*) - G({w_B}^*)]\,(1 - G({w_A}^*)) - \int_0^{{w_A}^*} \int_0^{{w_B}^* - {w_A}^* + a} \mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ (v_B - {w_B}^* - (v_A - {w_A}^*)) \int_0^{{w_A}^*} g({w_B}^* - {w_A}^* + a)\,\mathrm{d}G(a) = 0\end{aligned} \tag{18}$$

Notice the asymmetry in the two first-order condition. This is because $w_A \leq w_B$. Setting $w_A = 0$, equation (18) reduces to the standard first-order condition we obtain in a single first-price auction in our setup, but the analogue is not possible with equation (17).

In order to prove that equations (15) and (16) indeed characterise a global optimum, we show that the maximum cannot be on the boundary of $[0, v_A] \times [0, v_B]$ and therefore must be a stationary point of $\Pi^F_{P1|v_A,v_B}$, characterised by the two equations above. First, we exploit symmetry properties of figure 1 for the case where $w_A \geq w_B$ to write bidder F's payoff function:

$$\begin{aligned}\overline{\Pi}^F_{P1|v_A,v_B}(w_A, w_B) = &\left[\int_0^{w_B} \int_0^{w_A - w_B + b} \mathrm{d}G(a)\,\mathrm{d}G(b) + \int_{w_B}^{\overline{v}} \int_0^{w_A} \mathrm{d}G(a)\,\mathrm{d}G(b)\right](v_A - w_A) \\ &+ \left[\int_0^{w_B} \int_{w_A - w_B + b}^{\overline{v}} \mathrm{d}G(a)\,\mathrm{d}G(b)\right](v_B - w_B)\end{aligned}$$

Consider the line segment $\{w_A \in [0, v_A], w_B = 0\}$. Trivially, the local optimum is at $({w_A}^*, 0)$, with ${w_A}^* = v_A - \frac{G({w_A}^*}{g({w_A}^*)}$, i.e. bidder F's optimal bid in a single first-price auction. However, this cannot be a global optimum because of Lemma 2. For the line segment $\{w_A = 0, w_B \in [0, v_B]\}$ we obtain $\Pi^F_{P1|v_A,v_B}(0, {w_B}^*) = G({w_B}^*)(v_B - {w_B}^*)$. For the line segment $\{w_A = v_A, w_B \in [0, v_B]\}$ we can also derive a first-order condition characterising a local maximum, denoted by $\widetilde{w_B}^*$ (clearly $(v_A, 0)$ and (v_A, v_B) are not maximal on this line segment). Similarly, we find a local maximum $\widetilde{w_A}^*$ on the line segment $\{w_A \in [0, v_A], w_B = v_B\}$. We show

$$\Pi^F_{P1|v_A,v_B}(v_A - \epsilon, \widetilde{w_B}^*) \quad > \quad \Pi^F_{P1|v_A,v_B}(v_A, \widetilde{w_B}^*) \tag{19}$$

$$\Pi^F_{P1|v_A,v_B}(\widetilde{w_A}^*, v_B - \epsilon) \quad > \quad \Pi^F_{P1|v_A,v_B}(\widetilde{w_A}^*, v_B) \tag{20}$$

$$\Pi^F_{P1|v_A,v_B}(\epsilon, {w_B}^*) \quad > \quad \Pi^F_{P1|v_A,v_B}(0, {w_B}^*) \tag{21}$$

To show (19), we simply graph the areas describing the probabilities of winning good A and good B respectively and consider an small change in one of the bids. In figure 11, bidder F receives a positive payoff only in the lighter area. With an small deviation ϵ this area remains unchanged while at the same time a strictly positive payoff is gained in the dark area below the dashed line, and therefore (19) holds. Figure 12 relates analogously to equation (20) (positive payoff is initially made only in the darker area).

Finally, we show equation (21). In figure 13, the ϵ-deviation causes the probability of winning good B to decrease (light area shrinks by dark triangle), and the probability of winning good A to increase (dark triangle plus dark rectangle). The profits made on good A and good B are $v_A - \epsilon$ and $v_B - \widetilde{w_B}^*$, respectively. From the graph, it is easy to see that we can bound the changes in expected profits on good K, denoted by $\Delta\Pi_K^F$ for $K = A, B$, as follows:

$$\begin{aligned}\Delta\Pi_A^F &> (v_A - \epsilon)P(a < \epsilon)P(b > \widetilde{w_B}^*)\\ \Delta\Pi_B^F &< (v_B - \widetilde{w_B}^*)P(a < \epsilon)P(b \in [\widetilde{w_B}^*, \widetilde{w_B}^* - \epsilon])\end{aligned}$$

Thus, we have $\Delta\Pi_A^F - \Delta\Pi_B^F > 0$ for ϵ small. $\square$

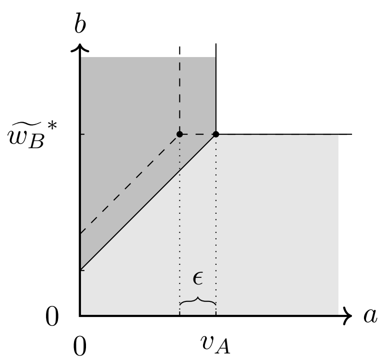

Figure 11: ϵ-deviation from $(v_A, \widetilde{w_B}^*)$

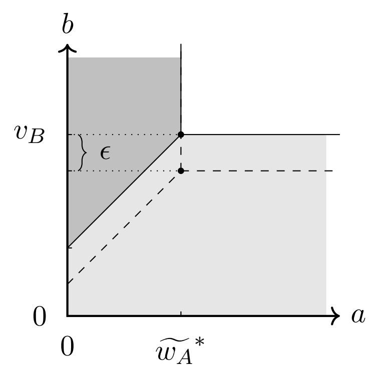

Figure 12: ϵ-deviation from $(\widetilde{w_A}^*, v_B)$

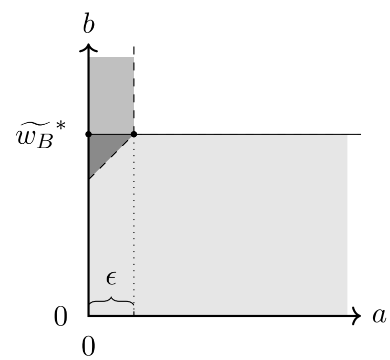

Figure 13: ϵ-deviation from $(0, \widetilde{w_B}^*)$

A.2 Sequential auctions

Proof of Proposition 3. (Good A sold first)
After having won the A-auction, bidder F's expected payoff function in the B-auction is

$$\begin{aligned}\Pi_{SeqA2|v_A,v_B}^{F|\text{won A}}(w_B) =& P\left(\text{F wins B}\right)\left(v_B - \mathrm{E}[b \mid \text{F wins B}]\right) + P(\text{F loses B})v_A - a\\ =& G(w_B)v_B - \int_0^{w_B} b\,\mathrm{d}G(b) + (1 - G(w_B))v_A - a\end{aligned}$$

The first-order condition yields $w_B^* = v_B - v_A$. The second-order condition $-g(w_B^*) < 0$ is sufficient for a local maximum, and global optimality can be shown by an ϵ-deviation from $w_B = v_B$ and $w_B = 0$. Bidder F's conditional expected second stage payoff is $\Pi_{SeqA2|v_A,v_B}^{F|\text{won A}}(w_B^*) = \int_0^{v_B - v_A} G(b)\,\mathrm{d}b + v_A - a$. After having lost the A-auction, bidder F bids her true valuation, i.e. $\overline{w_B}^* = v_B$ in the B-auction with a conditional expected second stage payoff of $\Pi_{SeqA2|v_A,v_B}^{F|\text{lost A}}(\overline{w_B}^*) =$

$\int_0^{v_B} G(b)\,\mathrm{d}b$. Now we can analyse the first stage auction (A-auction). Bidder F's expected payoff is

$$\begin{aligned}\Pi^F_{SeqA2|v_A,v_B}(w_A) =& P\,(\text{F wins A})\,\mathrm{E}[\Pi^{F|\text{won A}}_{SeqA2|v_A,v_B}({w_B}^*) \mid \text{F wins A}]\\ &+ P\,(\text{F loses A})\,\mathrm{E}[\Pi^{F|\text{lost A}}_{SeqA2|v_A,v_B}(\overline{w_B}^*) \mid \text{F loses A}]\\ =& \int_0^{w_A} \left(\int_0^{v_B-v_A} G(b)\,\mathrm{d}b + v_A - a\right)\mathrm{d}G(a) + \int_{w_A}^{\overline{v}} \left(\int_0^{v_B} G(b)\,\mathrm{d}b\right)\mathrm{d}G(a)\\ =& G(w_A)\left(v_A - w_A - \int_{v_B-v_A}^{v_B} G(b)\,\mathrm{d}b\right) + \int_0^{w_A} G(a)\,\mathrm{d}a + \int_0^{v_B} G(b)\,\mathrm{d}b\end{aligned}$$

The first-order condition yields ${w_A}^* = v_A - \int_{v_B-v_A}^{v_B} G(b)\,\mathrm{d}b$. It is not hard to see that the RHS is always weakly greater than zero, and local second-order conditions and global optimality are satisfied. Bidder F bids her true first stage valuation for the good, that is the difference in the expected second stage payoffs conditional on winning or losing in the first auction, respectively, and not accounting for the sunk cost a. □

Proof of Proposition 4. (Good B is sold first)
After having won the B-auction, the flexible bidder cannot improve on her allocation anymore, so trivially ${w_A}^* = 0$. After having lost the B-auction, bidder F simply bids her true valuation, i.e. $\overline{w_A}^* = v_A$ in the A-auction with a conditional expected second stage payoff of $\Pi^{F|\text{lost B}}_{SeqB2|v_A,v_B}(\overline{w_A}^*) = \int_0^{v_A} G(a)\,\mathrm{d}a$. Now we can analyse the first stage auction (B-auction). Bidder F's payoff is

$$\begin{aligned}\Pi^F_{SeqB2|v_A,v_B}(w_B) =& P\,(\text{F wins B})\,\mathrm{E}[\Pi^{F|\text{won B}}_{SeqB2|v_A,v_B}({w_A}^*) \mid \text{F wins B}]\\ &+ P\,(\text{F loses B})\,\mathrm{E}[\Pi^{F|\text{lost B}}_{SeqB2|v_A,v_B}(\overline{w_A}^*) \mid \text{F loses B}]\\ =& \int_0^{w_B} (v_B - b)\,\mathrm{d}G(b) + \int_{w_B}^{\overline{v}} \int_0^{v_A} G(a)\,\mathrm{d}a\,\mathrm{d}G(b)\\ =& G(w_B)\left(v_B - w_B - \int_0^{v_A} G(a)\,\mathrm{d}a\right) + \int_0^{w_B} G(b)\,\mathrm{d}b + \int_0^{v_A} G(a)\,\mathrm{d}a\end{aligned}$$

The first-order condition yields ${w_B}^* = v_B - \int_0^{v_A} G(a)\,\mathrm{d}a$. The RHS is always weakly greater than zero, and local second-order condition and global optimality are satisfied. Again, the flexible bidder bids her true first stage valuation for the good, that is the difference in the expected second stage payoffs conditional on winning or losing in the first auction, respectively, and not accounting for the sunk cost b. □

Proof of Proposition 5. (Good A is sold first)
After having won the A-auction, bidder F's expected payoff function in the B-Auction is

$$\begin{aligned}\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}(w_B) =& P\,(\text{F wins B})\,(v_B - w_B) + P(\text{F loses B})v_A - w_A\\ =& G(w_B)\,(v_B - w_B) + (1 - G(w_B))v_A - w_A\end{aligned}$$

The first-order condition yields ${w_B}^* + \frac{G({w_B}^*)}{g({w_B}^*)} = v_B - v_A$, the RHS of which is strictly positive. The function $H(x) := x + \frac{G(x)}{g(x)}$ is defined on $[0, v_B]$ and continuous, with $H(0) = 0$ and $H(v_B) > v_B$. By the intermediate value theorem and Assumption 1 (weakly decreasing reverse hazard rate), a solution $0 < {w_B}^* < v_B$ must exists, is unique, and is characterised by the first-order condition. Bidder F's conditional expected second stage payoff is $\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}({w_B}^*) = G({w_B}^*)(v_B - {w_B}^* - v_A) + v_A - w_A$. After having lost the A-auction, bidder F's expected payoff function in the B-Auction is

$$\begin{aligned}\Pi^{F|\text{lost A}}_{SeqA1|v_A,v_B}(w_B) =& P\,(\text{F wins B})\,(v_B - w_B)\\ =& G(w_B)\,(v_B - w_B)\end{aligned}$$

Bidder F's bid then corresponds to the optimal bid in a single good first-price auction, i.e. ${\overline{w_B}}^* = v_B - \frac{G({\overline{w_B}}^*)}{g({\overline{w_B}}^*)}$. With the same argument as above, the unique solution must satisfy $0 < {\overline{w_B}}^* < v_B$. Her conditional expected second stage payoff of $\Pi^{F|\text{lost J}}_{SeqA1|v_A,v_B}({\overline{w_B}}^*) = G({\overline{w_B}}^*)(v_B - {\overline{w_B}}^*)$. In the first stage auction (A-auction), bidder F's payoff is then

$$\begin{aligned}\Pi^{F}_{SeqA1|v_A,v_B}(w_A) =& P\,(\text{F wins A})\,\mathrm{E}[\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}({w_B}^*) \mid \text{F wins A}]\\ &+ P\,(\text{F loses A})\,\mathrm{E}[\Pi^{F|\text{lost A}}_{SeqA1|v_A,v_B}({\overline{w_B}}^*) \mid \text{F loses A}]\\ =& G(w_A)\,[G({w_B}^*)(v_B - {w_B}^* - v_A) + v_A - w_A]\\ &+ (1 - G(w_A))G({\overline{w_B}}^*)(v_B - {\overline{w_B}}^*)\end{aligned}$$

The first-order condition yields

$$ {w_A}^* + \frac{G({w_A}^*)}{g({w_A}^*)} = G({w_B}^*)(v_B - {w_B}^* - v_A) + v_A - G({\overline{w_B}}^*)(v_B - {\overline{w_B}}^*) \tag{22}$$

$$\Leftrightarrow \quad {w_A}^* + \frac{G({w_A}^*)}{g({w_A}^*)} = \frac{G({w_B}^*)^2}{g({w_B}^*)} - \frac{G({\overline{w_B}}^*)^2}{g({\overline{w_B}}^*)} + v_A \tag{23}$$

$H(x) := \frac{G(x)^2}{g(x)}$ is an increasing function because of Assumption 1. From above, it is clear that ${w_B}^* < {\overline{w_B}}^*$, so the RHS of equation 23 is weakly smaller than v_A. $\widetilde{G}(x) := x + \frac{G(x)}{g(x)}$ is defined on $[0, v_A]$, continuous, with $\widetilde{G}(0) = 0$ and $\widetilde{G}(v_A) > v_A$. So by the intermediate value theorem and Assumption 1, the unique solution ${w_A}^*$ must satisfy ${w_A}^* < v_A$. If the RHS of equation 23 is negative, however, the FOC does not yield a solution. In that case, the optimum must lie on the boundary of $[0, v_A]$. In a standard first-price auction, both alternatives, bidding zero and your value yield zero profit. Here, however, bidding zero is strictly better than bidding your value, because the first-stage payment may be a sunk cost in case bidder F wins in the second stage. Formally, we show the following lemma.

Lemma 5. *${w_A}^* = v_A$ is never optimal as bidder F's first stage auction bid.*

Proof.

$$
\begin{aligned}
&\Pi^F_{SeqA1|v_A,v_B}(v_A-\epsilon) = \\
&G(v_A-\epsilon)\left[G(w_B{}^*)(v_B-w_B{}^*-v_A)+\epsilon-G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*)\right] \\
&+G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*) \\
&=G(v_A-\epsilon)\left[\frac{G(w_B{}^*)^2}{g(w_B{}^*)}+\epsilon-\frac{G(\overline{w_B}{}^*)^2}{g(\overline{w_B}{}^*)}\right]+G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*) \\
&>G(v_A)\left[\frac{G(w_B{}^*)^2}{g(w_B{}^*)}-\frac{G(\overline{w_B}{}^*)^2}{g(\overline{w_B}{}^*)}\right]+G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*) \\
&=G(v_A)\left[G(w_B{}^*)(v_B-w_B{}^*-v_A)-G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*)\right] \\
&+G(\overline{w_B}{}^*)(v_B-\overline{w_B}{}^*)=\Pi^F_{SeqA1|v_A,v_B}(v_A)
\end{aligned}
$$

for small ϵ. The inequality holds because $\overline{w_B}{}^* > w_B{}^*$ and the reverse hazard rate is weakly decreasing. □

Together with equation (22) it is not hard to see that the following holds: fix v_B and let $\widehat{v_A} := G(\overline{w_B}{}^*(\widehat{v_A},v_B))(v_B-\overline{w_B}{}^*(\widehat{v_A},v_B)) - G(w_B{}^*(\widehat{v_A},v_B))(v_B-w_B{}^*(\widehat{v_A},v_B)-v_A)$. Then for all $v_A \leq \widehat{v_A}$, $w_A^* = 0$ (followed by $\overline{w_B}{}^*$ in the second stage auction). □

Proof of Proposition 6. (Good B sold first)
After having won the B-auction, bidder F's expected payoff function in the A-Auction is

$$
\begin{aligned}
\Pi^{F|\text{won B}}_{SeqB1|v_A,v_B}(w_A) =& P\,(\text{F wins A})\,(v_B-w_A)+P(\text{F loses A})v_B-w_B \\
=& G(w_A)\,(v_B-w_A)+(1-G(w_A))v_B-w_B
\end{aligned}
$$

Clearly bidder F cannot improve upon her payoff by bidding again in the A-auction after having won the B-auction, so $w_A{}^* = 0$. After having lost the B-auction, bidder F's bid corresponds to the optimal bid in a single good first-price auction, i.e. $\overline{w_A}{}^* = v_A - \frac{G(\overline{w_A}{}^*)}{g(\overline{w_A}{}^*)}$. Again, the unique solution must satisfy $0 < \overline{w_A}{}^* < v_A$. Her conditional expected second stage payoff of $\Pi^{F|\text{lost B}}_{SeqB1|v_A,v_B}(\overline{w_A}{}^*) = G(\overline{w_A}{}^*)(v_A-\overline{w_A}{}^*)$. In the first stage auction (B-auction), bidder F's payoff is then

$$
\begin{aligned}
\Pi^F_{SeqB1|v_A,v_B}(w_B) =& P\,(\text{F wins B})\,\mathrm{E}[\Pi^{F|\text{won B}}_{SeqB1|v_B,v_A}(w_A{}^*)\mid \text{F wins B}] \\
&+P\,(\text{F loses B})\,\mathrm{E}[\Pi^{F|\text{lost B}}_{SeqB1|v_B,v_A}(\overline{w_A}{}^*)\mid \text{F loses B}] \\
=& G(w_B)\,[v_B-w_B]+(1-G(w_B))G(\overline{w_A}{}^*)(v_A-\overline{w_A}{}^*)
\end{aligned}
$$

The first-order condition yields

$$
w_B{}^* + \frac{G(w_B{}^*)}{g(w_B{}^*)} = v_B - G(\overline{w_A}{}^*)(v_A-\overline{w_A}{}^*) \tag{24}
$$

With the same reasoning as above, a unique solution ${w_B}^*$ exists and must satisfy $0 < {w_B}^* < v_B$. $\square$

A.3 Simultaneous auctions

Proof of Proposition 7. Using simply the probabilities of winning and expected payments, we can define bidder F's expected payoff from bidding $w_A, w_B \geq 0$ in the A-auction and B-auction respectively as

$$\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(w_A, w_B) =& P\,(\text{F wins A, not B})\,\mathrm{E}[v_A - a \mid \text{F wins A}] \\
&+ P\,(\text{F wins B, not A})\,\mathrm{E}[v_B - b \mid \text{F wins B}] \\
&+ P\,(\text{F wins A and B})\,\mathrm{E}[v_B - a - b \mid \text{F wins A and B}] \\
=& G(w_A)\,(1 - G(w_B))\,(v_A - \mathrm{E}[a \mid w_A]) \\
&+ G(w_B)\,(1 - G(w_A))\,(v_B - \mathrm{E}[b \mid w_B]) \\
&+ G(w_A)G(w_B)\,(v_B - \mathrm{E}[a \mid w_A] - \mathrm{E}[b \mid w_B])
\end{aligned}$$

with $\mathrm{E}[x \mid w] = G(w)^{-1} \int_0^w x\,\mathrm{d}G(x)$ (the expected price conditional having won the good with a bid of $w > 0$). The payoff function can be simplified as shown in the main text and any stationary point of $\Pi^F_{Sim2|v_A,v_B}$ is characterised by

$$\boldsymbol{\nabla}\Pi^F_{Sim2|v_A,v_B}(w_A, w_B) = \begin{pmatrix} g(w_A)\,[v_A - w_A] - g(w_A)G(w_B)v_A \\ g(w_B)\,[v_B - w_B] - g(w_B)G(w_A)v_A \end{pmatrix} = 0, \tag{25}$$

which can be simplified to obtain

$$\begin{aligned}
{w_A}^* &= v_A(1 - G({w_B}^*)) &(26)\\
{w_B}^* &= v_B - v_A G({w_A}^*) &(27)
\end{aligned}$$

We prove that the necessary conditions are indeed sufficient to characterise the global maximum of bidder F's maximisation problem. Instead of a second-order approach, we argue by eliminating other equilibrium candidates on the boundary of bidder F's choice set. Note that the objective function is continuous and the admissible range a compact set. Therefore, the maximum is obtained on $\mathcal{M} := [0, v_A] \times [0, v_B]$. This can either happen at the boundary or at stationary point of $\Pi^F_{Sim2|v_A,v_B}$. Below, we demonstrate that no point on the boundary of $\mathcal{M}$ can be optimal, and therefore, the unique stationary point must coincide with the global maximum. We transform bidder F's payoff function once more with integration by parts and obtain

$$\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(w_A, w_B) =& G(w_A)\,[v_A - w_A] + \int_0^{w_A} G(a)\,\mathrm{d}a \\
&+ G(w_B)\,[v_B - w_B] + \int_0^{w_B} G(b)\,\mathrm{d}b - G(w_A)G(w_B)v_A
\end{aligned}$$

First, consider the line segment $\{w_A \in [0, v_A], w_B = 0\}$. Trivially, all points in this set are payoff-dominated by the point $(v_A, 0)$. We obtain $\Pi^F_{Sim2|v_A,v_B}(v_A, 0) = \mathcal{F}(v_A)$. Similarly, for the line segment $\{w_A = 0, w_B \in [0, v_B]\}$ we obtain $\Pi^F_{Sim2|v_A,v_B}(0, v_B) = \mathcal{F}(v_B)$. Now consider the set $\{w_A = v_A, w_B \in [0, v_B]\}$. From the first-order condition on w_B and $\frac{\partial^2 \Pi^F_{Sim2|v_A,v_B}}{\partial {w_B}^2}$ we find and confirm a local maximum at $\widetilde{w_B}^* = v_B - v_A G(v_A)$. We have $\Pi^F_{Sim2|v_A,v_B}(v_A, \widetilde{w_B}^*) = \mathcal{F}(v_A) + \mathcal{F}(\widetilde{w_B}^*)$. Similarly, in the set $\{w_A \in [0, v_A], w_B = v_B\}$, we find a local maximum at $\widetilde{w_A}^* = v_A - v_A G(v_B)$ with $\Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B) = \mathcal{F}(\widetilde{w_A}^*) + \mathcal{F}(v_B)$. Finally, we have to consider the point (v_A, v_B) with $\Pi^F_{Sim2|v_A,v_B}(v_A, v_B) = \mathcal{F}(v_A) + \mathcal{F}(v_B) - v_A G(v_A) G(v_B)$. $(v_A, 0)$ and $(0, v_B)$ are obviously payoff-dominated.

We proceed to show

$$
\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, \widetilde{w_B}^*) &> \Pi^F_{Sim2|v_A,v_B}(v_A, \widetilde{w_B}^*) &(28)\\
\Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B - \epsilon) &> \Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B) &(29)\\
\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, v_B) &> \Pi^F_{Sim2|v_A,v_B}(v_A, v_B) &(30)
\end{aligned}
$$

To show (28) to (30) we use linear and quadratic approximations of $G(v_k)$ and $g(v_k)$ at $v_k - \epsilon, k = A$ or B, respectively. We have

$$
\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, \widetilde{w_B}^*) =& G(v_A - \epsilon)\epsilon + \int_0^{v_A - \epsilon} G(a)\,\mathrm{d}a + \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b \\
&+ G(\widetilde{w_B}^*) v_A G(v_A) - G(\widetilde{w_B}^*) v_A G(v_A - \epsilon) \\
=& G(v_A - \epsilon)\epsilon + \int_0^{v_A - \epsilon} G(a)\,\mathrm{d}a + \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b \\
&+ G(\widetilde{w_B}^*) v_A \left[G(v_A - \epsilon) + g(v_A - \epsilon)\epsilon + o(\epsilon^2) - G(v_A - \epsilon)\right] \\
>& \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b + \int_0^{v_A - \epsilon} G(a)\,\mathrm{d}a + G(v_A - \epsilon)\epsilon + \frac{\epsilon^2}{2} g(v_A - \epsilon) + o(\epsilon^3) \\
=& \Pi^F_{Sim2|v_A,v_B}(v_A, \widetilde{w_B}^*)
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, v_B) =& G(v_A - \epsilon)\epsilon + \int_0^{v_A - \epsilon} G(a)\,\mathrm{d}a + \int_0^{v_B} G(b)\,\mathrm{d}b - G(v_A - \epsilon) G(v_B) v_A \\
>& \int_0^{v_B} G(b)\,\mathrm{d}b + \int_0^{v_A - \epsilon} G(a)\,\mathrm{d}a + G(v_A - \epsilon)\epsilon + \frac{\epsilon^2}{2} g(v_A - \epsilon) + o(\epsilon^3) \\
&- v_A G(v_B) \left[G(v_A - \epsilon) + g(v_A - \epsilon)\epsilon + o(\epsilon^2)\right] \\
=& \Pi^F_{Sim2|v_A,v_B}(v_A, v_B)
\end{aligned} \tag{32}
$$

Showing (29) goes analogous to (31). Equations (31) and (32) hold for small deviations ϵ, hence, the global maximum cannot be obtained at the boundary of $\mathcal{M}$. □

Proof of Proposition 8. The flexible bidder's expected payoff from bidding $w_A, w_B \geq 0$ in the A-auction and B-auction is

$$\begin{aligned}
\Pi^F_{Sim1|v_A,v_A}(w_A, w_B) &= P\left(\text{F wins A, not B}\right)(v_A - w_A) \\
&+ P\left(\text{F wins B, not A}\right)(v_B - w_B) \\
&+ P\left(\text{F wins A and B}\right)(v_B - w_A - w_B) \\
&= G(w_A)\left(1 - G(w_B)\right)(v_A - w_A) \\
&+ G(w_B)\left(1 - G(w_A)\right)(v_B - w_B) \\
&+ G(w_A)G(w_B)\left(v_B - w_A - w_B\right)
\end{aligned}$$

The first-order conditions are

$$\boldsymbol{\nabla}\Pi^F_{Sim1|v_A,v_A}(w_A, w_B) = \begin{pmatrix} g(w_A)\left[v_A - w_A - G(w_B)v_A\right] - G(w_A) \\ g(w_B)\left[v_B - w_B - G(w_A)v_A\right] - G(w_B) \end{pmatrix} = 0, \tag{33}$$

and can be simplified to

$$w_A{}^* = v_A(1 - G(w_B{}^*)) - \frac{G(w_A{}^*)}{g(w_A{}^*)} \tag{34}$$

$$w_B{}^* = v_B - v_A G(w_A{}^*) - \frac{G(w_B{}^*)}{g(w_B{}^*)} \tag{35}$$

Again, we prove that none of the points on the boundary of bidder F's choice set $\mathcal{M} := [0, v_A] \times [0, v_B]$ is optimal, and hence the unique stationary point must coincide with the global maximum.

Consider the line segment $\{w_A \in [0, v_A], w_B = 0\}$. The local optimum is at $(w_A{}^*, 0)$, with $w_A{}^* = v_A - \frac{G(w_A{}^*)}{g(w_A{}^*)}$ (bidder F's optimal bid in a single first-price auction) and payoff $\Pi^F_{P1|v_A,v_B}(0, w_A{}^*) = \frac{G(w_A{}^*)^2}{g(w_A{}^*)}$. Similarly, for the line segment $\{w_A = 0, w_B \in [0, v_B]\}$ we obtain $\Pi^F_{P1|v_A,v_B}(0, w_B{}^*) = \frac{G(w_B{}^*)^2}{g(w_B{}^*)}$. Hence the first local optimum is dominated by the latter. On the line segment $\{w_A = v_A, w_B \in [0, v_B]\}$ the first-order condition characterises a unique stationary point $\widetilde{w_B}^*$. bidder F's payoff at this stationary point is $\Pi^F_{P1|v_A,v_B}(v_A, \widetilde{w_B}^*) = \frac{G(\widetilde{w_B}^*)^2}{g(\widetilde{w_B}^*)}$, which is dominated by $(0, w_B{}^*)$ because $\widetilde{w_B}^* < w_B{}^*$. Similarly, we find the stationary point $\widetilde{w_A}^*$ on the line segment $\{w_A \in [0, v_A], w_B = v_B\}$ also to be dominated by $(0, w_B{}^*)$. Therefore, we only show

$$\Pi^F(\epsilon, w_B{}^*) \quad > \quad \Pi^F(0, w_B{}^*) \tag{36}$$

We have

$$\begin{aligned}\Pi^F(\epsilon, {w_B}^*) =& G({w_B}^*)(v_B - {w_B}^*) + G(\epsilon)(v_A(1 - G({w_B}^*)) - \epsilon) \\ >& G({w_B}^*)(v_B - {w_B}^*) = \Pi^F(0, {w_B}^*)\end{aligned} \tag{37}$$

(37) holds for small deviations ϵ. Overall, the global maximum cannot be obtained at the boundary of $\mathcal{M}$. $\square$

Proof of Proposition 9. First we show the result for the second-price rule. Note that if G is convex or concave and weakly increasing, the flat parts of the function can only occur at the left or right end of the interval $[0, \bar{v}]$, and include the endpoints respectively. Thus, the G can be restricted to a strictly increasing function on a subset of $[0, \bar{v}]$. For simplicity we argue only for the case where G is strictly increasing on $[0, \bar{v}]$. The inverse of G is defined on this interval and denoted G^{-1}. We substitute (26) into (27) and obtain

$$G^{-1}\left(\frac{v_B - {w_B}^*}{v_A}\right) = v_A(1 - G({w_B}^*))$$

Let $g({w_B}^*) := G^{-1}\left(\frac{v_B - {w_B}^*}{v_A}\right)$ and $h({w_B}^*) := v_A(1 - G({w_B}^*))$. g and h are both strictly decreasing functions and from standard composition rules it follows that whenever G is convex (concave), g and h are concave (convex). Two single valued convex (concave) functions may intersect at most twice on a given interval $[\underline{x}, \bar{x}]$. If they intersect twice, the ordering of the functions' values has to be the same at $\underline{x}$ and $\bar{x}$. g is defined on $G := [v_B - v_A, v_B]$ and h is defined on $H := [0, \bar{v}]$. Trivially, $G \subset H$. Because $g(v_B - v_A) = \bar{v} > v_A > h(v_B - v_A)$ and $g(v_B) = 0 = h(\bar{v}) < h(v_B)$, g and h intersect exactly once on $[v_B - v_A, v_B]$. Therefore, ${w_B}^*$ is uniquely defined, and so is ${w_A}^*$ by (26).

Under the first-price rule, the argument is similar. Remember that the reverse hazard rate is assumed to be weakly decreasing. As before we can restrict G to be a strictly increasing function on a subset of $[0, \bar{v}]$ and denote the inverse G^{-1} We simply define $\lambda(x) := x + \frac{G(x)}{g(x)}$, which is a strictly increasing function with inverse λ^{-1}. Substituting (34) into (35) we obtain

$$G^{-1}\left(\frac{v_B - \lambda({w_B}^*)}{v_A}\right) = \lambda^{-1}\left(v_A(1 - G({w_B}^*))\right) \tag{38}$$

Again, define the LHS of (38) as $g({w_B}^*)$ and the RHS as $h({w_B}^*)$. Further note that $\lambda^{-1}(v_B) < v_B < \bar{v}$ because λ^{-1} is below the 45-degree line. Then, by the same argument as before, g and h intersect exactly once on $[\lambda^{-1}(v_B - v_A), \lambda^{-1}(v_B)]$. The proof the second-price rule contains the full argument for $\lambda(x) = x$. $\square$

B Example: 2-good case with uniform distributions of a and b

B.1 First-price PMA

Proof of Proposition 10. The flexible bidder's profit function simplifies to

$$\Pi^F_{P1|v_A,v_B}(w_A,w_B) =$$
$$-\frac{1}{2}{w_A}^3 + {w_A}^2\left(\frac{1}{2}v_A - \frac{1}{2}v_B + \frac{3}{2}w_B - 1\right) + v_B w_B - {w_B}^2 + v_A w_A(1-w_B)$$

First-order conditions are

$$-\frac{3}{2}{w_A}^2 + w_A\left(-2 + 3w_B + v_A - v_B\right) + v_A(1-w_B) = 0 \tag{39}$$
$$\frac{3}{2}{w_A}^2 + v_B - 2w_B - v_A w_A = 0 \tag{40}$$

Substituting (40) into (39) yields

$$9{w_A}^3 - \left(9v_A + 6\right){w_A}^2 + \left(2{v_A}^2 + 4v_A + 2v_B - 8\right)w_A - 2v_A v_B + 4v_A = 0 \tag{41}$$

If an interior solution exists, it must be a stationary point of $\Pi^F_{P1|v_A,v_B}(w_A,w_B)$, i.e. the optimal w_A must be a root of equation (41). Let $p(w_A) := \alpha {w_A}^3 + \beta {w_A}^2 + \gamma w_A + \delta$ denote the polynomial on the left hand side of (41), where

$$\begin{aligned}
\alpha :=& 9\\
\beta :=& -\left(9v_A + 6\right)\\
\gamma :=& 2{v_A}^2 + 4v_A + 2v_B - 8\\
\delta :=& -2v_A v_B + 4v_A
\end{aligned}$$

Using *Mathematica*, one can easily verify that the discriminant of $p(w_A)$

$$\Delta_p = 18\alpha\beta\gamma\delta - 4\beta^3\delta + \beta^2\gamma^2 - 4\alpha\gamma^3 - 27\alpha^2\delta^2$$

is strictly positive for $v_A, v_B < 2$, hence $p(w_A)$ possesses three distinct real roots. Existence of at least one real root follows from the argument given below for the existence of an interior solution. Furthermore, we know that $p(w_A) \to -\infty$ as $w_A \to -\infty$ and $p(w_A) \to \infty$ as $w_A \to \infty$. It is also easy to check that

$$\begin{aligned}
p(0) =& \delta && > 0 && \text{if } v_B < 2 \text{ and}\\
p(1) =& -5 - v_A + 2{v_A}^2 + 2v_B - 2v_A v_B && < -5 + 2v_B < 0 && \text{if } v_B < 2
\end{aligned}$$

Thus, by the intermediate value theorem, $p(w_A)$ has exactly one real root on $(0,1)$. Moreover, one can easily check that $p(v_A) < 0$ for $v_A < 2$, hence the solution is such that $w_A < v_A$. From equation

(40) one can derive w_B as a function of w_A. Therefore, if w_A is uniquely defined, and so is w_B. Global optimality is immediate from Proposition 2. □

Comparative statics. The implicit function theorem applied to the first-order conditions yields

$$\frac{\partial w_A}{\partial v_A} = \frac{\frac{1}{2}w_A(3w_A - v_A) + w_B - w_A - 1}{\frac{1}{2}(3w_A - v_A)^2 + 3(w_B - w_A) + v_A - v_B - 2}$$
$$\frac{\partial w_A}{\partial v_B} = \frac{\frac{1}{2}(v_A - w_A)}{\frac{1}{2}(3w_A - v_A)^2 + 3(w_B - w_A) + v_A - v_B - 2}$$
$$\frac{\partial w_B}{\partial v_A} = \frac{\partial w_A}{\partial v_A}\frac{3w_A - v_A}{2} - \frac{w_A}{2}$$
$$\frac{\partial w_B}{\partial v_B} = \frac{\partial w_A}{\partial v_B}\frac{3w_A - v_A}{2} + \frac{1}{2}$$

B.2 Simultaneous first-price auction

Comparative statics.

We have $\frac{\partial w_A}{\partial v_A} = \frac{(1-v_B)(1+v_A{}^2)}{(1-v_A{}^2)} \geq 0$, $\frac{\partial w_A}{\partial v_B} = \frac{-v_A(1-v_A{}^2)}{(1-v_A{}^2)} < 0$, $\frac{\partial w_B}{\partial v_A} = \frac{2v_A(v_B-1)}{(1-v_A{}^2)} \leq 0$, $\frac{\partial w_B}{\partial v_B} = 1 > 0$.

We have $\frac{\partial w_A}{\partial v_A} = \frac{(2-v_B)(4+v_A{}^2)}{(4-v_A{}^2)} \geq 0$, $\frac{\partial w_A}{\partial v_B} = \frac{-v_A(4-v_A{}^2)}{(4-v_A{}^2)} < 0$, $\frac{\partial w_B}{\partial v_A} = \frac{4v_A(v_B-2)}{(4-v_A{}^2)} \leq 0$, $\frac{\partial w_B}{\partial v_B} = 2 > 0$.

C Proofs for the general model

C.1 Standard PMA

Proof of Proposition 11. The flexible bidder's preference for her ex-post allocation coincides with the efficient allocation: she wants to win good j if and only if

$$v_j > x_j \qquad \text{and} \tag{42}$$
$$v_j - x_j = \max_{k\in\mathcal{J}} v_k - x_k \tag{43}$$

Suppose a realisation of $x_k, k = 1,..,M$ is such that bidder F wins good j. Then $w_j - x_j > w_k - x_k\ \forall k \in \mathcal{J}$ by the allocation rule. Substituting the optimal bids for w_j and w_k, we obtain $v_j - x_j > v_k - x_k\ \forall k \in \mathcal{J}$, i.e. bidder F prefers to win good j. As bidder F wins for every realisation precisely what would be best for her to win, no profitable deviation is possible. Note that when $v_M - v_j \geq \bar{v}$, bidder F always wants to win good M because the maximal difference in payments is $\bar{v}$. In these cases, it may be optimal to bid ${w_j}^* = 0$ for some $j \leq \tilde{j}, j \neq M$, if ${w_M}^*$ is chosen not too large. □

C.2 First-price PMA

Proof of Lemma 4. We note again equation 6, which is key to this lemma, together with the fact that for any two goods $k, l \in \mathcal{J}$, x_k and x_l are *iid* distributed.

$$P_k(\mathbf{w}) = Prob(w_k > x_k \text{ and } w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j) \tag{44}$$

Because of the above equation, and because x_k and x_l are *iid*, we have

$$\begin{aligned}&P_k(w_1, ..., w_{l-1}, w_l, w_{l+1}, ..., w_{k-1}, w_k, w_{k+1}, ...w_M)\\=&P_l(w_1, ..., w_{l-1}, w_k, w_{l+1}, ..., w_{k-1}, w_l, w_{k+1}, ...w_M)\end{aligned}$$

Now suppose $w_l > w_k$ for some $l < k$ (so $v_l \leq v_k$). Then we can easily show that bidding $\widetilde{w_l} = w_k$ and $\widetilde{w_k} = w_l$ is a profitable deviation. Let $\widetilde{\mathbf{w}}$ denote the vector $(w_1, ..., w_M)$ where w_k is substituted by $\widetilde{w_k}$ and w_l is substituted by $\widetilde{w_l}$. Let $\mathbf{w'}$ denote the vector $(w_1, ..., w_M)$ where w_k is swapped with w_l.

$$\begin{aligned}\Pi^F(\widetilde{\mathbf{w}}) &= \sum_{j \neq k,l} P_j(\widetilde{\mathbf{w}})(v_j - w_j) + P_k(\widetilde{\mathbf{w}})(v_k - \widetilde{w_k}) + P_l(\widetilde{\mathbf{w}})(v_l - \widetilde{w_l})\\&= \sum_{j \neq k,l} P_j(\widetilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w'})(v_k - w_l) + P_l(\mathbf{w'})(v_l - w_k)\\&= \sum_{j \neq k,l} P_j(\widetilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w})(v_l - w_k) + P_l(\mathbf{w})(v_k - w_l)\\&> \sum_{j \neq k,l} P_j(\widetilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w})(v_k - w_k) + P_l(\mathbf{w})(v_l - w_l)\end{aligned}$$

The last inequality holds because $P_l(\mathbf{w}) > P_k(\mathbf{w})$ for $w_l > w_k$ (this follows immediately from equation (44). The procedure to find profitable deviations can be applied repeatedly to restore the order $w_M \geq w_{M-1} \geq ... \geq w_1$. □

Proof of Proposition 12. To prove that bidder F's maximisation problem possesses indeed an interior solution, we simply show that the maximum cannot be obtained on the boundary of $[0, v_1] \times ... \times [0, v_M]$, and therefore must be a stationary point of $\Pi^F_{P1|\mathbf{v}}$, characterised by the system of first-order conditions $\boldsymbol{\nabla}\Pi^F_{P1|\mathbf{v}} = 0$. Consider the border surfaces of $\mathcal{C}_M$. These surfaces are q-dimensional hyperplanes, with $q \in \{1, ..., M-1\}$. They can be described by vectors $(w_1, ..., w_M)$ where the j^{th} entry may be substituted by a constant $c \in \{0, v_j\}$, and overall $M - q$ elements of $\mathbf{w}$ are substituted when describing a q-dimensional hyperplane.

It is sufficient to consider $(M-1)$-dimensional hyperplanes because k-dimensional border surfaces are encompassed in $k+1$-dimensional border surfaces. If we do not put restrictions on the elements of $\mathbf{w}_{-j}$, which are not substituted by a constant, it is also sufficient to consider points on the hyperplane in which only the j^{th} entry is substituted by a constant $c \in \{0, v_j\}$.

First, let $c = v_j$. In this is case, it is obvious that an ϵ-deviation orthogonal to the hyperplane increases the flexible bidder's profits, i.e.

$$\Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) > \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)$$

The flexible bidder makes zero profit on good j if she wins, so the deviation strictly increases her profits in those cases where she wins, while at the same time the probability of winning another good does not decrease. More formally, note that $\frac{dP_k(\mathbf{w})}{dw_j} < 0$ because $P_k(\mathbf{w}) = Prob(w_k > x_k$ and $w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j)$. Then, it is immediate that

$$\begin{aligned}
&\Pi^F_{P1|\mathbf{v}}(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) \\
&= \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_k - w_k] + P_j(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_j - v_j + \epsilon] \\
&> \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_k - w_k] \\
&\geq \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)
\end{aligned}$$

Second, we consider $(M-1)$-dimensional hyperplanes with $w_j = c = 0$. We need to establish that

$$\Pi^F(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) > \Pi^F(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$$

Let

$$\begin{aligned}
&\Delta P_k = [P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) \text{ and} \\
&\Delta P_j = P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M)
\end{aligned}$$

Then

$$\begin{aligned}
\Delta P_k =& Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j) \\
&- Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
=& Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j, \epsilon - x_j > 0) \\
&+ Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j, \epsilon - x_j < 0) \\
&- Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
>& Prob(w_k - x_k > \epsilon, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
&+ Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j < 0) \\
&- Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
=& Prob(w_k - x_k > \epsilon, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
&- Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
=& - Prob(x_j < \epsilon) Prob(x_k \in [w_k, w_k - \epsilon], w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j)
\end{aligned}$$

and

$$
\begin{aligned}
\Delta P_j =& P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) \\
>& Prob(x_j < \epsilon) Prob(x_1 > w_1) ... Prob(x_{j-1} > w_{j-1}) Prob(x_{j+1} > w_{j+1}) ... Prob(x_M > w_M)
\end{aligned}
$$

Using the above, we obtain

$$
\begin{aligned}
& \Pi^F_{P1|\mathbf{v}}(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - \Pi^F(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) \\
=& \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M)\,[v_k - w_k] + P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M)\,[v_j - \epsilon] \\
-& \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)\,[v_k - w_k] \\
=& \sum_{k \neq j} [P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)]\,[v_k - w_k] \\
+& P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M)\,[v_j - \epsilon] \\
=& \sum_{k \neq j} \Delta P_k\,[v_k - w_k] + \Delta P_j\,[v_j - \epsilon] \\
>& Prob(x_j < \epsilon) \Bigg[- \sum_{k \neq j} Prob(x_k \in [w_k, w_k - \epsilon], w_k - x_k > w_{k'} - x_{k'} \; \forall k' \neq j)\,[v_k - w_k] \\
+& Prob(x_1 > w_1) ... Prob(x_{j-1} > w_{j-1}) Prob(x_{j+1} > w_{j+1}) ... Prob(x_M > w_M)\,[v_j - \epsilon] \Bigg]
\end{aligned}
$$

The last expression is greater than zero for ϵ small, and we are done. □

C.3 Simultaneous second-price auction

Proof of Proposition 13. We need to show than on every border surface of the parallelepiped $\mathcal{C}_M$, Π^F is not maximal. We use the notational convention that $\prod_k^p x_k = 1$ if $k > p$. First, we transform the expected payoff function with integration by parts and obtain

$$
\Pi^F(\mathbf{w}) = \sum_{k=1}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left(\prod_{l=k+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]
$$

Now consider the border surfaces of $\mathcal{C}_M$. These surfaces are q-dimensional hyperplanes, with $q \in \{1, ..., M-1\}$. They can be described by vectors $\mathbf{w} = (w_1, ..., w_M)$ where the jth entry may be substituted by a constant $c \in \{0, v_j\}$, and overall $M - q$ elements of $\mathbf{w}$ are substituted when describing a q-dimensional hyperplane. We consider $(M-1)$-dimensional hyperplanes and substitute *only* the jth entry by a constant $c \in \{0, v_j\}$. This is sufficient because we make no restriction on $\mathbf{w}_{-j}$.

First, let $w_j = c = 0$. To show that the flexible bidder can do better submitting a strictly positive bid, we first show the following lemma.

Lemma 6. *Let* $\mathbf{w} = (w_{-j}, w_j = 0)$ *be the flexible bidder's bid. Then she can improve her expected payoff by bidding* $\mathbf{w}' = (w_{-j,-(j-1)}, w'_{j-1} = 0, w'_j = w_{j-1})$.

Proof. We have

$$\begin{aligned}
\Pi^F(\mathbf{w}) = &\sum_{k=1}^{j-2}\left[v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x\right] \\
&+ v_{j-1}G(w_{j-1})\prod_{l=j}^{M}(1-G(w_l)) - \int_0^{w_{j-1}} x g(x)\,\mathrm{d}x \\
&+ \sum_{k=j+1}^{M}\left[v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x\right] \text{ and} \\
\Pi^F(\mathbf{w}') = &\sum_{k=1}^{j-2}\left[v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x\right] \\
&+ v_j G(w_{j-1})\prod_{l=j+1}^{M}(1-G(w_l)) - \int_0^{w_{j-1}} x g(x)\,\mathrm{d}x \\
&+ \sum_{k=j+1}^{M}\left[v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x\right]
\end{aligned}$$

Because of the substitution from $\mathbf{w}$ to $\mathbf{w}'$ the first lines of $\Pi^F(\mathbf{w})$ and $\Pi^F(\mathbf{w}')$ are identical, and trivially the third lines are identical too. Because $w_j = 0$ and $v_j \geq v_{j-1}$, the second line of $\Pi^F(\mathbf{w}')$ is weakly greater than the second line of $\Pi^F(\mathbf{w})$. $\square$

From the lemma above it follows that every bid $(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$ can be replaced by a bid $(0, w_1, w_2, ..., w_{j-1}, w_{j+1}, ..., w_M)$. Therefore, we only need to show that $\Pi^F(\widetilde{w_1}, w_2, ..., w_M) > \Pi^F(0, w_2, ..., w_M)$. We simply choose $\widetilde{w_1} = v_1 \prod_{l=2}^{M}(1 - G(w_l))$. Then we have

$$\begin{aligned}
\Pi^F(0, w_2, ..., w_M) &= \sum_{k=2}^{M}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) + \int_0^{w_k} G(x)\,\mathrm{d}x\right] \text{ and} \\
\Pi^F(\widetilde{w_1}, w_2, ..., w_M) &= \int_0^{\widetilde{w_1}} G(x)\,\mathrm{d}x + \sum_{k=2}^{M}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\prod_{l=k+1}^{M}(1-G(w_l)) + \int_0^{w_k} G(x)\,\mathrm{d}x\right]
\end{aligned}$$

Now consider hyperplanes where $c = v_j$. Then, we have

$$
\begin{aligned}
&\Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M) = \\
&\sum_{k=1}^{j-1}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\left((1 - G(v_j))\prod_{\substack{l=k+1,\\ l\neq j}}^{M}(1 - G(w_l)) - 1\right) + \int_0^{w_k} G(x)\,\mathrm{d}x\right] \\
&+\sum_{k=j+1}^{M}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\left(\prod_{l=k+1}^{M}(1 - G(w_l)) - 1\right) + \int_0^{w_k} G(x)\,\mathrm{d}x\right] \\
&+v_j G(v_j)\left(\prod_{l=j+1}^{M}(1 - G(w_l)) - 1\right) + \int_0^{v_j} G(x)\,\mathrm{d}x
\end{aligned}
$$

For an ϵ-deviation orthogonal to the hyperplane we obtain

$$
\begin{aligned}
&\Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) = \\
&\sum_{k=1}^{j-1}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\left((1 - G(v_j - \epsilon))\prod_{\substack{l=k+1,\\ l\neq j}}^{M}(1 - G(w_l)) - 1\right) + \int_0^{w_k} G(x)\,\mathrm{d}x\right] \\
&+\sum_{k=j+1}^{M}\left[G(w_k)[v_k - w_k] + v_k G(w_k)\left(\prod_{l=k+1}^{M}(1 - G(w_l)) - 1\right) + \int_0^{w_k} G(x)\,\mathrm{d}x\right] \\
&+G(v_j - \epsilon)\epsilon + v_j G(v_j - \epsilon)\left(\prod_{l=j+1}^{M}(1 - G(w_l)) - 1\right) + \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x
\end{aligned}
$$

Now define

$$
\psi := \sum_{k=1}^{j-1}\left[v_k G(w_k)\prod_{\substack{l=k+1,\\ l\neq j}}^{M}(1 - G(w_l))\right] - v_j\left(\prod_{l=j+1}^{M}(1 - G(w_l)) - 1\right).
$$

ψ is strictly positive if at least one $w_k, k \in \mathcal{J} \setminus \{j\}$ is strictly positive, and we demonstrated above below that indeed all $w_j, j \in \mathcal{J}$ are strictly positive. Denote the difference in payoffs as $\Delta := \Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) - \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)$. Then we can simplify Δ to obtain

$$
\Delta = \psi\left(G(v_j) - G(v_j - \epsilon)\right) + G(v_j - \epsilon)\epsilon + \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x - \int_0^{v_j} G(x)\,\mathrm{d}x
$$

A simple linear approximation of $G(v_j)$ and a quadratic approximation of $\int_0^{v_j} G(x)\,\mathrm{d}x$ (after Taylor) around $v_j - \epsilon$ yields

$$\Delta = \psi\left(G(v_j-\epsilon) + g(v_j-\epsilon)\epsilon + o(\epsilon^2) - G(v_j-\epsilon)\right) + G(v_j-\epsilon)\epsilon + \int_0^{v_j-\epsilon} G(x)\,\mathrm{d}x$$
$$- \int_0^{v_j-\epsilon} G(x)\,\mathrm{d}x - G(v_j-\epsilon)\epsilon - g(v_j-\epsilon)\frac{\epsilon^2}{2} - o(\epsilon^3),$$

which is strictly positive as $\epsilon \to 0$ as long as $\psi > 0$, i.e. bidding truthfully in any of the M simultaneous auction is not optimal if the flexible bidder makes at least one other strictly positive bid for a another good. The continuous function Π^F must attain its maximum on the compact set $\mathcal{C}_M$ by the extreme value theorem. Let $\overline{\mathcal{C}_\mathcal{M}}$ denote the open set defined by $\overline{\mathcal{C}_\mathcal{M}} := (0, v_1) \times ... \times (0, v_M)$. The global maximum must also be a local maximum in an ϵ-neighbourhood on $\overline{\mathcal{C}_\mathcal{M}}$. Since Π^F is also differentiable everywhere on $\overline{\mathcal{C}_\mathcal{M}}$, by Fermat's theorem, the global maximum must be a stationary point of Π^F, i.e. it is characterised by the first-order conditions. □

Calculations for example 1. Let $M = 2$ and $v_1 = v_2 = v < 1$. We assume a probability distribution with support $[0, 1]$ and a uniform spike around $x \in [0, 1]$, where $x < v$. Let $h := \frac{1-\epsilon+2\epsilon^2}{2\epsilon}$ and $\epsilon < x$. Formally, the probability density function is

$$g(t) = \begin{cases} \epsilon & \text{if} \quad t < x - \epsilon \\ h & \text{if} \quad t \in [x-\epsilon, x+\epsilon) \\ \epsilon & \text{if} \quad t \geq x + \epsilon \end{cases}$$

and the probability distribution function is

$$G(t) = \begin{cases} \epsilon t & \text{if} \quad t < x - \epsilon \\ (\epsilon - h)(x-\epsilon) + ht & \text{if} \quad t \in [x-\epsilon, x+\epsilon) \\ 1 - \epsilon + \epsilon t & \text{if} \quad t \geq x + \epsilon \end{cases}$$

To find the solution one has to distinguish six different cases. $\mathbf{H}\Pi^F$ denotes the hessian of Π^F.

Case 1: $w_A, w_B < x - \epsilon$

Solving equations (26) and (27) on this domain gives us ${w_A^1}^* = {w_B^1}^* = \frac{v}{1+v\epsilon}$, which is a global maximum (on this domain) because $\mathbf{H}\Pi^{F,1}(w_1, w_2)$ is negative definite. We obtain $\Pi^{F,1}({w_A^1}^*, {w_B^1}^*) = \frac{v^2\epsilon}{1+v\epsilon}$.

Case 2: $w_A, w_B \in [x-\epsilon, x+\epsilon)$

Solving equations (26) and (27) on this domain gives us ${w_A^2}^* = {w_B^2}^* = \frac{v(1+h(x-\epsilon)-x\epsilon+\epsilon^2}{1+hv}$, but $\mathbf{H}\Pi^{F,2}(w_1, w_2)$ is only negative definite if $h < v^{-1}$, i.e. if the spike is not too prominent.

Case 3: $w_A, w_B \geq x + \epsilon$

Solving equations (26) and (27) on this domain gives us ${w_A^3}^* = {w_B^3}^* = \frac{v\epsilon}{1+v\epsilon}$, which, again, is a global maximum because $\mathbf{H}\Pi^{F,3}(w_1, w_2)$ is negative definite. We obtain $\Pi^{F,3}({w_A^3}^*, {w_B^3}^*) =$

$\epsilon(2\epsilon(2h-2\epsilon+1)-3)-\frac{1}{v^2\epsilon+v}+v+\frac{1}{v}+2x(\epsilon-1)$.

Case 4: $w_A < x-\epsilon, w_B \in [x-\epsilon, x+\epsilon)$

Solving equations (26) and (27) on this domain gives us ${w_A^4}^* = \frac{v\left(h(v-x+\epsilon)+x\epsilon-\epsilon^2-1\right)}{hv^2\epsilon-1}$ and ${w_B^4}^* = \frac{v\left(v\epsilon\left(h(x-\epsilon)-x\epsilon+\epsilon^2+1\right)-1\right)}{hv^2\epsilon-1}$, but, again, $\mathbf{H}\Pi^{F,4}(w_1, w_2)$ is only negative definite if $h < (v^2\epsilon)^{-1}$, i.e. if the spike is not too prominent.

Case 5: $w_A < x-\epsilon, w_B \geq x+\epsilon$

Solving equations (26) and (27) on this domain gives us ${w_A^5}^* = \frac{(v-1)v\epsilon}{v^2\epsilon^2-1}$ and ${w_B^5}^* = \frac{v(v\epsilon^2-1)}{v^2\epsilon^2-1}$, which is a global maximum because $\mathbf{H}\Pi^{F,5}(w_1, w_2)$ is negative definite. We obtain $\Pi^{F,5}({w_A^5}^*, {w_B^5}^*) = \frac{v^2\epsilon\left(\epsilon^2(2\epsilon(2h-2\epsilon+1)-3)+2x(\epsilon-1)\epsilon-1\right)-2\epsilon(\epsilon(2h-2\epsilon+1)+x)+2v^3\epsilon^2+2v(\epsilon-1)+2(x+\epsilon)}{2v^2\epsilon^2-2}$.

Case 6: $w_A \in [x-\epsilon, x+\epsilon), w_B \geq x+\epsilon$

Solving equations (26) and (27) on this domain gives us ${w_A^6}^* = \frac{v\epsilon\left(v\left(h(x-\epsilon)-x\epsilon+\epsilon^2+1\right)-1\right)}{hv^2\epsilon-1}$ and ${w_B^6}^* = \frac{v\left(h(v\epsilon-x+\epsilon)+x\epsilon-\epsilon^2-1\right)}{hv^2\epsilon-1}$, and $\mathbf{H}\Pi^{F,6}(w_1, w_2)$ is only negative definite if $h < (v^2\epsilon)^{-1}$, i.e. if the spike is not too prominent. We obtain $\Pi^{F,6}({w_A}^*, {w_B}^*) =$

$$\frac{2hv^3\epsilon+2v\left(-\epsilon^2(h+x)+hx\epsilon+\epsilon^3+\epsilon-1\right)-\epsilon^2(5h+2x+2)+\epsilon(x(2h+x-2)+2)+x(2-hx)+5\epsilon^3}{2hv^2\epsilon-2}$$
$$\frac{-v^2\epsilon\left(-4h^2\epsilon^2+h\left(-x^2\epsilon+2x\left(\epsilon^2-\epsilon+2\right)+\epsilon\left(3\epsilon^2-2\epsilon+1\right)\right)+\left(-x\epsilon+\epsilon^2+1\right)^2\right)}{2hv^2\epsilon-2}$$

Now let $\epsilon = 0.1$ (implying $h = 4.6$), $v = 0.7$, and $x = 0.5$. The six cases give the following candidates for the optimal bid:

$({w_A^1}^*, {w_B^1}^*) \approx (0.65, 0.65)$, which is not admissible on $[0, x-\epsilon]$. Case 2 does not fulfil the sufficient condition for a global maximum; the stationary point is $({w_A^2}^*, {w_B^2}^*) \approx (0.464, 0.464)$ and yields an expected payoff of 0.120. The boundary point $(x-\epsilon, x+\epsilon)$ yields a higher payoff of 0.197. $({w_A^3}^*, {w_B^3}^*) \approx (0.065, 0.065)$ fulfils the second-order conditions, but is not admissible on $[x+\epsilon, 1]$. $({w_A^4}^*, {w_B^4}^*) \approx (-0.380, 0.727)$, which also fulfils the second-order conditions, but is not admissible on $[0, x-\epsilon) \times [x-\epsilon, x+\epsilon)$. We obtain $({w_A^5}^*, {w_B^5}^*) \approx (0.021, 0.699)$, which is a global optimum and yields a payoff of 0.205. $({w_A^6}^*, {w_B^6}^*) \approx (-0.087, 2.239)$ also fulfils the second-order conditions, but is again not admissible on $[x-\epsilon, x+\epsilon) \times [x+\epsilon, 1]$.

Clearly, the global maximum is $({w_A^5}^*, {w_B^5}^*) \approx (0.021, 0.699)$. For the general case, one can easily verify that ${w_A^5}^* \to 0$, ${w_B^5}^* \to v$, and $\Pi^{F,5}({w_A^5}^*, {w_B^5}^*) \to v - x$ as $\epsilon \to 0$.

However, depending on how much probability mass concentrates around the spike, we may also obtain a symmetric equilibrium. For example, let $\epsilon = 0.4$ (implying $h = 1.15$), $v = 0.7$, and $x = 0.5$. The six cases give the following candidates for the optimal bid:

$({w_A^1}^*, {w_B^1}^*) \approx (0.547, 0.547)$, which is not admissible on $[0, x-\epsilon]$. Case 2 fulfils the sufficient condition for a global maximum; the stationary point is $({w_A^2}^*, {w_B^2}^*) \approx (0.417, 0.417)$ and yields an expected payoff of 0.259. $({w_A^3}^*, {w_B^3}^*) \approx (0.219, 0.219)$ fulfils the second-order conditions, but is not admissible on $[x+\epsilon, 1]$. $({w_A^4}^*, {w_B^4}^*) \approx (0.244, 0.632)$, which also fulfils the second-order conditions, but is not admissible on $[0, x-\epsilon) \times [x-\epsilon, x+\epsilon)$. We obtain $({w_A^5}^*, {w_B^5}^*) \approx (0.091, 0.674)$, which is a global optimum and yields a payoff of 0.220. $({w_A^6}^*, {w_B^6}^*) \approx (0.089, 0.680)$ also fulfils the second-order conditions, but is again not admissible on $[x-\epsilon, x+\epsilon) \times [x+\epsilon, 1]$.

The global maximum here is $({w_A^2}^*, {w_B^2}^*) \approx (0.417, 0.417)$.

C.4 Simultaneous first-price auction

Proof of Proposition 14. It is obvious that under the first-price rule bidding one's true value on any good can never be optimal, because expected profit on that good would always be less than zero. So we only show that any bid $(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$ cannot be optimal. With probability $P := \prod_{l=1, l\neq j}(1-G(w_l))$ bidder F wins nothing in the auctions she participates in. Now simply choose $\widetilde{w_j} = \alpha P v_j$, with $0 < \alpha < 1$. Then we have

$$\begin{aligned}
\Pi^F((w_1, ..., w_{j-1}, \widetilde{w_j}, w_{j+1}, ..., w_M) &\geq \Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P(v_j - \widetilde{w_j}) - (1-P)\widetilde{w_j} \\
&= \Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P(v_j - \alpha P v_j) - (1-P)\alpha P v_j \\
&= \Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P v_j(1 - \alpha P - \alpha(1-P)) \\
&> \Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)
\end{aligned}$$

□

C.5 Sequential second-price auction

Proof of Proposition 15. In the last auction for good M, she knows the past history of outcomes of auctions $1, ..., M-1$. Denote by v_{M-1}^{max} the value of the highest-value object bidder F obtained in the history H_{M-1}. Then, her payoff in the last auction for good M is

$$\Pi^F_{M, v_{M-1}^{max}}(w_M) = G(w_M)v_M - \int_0^{w_M} x \,\mathrm{d}G(x) + (1 - G(w_M))v_{M-1}^{max}$$

She bids ${w_{M, v_{M-1}^{max}}}^* = v_M - v_{M-1}^{max}$ and obtains $\Pi^{F*}_{M, v_{M-1}^{max}} = v_{M-1}^{max} + \int_0^{v_M - v_{M-1}^{max}} G(x)\,\mathrm{d}x$. In the $M-1$-auction, the flexible bidder takes possible future gains into account. She maximises

$$\Pi^F_{M-1, v_{M-2}^{max}}(w_{M-1}) = G(w_{M-1})\Pi^{F*}_{M, v_{M-1}} - \int_0^{w_{M-1}} x \,\mathrm{d}G(x) + (1 - G(w_{M-1}))\Pi^{F*}_{M, v_{M-2}^{max}}$$

In the auction for good $j < M$, given history H_{j-1}, and thus v_{j-1}^{max}, she maximises

$$\Pi^F_{j, v_{j-1}^{max}}(w_j) = G(w_j)\Pi^{F*}_{j+1, v_j} - \int_0^{w_j} x \,\mathrm{d}G(x) + (1 - G(w_j))\Pi^{F*}_{j+1, v_{j-1}^{max}}$$

and her optimal bid is given by

$${w_{j, v_{j-1}^{max}}}^* = \max\left\{\Pi^{F*}_{j+1, v_j} - \Pi^{F*}_{j+1, v_{j-1}^{max}}, 0\right\} \tag{45}$$

We then have $\Pi^{F*}_{j,v^{max}_{j-1}} = \Pi^{F*}_{j+1,v^{max}_{j-1}} + \int_0^{\max\{\Pi^{F*}_{j+1,v_j} - \Pi^{F*}_{j+1,v^{max}_{j-1}},0\}} G(x)\,\mathrm{d}x$. Clearly, bidding truthfully cannot be optimal because of the opportunity to improve one's payoff in the auction for good $j+1$. So the maximum must be characterised by the first-order condition given above. $\square$

Equilibrium bidding for sales order $M,...,1$**.** In the last auction of good 1, the flexible bidder trivially bids her valuation $w_1{}^* = v_1$ and obtains $\Pi^{F*}_1 = \int_0^{v_1} G(x)\,\mathrm{d}x$. In the auction of good $j>1$, the flexible bidder bids w_j to maximise her expected stage-payoff

$$\Pi^F_j(w_j) = G(w_j)v_j - \int_0^{w_j} x\,\mathrm{d}G(x) + (1-G(w_j))\,\Pi^{F*}_{j+1}$$

The first-order condition yields $w_j{}^* = v_j - \Pi^F_{j+1}$. This is the optimal bid if the flexible bidder did not win the auction for good $j-1$, and she bids $w^*_j = 0$ in all subsequent auctions for good j if she won good $j-1$. The second-order condition guarantees a local maximum at $w_j{}^*$ and it can indeed be verified that this is also a global maximum (neither $w_j = 0$ nor $w_j = v$ are optimal as can be shown by a simple ϵ-deviation). Simplifying the expected equilibrium profit, we obtain $\Pi^{F*}_j := \Pi^F_j(w_j{}^*) = \Pi^{F*}_{j+1} + \int_0^{w_j{}^*} G(x)\,\mathrm{d}x$.

C.6 Sequential first-price auction

Proof of Proposition 16. In the last auction for good M, she knows the past history of outcomes of auctions $1,...,M-1$. Denote by v^{max}_{M-1} the value of the highest-value object bidder F obtained in the history H_{M-1}. Then, her payoff in the last auction for good M is

$$\Pi^F_{M,v^{max}_{M-1}}(w_M) = G(w_M)(v_M - w_M) + (1-G(w_M))v^{max}_{M-1}$$

The first-order condition characterises $w_{M,v^{max}_{M-1}}{}^* = v_M - v^{max}_{M-1} - \frac{G(w_M)}{g(w_M)}$ and obtains $\Pi^{F*}_{M,v^{max}_{M-1}} = v^{max}_{M-1} + \frac{G(w_{M,v^{max}_{M-1}}{}^*)^2}{g(w_{M,v^{max}_{M-1}}{}^*)}$. In the $M-1$-auction, the flexible bidder takes possible future gains into account. She maximises

$$\Pi^F_{M-1,v^{max}_{M-2}}(w_{M-1}) = G(w_{M-1})\left(\Pi^{F*}_{M,v_{M-1}} - w_{M-1}\right) + (1-G(w_{M-1}))\Pi^{F*}_{M,v^{max}_{M-2}}$$

In the auction for good $j<M$, given history H_{j-1}, and thus v^{max}_{j-1}, she maximises

$$\Pi^F_{j,v^{max}_{j-1}}(w_j) = G(w_j)\left(\Pi^{F*}_{j+1,v_j} - w_j\right) + (1-G(w_j))\Pi^{F*}_{j+1,v^{max}_{j-1}}$$

and her optimal bid is given by

$$w_{j,v^{max}_{j-1}}{}^* = \max\left\{\Pi^{F*}_{j+1,v_j} - \Pi^{F*}_{j+1,v^{max}_{j-1}} - \frac{G(w_j)}{g(w_j)}, 0\right\} \tag{46}$$

and we have $\Pi^{F*}_{j,v^{max}_{j-1}} = \Pi^{F*}_{j+1,v^{max}_{j-1}} + \frac{G(w_{j,v^{max}_{j-1}}{}^{*})^2}{g(w_{j,v^{max}_{j-1}}{}^{*})}$. Of course, truthful bidding truthfully cannot be optimal under the first-price payment rule. So the maximum must be characterised by the first-order condition. □

C.7 Identical values

Assuming symmetric equilibria, we obtain the following equilibrium characterisations:

$$\begin{aligned}
w^*_{P2} &= v \\
w^*_{P1} &= v - \frac{\left(1 - (1 - G(w^*_{P1}))^M\right)}{M\left(1 - G(w^*_{P1})\right)^{M-1} g(w^*_{P1})} \\
w^*_{Sim2} &= v\left(1 - G(w^*_{Sim2})\right)^{M-1} \\
w^*_{Sim1} &= v\left(1 - G(w^*_{Sim1})\right)^{M-1} - \frac{G(w^*_{Sim1})}{g(w^*_{Sim1})} \\
w_{j,Seq2}{}^* &= v - \Pi^F_{Seq2,j+1} \\
w_{j,Seq1}{}^* &= v - \frac{G(w_{j,Seq1}{}^*)}{g(w_{j,Seq1}{}^*)} - \Pi^F_{Seq1,j+1}
\end{aligned}$$

where $\Pi^F_{Seq2,M+1} = \Pi^F_{Seq1,M+1} = 0$. Note that if the flexible bidder's value $v = 1$ and the inflexible bidders' values are uniformly distributed on $[0, 1]$, i.e. $G(x) = x$ $v = 1$, we find closed form expression for the equilibrium bid in the first-price PMA; we have $w^*_{P1} = 1 - \exp\left(\frac{\log\left(\frac{1}{1+M}\right)}{M}\right)$.

Product-Mix auctions. In the standard PMA, the flexible bidder's payoff can be written as

$$\Pi^F_{P2}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right) v - \int_0^w xM\left(1 - G(x)\right)^{M-1} g(x)\,\mathrm{d}x$$

The first-order conditions yield $w^* = v$, which is of course the dominant strategy.

In the first-price PMA, assuming a symmetric equilibrium $w_j = w\ \forall j \in \mathcal{J}$, we have

$$\Pi^F_{P1}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right)(v - w)$$

The first-order conditions yield $(v - w^*)M\left(1 - G(w^*)\right)^{M-1} g(w^*) - \left(1 - (1 - G(w^*))^M\right) = 0$.

To state the auctioneer's revenue, let $x_{(r)}$ denote the $r-$statistic of the M random variables x, which are all iid distributed with probability distribution function G.[42]

Lemma 7. *The auctioneer's revenue in the first-price PMA with identical values in a symmetric equilibrium is given by*

$$R_{P1}(\mathbf{w}) = \int_0^w (w - x)M(1 - G(x))^{M-1} g(x)\,\mathrm{d}x + M\int_0^{\bar{v}} x\,\mathrm{d}G(x)$$

[42] We denote by $x_{(1)}$ the minimum of the M RVs, and by $x_{(M)}$ the maximum of the M RVs.

Proof. The probability density function of $x_{(r)}$ is given by $\frac{M!}{(r-1)!(M-r)!}g(x)G(x)^{r-1}(1-G(x))^{M-r}$. Let $b_{(r)}$ denote the $r-$statistic of the $M+1$ bids the auctioneer receives. Then her expected revenue, given a bid $\mathbf{w}$ from the flexible bidder, is

$$\begin{aligned}
R_{P1}(\mathbf{w}) &= \sum_{r=1}^{M} \mathbb{E}\left[b_{(r)}\right] \\
&= \int_0^w wM(1-G(x))^{M-1}g(x)\,\mathrm{d}x + \int_w^{\bar{v}} xM(1-G(x))^{M-1}g(x)\,\mathrm{d}x + \sum_{r=2}^{M}\mathbb{E}\left[x_{(r)}\right] \\
&= \int_0^w wM(1-G(x))^{M-1}g(x)\,\mathrm{d}x + \int_w^{\bar{v}} xM(1-G(x))^{M-1}g(x)\,\mathrm{d}x + M\mathbb{E}\left[x\right] - \mathbb{E}\left[x_{(1)}\right] \\
&= \int_0^w (w-x)M(1-G(x))^{M-1}g(x)\,\mathrm{d}x + M\mathbb{E}\left[x\right]
\end{aligned}$$

□

Simultaneous auctions. Assuming a symmetric equilibrium $w_j = w\ \forall j \in \mathcal{J}$, the flexible bidder's payoff in the simultaneous second-price auction is

$$\Pi^F_{Sim2}(\mathbf{w}) = \left(1-(1-G(w))^M\right)v - M\int_0^w xg(x)\,\mathrm{d}x$$

The first-order conditions yield $w^* = v\left(1-G(w^*)\right)^{M-1}$.

In the simultaneous first-price auction, again assuming a symmetric equilibrium $w_j = w\ \forall j \in \mathcal{J}$, we have

$$\Pi^F_{Sim1}(\mathbf{w}) = \left(1-(1-G(w))^M\right)v - MG(w)w$$

The first-order conditions yield $w^* = v\left(1-G(w^*)\right)^{M-1} - \frac{G(w^*)}{g(w^*)}$.

The auctioneer's expected revenue is straightforward to compute. Given a bid $\mathbf{w}$ from the flexible bidder, we have

$$\begin{aligned}
R_{Sim1}(\mathbf{w}) =& M\mathbb{E}\left[\max\{w,x\}\right] \\
=& M\left[\int_0^w wg(x)\,\mathrm{d}x + \int_w^{\bar{v}} xg(x)\,\mathrm{d}x\right] \\
=& \int_0^w (w-x)Mg(x)\,\mathrm{d}x + M\mathbb{E}[x]
\end{aligned}$$

Sequential auctions. The sequential auctions with identical values can be seen as a special case of the sales order $M, ..., 1$. The first-order condition characterises ${w_j}^* = v - \Pi^F_{j+1}$, where $\Pi^F_{M+1} = 0$. This is the optimal bid if the flexible bidder did not win the auction for good $j-1$, and she bids $w^*_f = 0$ in all subsequent auctions if she won good $j-1$.

The sequential first-price auction is solved analogously. The first-order condition yields ${w_j}^* = v - \frac{G({w_j}^*)}{g({w_j}^*)} - \Pi^F_{j+1}$, where $\Pi^F_{M+1} = 0$. This is the optimal bid if the flexible bidder did not win the auction for good $j-1$, and she bids $w_f^* = 0$ in all subsequent auctions if she won good $j-1$.

The auctioneer's revenue is also computed recursively. We have

$$R_j(w_j) = G(w_j)\left[w_j + (M-j)\mathbb{E}[x]\right] + \int_{w_j}^{\overline{v}} x g(x) + (1-G(w_j))R_{j+1}(w_{j+1})$$

D Formulas and closed form solutions for flexible bidder's payoffs, revenue, and efficiency, 2-good case with uniform distributions of a and b

The closed form solutions for the various auction types (whenever available) are listed below. We compare them using the functions *Reduce* and *Simplify* in *Mathematica* and find the ordering posited in section 4.

$$\begin{aligned}
\Pi^F_{P2|v_A,v_B} &= \frac{1}{6}({v_A}^3 + 3{v_A}^2(1-v_B) + 3{v_B}^2) \\
\Pi^F_{Sim2|v_A,v_B} &= \frac{{v_A}^2 - 2{v_A}^2 v_B + {v_B}^2}{2-2{v_A}^2} \\
\Pi^F_{SeqA2|v_A,v_B} &= \frac{1}{8}\left({v_A}^4 - 4{v_A}^3(v_B-1) + 4{v_A}^2(v_B-1)^2 + 4{v_B}^2\right) \\
\Pi^F_{SeqB2|v_A,v_B} &= \frac{1}{8}\left({v_A}^4 - 4{v_A}^2(v_B-1) + 4{v_B}^2\right) \\
\Pi^F_{Sim1|v_A,v_B} &= \frac{{v_A}^2 - {v_A}^2 v_B + {v_B}^2}{4-{v_A}^2} \\
\Pi^F_{SeqA1|v_A,v_B} &= \frac{1}{64}\left({v_A}^2(4+v_A)^2 - 4{v_A}^2(4+v_A)v_B + 4(4+{v_A}^2){v_B}^2\right) \\
\Pi^F_{SeqB1|v_A,v_B} &= \frac{1}{64}\left({v_A}^4 - 8{v_A}^2(v_B-2) + 16{v_B}^2\right)
\end{aligned}$$

$$\begin{aligned}
R_{P1|v_A,v_B}(w_A,w_B) = &\int_0^{w_A}\int_{w_B-w_A+a}^{1}(w_A+b)\,\mathrm{d}G(b)\,\mathrm{d}G(a) + \int_0^{w_A}\int_0^{w_B-w_A+a}(w_B+a)\,\mathrm{d}G(b)\,\mathrm{d}G(a) \\
&+ \int_{w_A}^{1}\int_0^{w_B}(w_B+a)\,\mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{1}\int_{w_B}^{1}(a+b)\,\mathrm{d}G(b)\,\mathrm{d}G(a)
\end{aligned}$$

$$
\begin{aligned}
R_{Sim1|v_A,v_B} =& G(w_A)\,(1-G(w_B))\,\mathrm{E}[w_A+b \mid \text{F wins A}]\\
&+G(w_B)\,(1-G(w_A))\,\mathrm{E}[w_B+a \mid \text{F wins B}]\\
&+G(w_A)G(w_B)\,\mathrm{E}[w_A+w_B \mid \text{F wins A and B}]\\
&+(1-G(w_A))\,(1-G(w_B))\,\mathrm{E}[a+b \mid \text{F wins nothing}]
\end{aligned}
$$

$$
\begin{aligned}
R_{SeqJ1|v_A,v_B} =& G({w_J}^*)\Big(G({w_K}^*)\,\mathrm{E}[{w_J}^*+{w_K}^* \mid \text{F wins J and K}]\\
&+(1-G({w_K}^*))\,\mathrm{E}[{w_J}^*+k \mid \text{F wins J, not K}]\Big)\\
&+(1-G({w_J}^*))\Big(G(\overline{w_K}^*)\,\mathrm{E}[j+\overline{w_K}^* \mid \text{F wins K, not J}]\\
&+(1-G(\overline{w_K}^*))\,\mathrm{E}[j+k \mid \text{F wins nothing}]\Big)
\end{aligned}
$$

$$
\begin{aligned}
R_{P2|v_A,v_B} &= R_{Sim2|v_A,v_B} = R_{SeqA2|v_A,v_B} = R_{SeqB2|v_A,v_B} = 1\\
R_{Sim1|v_A,v_B} &= \frac{3{v_A}^4+4(8+{v_B}^2)+{v_A}^2({v_B}^2-8v_B-12)}{2({v_A}^2-4)^2}\\
R_{SeqA1|v_A,v_B} &= \frac{1}{128}(3{v_A}^4-4{v_A}^3(-4+3v_B)+16(8+{v_B}^2)+4{v_A}^2(4-8v_B+3{v_B}^2))\\
R_{SeqB1|v_A,v_B} &= \frac{1}{128}(3{v_A}^4-16{v_A}^2(-1+v_B)+16(8+{v_B}^2))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{P2|v_A,v_B} &= \frac{1}{6}({v_A}^3+3{v_A}^2(1-v_B)+3(2+{v_B}^2))\\
\mathcal{W}_{Sim2|v_A,v_B} &= \frac{2+{v_B}^2-{v_A}^2(1+2v_B)}{2-2{v_A}^2}\\
\mathcal{W}_{SeqA2|v_A,v_B} &= \frac{1}{8}\left({v_A}^4-4{v_A}^3(v_B-1)+4{v_A}^2(v_B-1)^2+4{v_B}^2+8\right)\\
\mathcal{W}_{SeqB2|v_A,v_B} &= \frac{1}{8}\left({v_A}^4-4{v_A}^2(v_B-1)+4{v_B}^2+8\right)\\
\mathcal{W}_{Sim1|v_A,v_B} &= \frac{{v_A}^4(1+2v_B)-{v_A}^2({v_B}^2+16v_B+4)+4(8+3{v_B}^2)}{2({v_A}^2-4)^2}\\
\mathcal{W}_{SeqA1|v_A,v_B} &= \frac{1}{128}(5{v_A}^4-4{v_A}^3(-8+5v_B)+16(8+3{v_B}^2)+4{v_A}^2(12-16v_B+5{v_B}^2))\\
\mathcal{W}_{SeqB1|v_A,v_B} &= \frac{1}{128}(128+5{v_A}^4+48{v_B}^2-16{v_A}^2(-3+2v_B))
\end{aligned}
$$

E Tables

Table 2: Equilibrium bids and interim payoffs (flexible bidder), revenue, and welfare; the multi-column for the sequential auction displays the second-stage bid conditional on winning or losing the first stage.

	(P2)	(Sim2)	(SeqA2)	(SeqB2)	(P1)	(Sim1)	(SeqA1)	(SeqB1)
				$v_A = 0, v_B = 0.5$				
${w_A}^*/\overline{w_A}^*$	0.000	0.000	0.000	0.000 0.000	0.000	0.000	0.000	0.000 0.000
${w_B}^*/\overline{w_B}^*$	0.500	0.500	0.500 0.500	0.500	0.250	0.250	0.250 0.250	0.250
Π_M^F	0.125	0.125	0.125	0.125	0.063	0.063	0.063	0.063
R_M	1.000	1.000	1.000	1.000	1.031	1.031	1.031	1.031
W_M	1.125	1.125	1.125	1.125	1.094	1.094	1.094	1.094
				$v_A = 0.5, v_B = 0.5$				
${w_A}^*/\overline{w_A}^*$	0.500	0.333	0.375	0.000 0.500	0.232	0.200	0.219	0.000 0.250
${w_B}^*/\overline{w_B}^*$	0.500	0.333	0.000 0.500	0.375	0.232	0.200	0.000 0.250	0.219
Π_M^F	0.208	0.167	0.195	0.195	0.110	0.100	0.110	0.110
R_M	1.000	1.000	1.000	1.000	1.050	1.040	1.048	1.048
W_M	1.208	1.167	1.195	1.195	1.160	1.140	1.159	1.159
				$v_A = 0.5, v_B = 1$				
${w_A}^*/\overline{w_A}^*$	0.500	0.000	0.125	0.000 0.500	0.193	0.133	0.156	0.000 0.250
${w_B}^*/\overline{w_B}^*$	1.000	1.000	0.500 1.000	0.875	0.480	0.467	0.250 0.500	0.469
Π_M^F	0.521	0.500	0.508	0.508	0.276	0.267	0.274	0.282
R_M	1.000	1.000	1.000	1.000	1.126	1.118	1.123	1.126
W_M	1.521	1.500	1.508	1.508	1.402	1.384	1.397	1.409
				$v_A = 0.9, v_B = 1$				
${w_A}^*/\overline{w_A}^*$	0.900	0.000	0.405	0.000 0.900	0.375	0.282	0.326	0.000 0.450
${w_B}^*/\overline{w_B}^*$	1.000	1.000	0.100 1.000	0.595	0.437	0.373	0.050 0.500	0.399
Π_M^F	0.622	0.500	0.582	0.582	0.354	0.313	0.356	0.362
R_M	1.000	1.000	1.000	1.000	1.144	1.109	1.138	1.140
W_M	1.622	1.500	1.582	1.582	1.498	1.423	1.494	1.502

Table 3: Relative differences for first-price PMA vs. first-price simultaneous auction

pma/sim		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	0.092	1	1	0.049	0.94	0.94	0.138	0.56	0.56
	$\widetilde{g}_3$	0.092	1	1	0.051	1	1	0.142	0.74	0.74
	$\widetilde{g}_2$	0.087	1	1	0.049	1	1	0.150	1	1
	uniform	0.065	1	1	0.038	1	1	0.155	1	1
	g_2	0.061	1	1	0.035	1	1	0.260	1	1
	g_3	0.055	1	1	0.031	1	1	0.351	1	1
	g_4	0.049	1	1	0.027	1	1	0.433	1	1
avg diff	$\widetilde{g}_4$	0.023			0.010			0.043		
	$\widetilde{g}_3$	0.023			0.011			0.047		
	$\widetilde{g}_2$	0.020			0.010			0.052		
	uniform	0.012			0.006			0.052		
	g_2	0.007			0.004			0.068		
	g_3	0.004			0.002			0.072		
	g_4	0.002			0.001			0.072		

Table 4: Relative differences for first-price PMA vs. first-price sequential auction, good A first

pma/seqA		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	0.010	0.5	0.72	0.022	0.92	1	-0.023	1	1
	$\widetilde{g}_3$	0.009	0.56	0.8	0.016	0.88	1	-0.023	1	1
	$\widetilde{g}_2$	0.008	0.62	0.9	0.011	0.86	1	-0.021	1	1
	uniform	0.005	0.64	0.94	0.005	0.84	1	-0.015	1	1
	g_2	0.003	0.7	0.98	0.003	1	1	-0.014	1	1
	g_3	0.002	0.76	0.98	0.003	1	1	-0.014	1	1
	g_4	0.001	0.96	0.96	0.002	1	1	-0.014	1	1
avg diff	$\widetilde{g}_4$	0.004			0.007			0.001		
	$\widetilde{g}_3$	0.004			0.005			0.002		
	$\widetilde{g}_2$	0.003			0.004			0.002		
	uniform	0.002			0.002			0.003		
	g_2	0.001			0.001			0.002		
	g_3	0.000			0.000			0.001		
	g_4	0.000			0.000			0.000		

Table 5: Relative differences for first-price PMA vs. first-price sequential auction, good B first

pma/seqB		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	-0.019	0.74	1	0.019	1	1	-0.042	0.74	1
	$\widetilde{g}_3$	-0.017	0.72	1	0.014	1	1	-0.040	0.74	1
	$\widetilde{g}_2$	-0.012	0.7	1	0.009	1	1	-0.036	0.72	1
	uniform	-0.005	0.64	1	0.004	1	1	-0.025	0.7	1
	g_2	0.001	0.86	0.86	0.003	1	1	-0.017	0.78	1
	g_3	0.001	0.92	0.92	0.003	0.98	1	-0.014	1	1
	g_4	0.001	0.94	1	0.002	0.98	1	-0.014	1	1
avg diff	$\widetilde{g}_4$	-0.008			0.003			-0.024		
	$\widetilde{g}_3$	-0.007			0.001			-0.023		
	$\widetilde{g}_2$	-0.004			0.001			-0.019		
	uniform	-0.001			0.000			-0.013		
	g_2	0.000			0.000			-0.007		
	g_3	0.000			0.000			-0.005		
	g_4	0.000			0.000			-0.003		

Table 6: Efficiency

W	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.825	0.806	0.821	0.832
$\widetilde{g}_3$	0.883	0.864	0.880	0.889
$\widetilde{g}_2$	0.988	0.969	0.985	0.993
g_1	1.219	1.205	1.217	1.221
g_2	1.449	1.439	1.447	1.449
g_3	1.571	1.565	1.571	1.571
g_4	1.649	1.645	1.648	1.648

Table 7: Revenue

R	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.496	0.491	0.492	0.494
$\widetilde{g}_3$	0.594	0.588	0.591	0.593
$\widetilde{g}_2$	0.755	0.747	0.752	0.754
g_1	1.069	1.062	1.067	1.069
g_2	1.378	1.373	1.377	1.378
g_3	1.530	1.527	1.529	1.530
g_4	1.621	1.619	1.621	1.621

Table 8: Bidder surplus

Π^F	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.329	0.316	0.329	0.337
$\widetilde{g}_3$	0.289	0.276	0.288	0.296
$\widetilde{g}_2$	0.234	0.222	0.233	0.238
g_1	0.150	0.143	0.150	0.152
g_2	0.070	0.066	0.070	0.071
g_3	0.041	0.038	0.041	0.041
g_4	0.027	0.025	0.027	0.027