Dynamic Stochastic General Equilibrium and Business Cycles

Lecture Notes for MPhil Course Macro IV, University of Oxford.

Florin O. Bilbiie[1]
Nuffield College, University of Oxford.

2 December 2005

ii

Contents

Why care? Aim of the course

This course is meant to achieve three objectives:

1. familiarize you with the way modern macroeconomics attempts to explain business cycle fluctuations, i.e. (co-) movement of macroeconomic time series, by using economic theory.

2. along the way to 1., learn state-of-the art modelling techniques, or tools that are of independent theoretical interest.

3. develop your economic intuition, which may easily seem of secondary importance when one struggles with 2; this is a trap we will try to avoid.

In a nutshell, we will use dynamic stochastic general equilibrium (DSGE) models in order to understand business cycles, i.e. the comovement of macroeconomic time series. Why do we need this framework in the first place?

Why **dynamic**? Real-world economic decisions are dynamic: just think of the consumption-savings decision, accumulation of wealth, etc. Why **stochastic**? The world is uncertain, and this fundamental uncertainty may be the source of macroeconomic volatility. There is a huge and endless debate as to what exactly are the sources of uncertainty, which we will touch upon. Why **general equilibrium**? GE theory imposes discipline. Agents' decisions are interrelated (the decision to consume by households interacts with households' decision to devote time to work, but also to the decision of firms to supply consumption goods, and to employ factors -such as labor- to produce these goods). These meaningful interactions take place in markets, and (under perfect competition at least) there will exist prices that make these markets clear. This begs questions about market power, the non-clearing of some markets, price setting, and various other frictions. On these issues (i.e. whether such frictions are important or not) there is again an enormous and endless debate that I will try to touch upon.

Most of the theory we will develop will concern the frictionless model in which 'technology shocks' are regarded as the main source of fluctuations: the **real**

business cycle (RBC) model originally due to Kydland and Prescott (1982)[1]. This is not to say that we should take this model at face value and refuse the idea of frictions (though many people do so), but because it constitutes an useful benchmark. I.e. we want to see how far can we go in explaining fluctuations by using the frictionless model and, importantly, what is it that we cannot explain, and which assumptions could we relax in order to explain which puzzling fact. We also want to have a vehicle for understanding basic theoretical concepts, develop our intuition and, importantly, talk about welfare.

Importantly, you should remember throughout this course that the techniques you learn here are being used in most branches of macroeconomics, pretty much independently of one's view regarding the presence -or lack thereof- of frictions (such as market power in the goods or labor markets, distortionary taxation, etc.) and the source fluctuations. To give just one prominent example, modern monetary policy analysis also uses DSGE models that nest the baseline frictionless models as a special case. These models[2], which you should see later in the year, augment the benchmark model by incorporating imperfect price adjustment and monopolistic competition. Two important implications of this are that, in contrast with the benchmark RBC model, 1. monetary policy influences the real allocation of resources; 2. other shocks than technology can play a crucial role in explaining fluctuations. A good starting point for understanding this literature is the excellent book 'Interest and Prices' by Michael Woodford (2003).

Read Lucas (2005), *Review of Economic Dynamics.*

---

[1] Something that is often overlooked or forgotten is that Kydland and Prescott did not try to show that 'real', technology shocks can explain the bulk of fluctuations. Indeed, their original model also featured nominal rigidities (wage stickiness) that allowed nominal disturbances to have real effects. The finding that technology shocks accounted for most of the observed fluctuations emerged from this larger model and was not imposed a priori.

[2] Dubbed by some 'New Keynesian', by others 'Neo-monetarist', 'Neo-Wicksellian', etc.

Chapter 1

What are we trying to explain? Stylised facts

READ King and Rebelo Section 2 and/or Cooley and Prescott Section 6.

The theory presented here attempts to explain business cycle fluctuations, i.e. movement of macroeconomic variables around a trend in which variables move together. This trend can be thought of as the 'balanced growth path', i.e. the equilibrium of the growth models which you have seen previously with Dr. Meeks. Business cycle theory uses the same model to study short-term fluctuations. It does this by documenting some statistics for macroeconomic time series and then building an 'artificial economy' (a business cycle model) in order to replicate them. Importantly, since stylised facts pertaining to both growth (long-term movements) and business cycles (short-term movements) are derived using the same data, business cycle theory consists of building one model that can explain both types of data features.

An important (and not free of consequences) step in analysing the data consists of deciding just how to extract the business-cycle information from a dataset, i.e. how to eliminate the trend. This is a whole 'industry', going from simply taking first-differences of logs of the raw data or using the *Hodrick-Prescott (HP) filter* to more sophisticated methods. The HP filter was first used by ... Hodrick and Prescott in 1980 to study empirical regularities in business cycles in quarterly post-war US data.

1
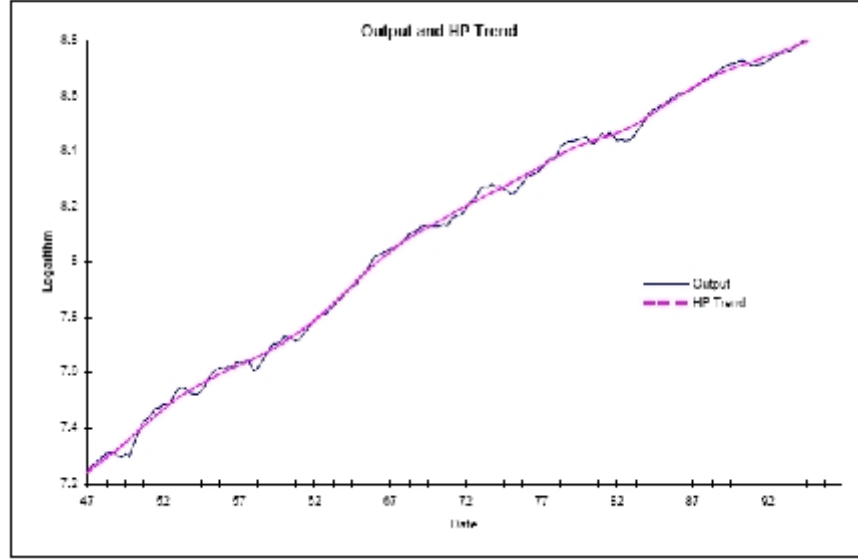
Source: KR (1999)

I briefly review what this method does. Consider a time series $\tilde{X}_t$ from which we want to extract the cyclical information. First take logs of this series (unless it is expressed as a 'rate'), $X_t = \ln \tilde{X}_t$. The HP filter decomposes this into a cyclical component X_t^C and a growth or secular (or trend) component X_t^G, where the latter is a weighted average of past, present and future observations. The cyclical component is hence:

$$X_t^C = X_t - X_t^G = X_t - \sum_{j=-J}^{J} \alpha_j X_{t-j}$$

The growth component is calculated by solving the optimization problem:

$$\min_{\{X_t^G\}_0^T} \sum_{t=1}^{T} \left(X_t^C\right)^2 + \lambda \sum_{t=1}^{T} \left[\left(X_{t+1}^G - X_t^G\right) - \left(X_t^G - X_{t-1}^G\right)\right]^2,$$

where λ is the smoothing parameter, whose conventionally chosen value for quarterly data is 1600. Note that when $\lambda \to \infty$ we get a linear trend as the growth component, while for λ the growth component is simply the series.

For our purposes it is enough to work with the stylised facts borrowed from King and Rebelo (1999) (which are instead based on an extensive study by Stock and Watson in the same volume of the Handbook of Macroeconomics).

Table 1: Business Cycle Statistics for U.S. Economy

Variable x	σ_x	σ_x/σ_y	$E\left[x_t x_{t-1}\right]$	$corr\left(x, y\right)$
y	1.81	1.00	0.84	1.00
c	0.35	0.74	0.8	0.88
i	5.30	2.93	0.87	0.80
l	1.79	0.99	0.88	0.88
Y/L	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Source: King and Rebelo, 1999

Most macroeconomists would know the main features of fluctuations by heart. Here are the main cyclical properties (i.e. co-movement of selected series with total output Y_t^C):

1. consumption C_t^C is less volatile than output

2. investment I_t^C is much more volatile than output (about three times)

3. hours worked L_t^C are as volatile as output

4. capital K_t^C is much less volatile than output

5. both labor productivity Y_t^C/L_t^C and real wage W_t^C are much less volatile than output

$$var\left(K_t^C\right) \sim var\left(W_t^C\right) < var\left(C_t^C\right) < var\left(Y_t^C\right) \sim var\left(L_t^C\right) << var\left(I_t^C\right)$$

Two additional observations, useful when we will consider variations of the baseline RBC model, are:

- (related to 4 above) although the capital stock is much less volatile than output, capital utilization is more volatile than output.

- (related to 3 above) hours per worker are much less volatile than output, suggesting that variations in total hours are accounted for by the extensive margin (employment).

In addition, all aggregates display substantial persistence as judged by the first-order autocorrelation.

Chapter 2

The Benchmark DSGE-RBC model

The stylised facts concerning fluctuations can be put together with the stylised facts on growth that you have seen a month or so ago with Dr. Meeks (just to remind you, the 'Kaldorian facts' include: the shares of income components and output components are roughly constant, the capital/output ratio is constant - both variables grow at the same rate, the consumption-to-output ratio is roughly constant, etc.). These facts imply that factors inducing permanent changes in the level of economic activity have proportional effects across series. Finally, hours worked per person are also constant, despite the real wage growing.

The real business cycle model does just this: it uses a 'stochastic' version of the growth model due to Brock and Mirman, therefore allowing for growth and fluctuations to be studied within the same model.

2.1 Environment

We assume that the economy is populated by an infinite number of atomistic households who are identical in all respects. Preferences of these households, defined over consumption C_t and hours worked L_t, are additively separable over time:

$$u\left(\{C_t, L_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right),$$

where $\beta \in (0,1)$ is the discount factor, and $U\left(.,.\right)$, the momentary felicity function, is continuously differentiable in both arguments and increasing and concave in C and decreasing and convex in L (i.e. increasing and concave in leisure).

The technology for producing the single good of this economy Y_t is described by the production function:

$$Y_t = A_t F\left(K_t, L_t\right),$$

where K_t is the stock of capital and L_t is labor. $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is increasing in both arguments, concave in each argument, continuously differentiable and homogenous of degree one. Moreover, $F(0,0) = F(0, L_t) = F(K_t, 0) = 0$ and the 'Inada conditions':

$$\lim_{K \to 0} F_K\,() = \infty; \lim_{K \to \infty} F_K\,() = 0.$$

2.2   Planner (centralised) economy

Suppose that the economy is governed by a benevolent social planner who chooses sequences $\{C_t\}_0^\infty, \{K_{t+1}\}_0^\infty, \{L_t\}_0^\infty$ to maximize the intertemporal objective,

$$\max E_t \sum_{i=0}^\infty \beta^i U\,(C_{t+i}, L_{t+i}) \tag{2.1}$$

subject to initial conditions for the stock of capital K_0 and technology A_0 and to the following constraints. Total output of this economy Y_t is produced using physical capital and labor:

$$Y_t = A_t F\,(K_t, L_t)\,, \tag{2.2}$$

where A_t is an exogenous productivity shifter, a 'technology shock' whose dynamics will be specified further. In this closed economy without government, output is used for two purposes: consumption and augmenting the capital stock, i.e. investment.

$$C_t + I_t = Y_t \tag{2.3}$$

The stock of capital K_t accumulates obeying the following dynamic equation (this is a rough approximation to the method used in practice, called the 'perpetual inventory method', to construct the capital stock) where we assumed that depreciation is constant and I_t the amount invested at t in the capital stock:

$$K_{t+1} = (1 - \delta)K_t + I_t \tag{2.4}$$

Finally, the amount of time spent working is bounded above by time endowment, normalized to unity: $L_t \leq 1$. We will assume an interior solution, such that some time is always devoted to leisure: $L_t < 1$.

To solve this problem consolidate all equality constraints into a single one and express consumption as a function of future capital, present capital and hours worked:

$$C_t + K_{t+1} = (1 - \delta)K_t + A_t F\,(K_t, L_t) \tag{2.5}$$

You can now solve our optimization problem using one of the techniques you learned. For example, using the Euler equation apparatus (see Appendix for a

refresher), we differentiate the following objective function with respect to next period's state[1] (capital) and hours worked:

$$\max_{\{K_{t+1}, L_t\}} E_t \sum_{i=0}^{\infty} \beta^i U\left[(1-\delta)K_{t+i} + A_{t+i}F\left(K_{t+i}, L_{t+i}\right) - K_{t+i+1}, L_{t+i}\right]$$

The first-order equilibrium conditions with respect to K_{t+1} and L_t respectively (together with the budget constraint above) are:

$$U_C\left(C_t, L_t\right) = \beta E_t \left\{U_C\left(C_{t+1}, L_{t+1}\right)\left[A_{t+1}F_K\left(K_{t+1}, L_{t+1}\right) + 1 - \delta\right]\right\} \quad (2.6)$$

$$-U_L\left(C_t, L_t\right) = U_C\left(C_t, L_t\right)A_t F_L\left(K_t, L_t\right) \quad (2.7)$$

The first equation states that the marginal cost of saving a unit of the consumption good today be equal to the expected marginal benefit of saving this tomorrow times the gross benefit of augmenting the capital stock by saving, where the latter is given by the marginal product of capital minus depreciation. The second equation states that the marginal disutility of working be equal to the marginal benefit of working, in utility terms. Alternatively, it equates the marginal rate of substitution between consumption and hours worked $-U_L\left(C_t, L_t\right)/U_C\left(C_t, L_t\right)$ to the marginal rate at which labor is transformed into the consumption good: $A_t F_L\left(K_t, L_t\right)$.

In terms of practical implementation, you will usually want to ensure (especially when you are dealing with much larger models) that you do have as many equations as variables in order to solve your model. In this simple example, we have the two first-order conditions plus the resource constraint for three variables C_t, K_{t+1}, L_t. (If you wanted to solve for investment and output you would simply use 2.3 and 2.4).

However, note a more subtle point related to finding the whole path of solutions for the variables of interest. Let's abstract from labor, for example by assuming that the household does not care about it at all, so we drop (2.7) and L_t from all equations. We need to find the entire paths for $\{C_t\}_0^\infty, \{K_{t+1}\}_0^\infty$ from 2.5 and the first equation of 2.6 and we have an initial condition for the capital stock K_0. You may be tempted to think that we are done, since 2.6 is a first-order difference equation, and we have one initial condition. This is misleading, since while 2.6 is a first-difference equation in C, it is a second-order difference equation in K, the variable for which we have the initial condition. In fact, we can substitute C_t from 2.5 into 2.6 to obtain a second-order difference equation in K_t, K_{t+1}, K_{t+2}. And we still have only one initial condition ... Why am I bothering you with this? Because you may now understand why we need an additional boundary condition on capital in order to solve for its entire optimal path $\{K_{t+1}\}_0^\infty$. This condition is the Transversality condition: $\lim_{i\to\infty} E_t\left[\beta^i U_C\left(C_{S,t+i}, .\right)K_{t+i}\right] = 0$.

Finally, note that since the model is stochastic, the decision rules are not found at time 0 and then remain unchanged; the model is stochastic, and a

---

[1]Differencing w.r.t. next period's state is equivalent to differentiating w.r.t. this period's control.

new realization of the shock each period changing agents' information set. This makes decision rules *state-contingent:* how much to consume, work, save etc., depends on the state of the economy in a given period. The state of this model is bi-dimensional: (A_t, K_t), where A is an exogenous state and K is an endogenous state. Therefore, formally 'decision rules' that solve the system of equilibrium conditions are best written as $C_t(A_t, K_t); K_{t+1}(A_t, K_t); L_t(A_t, K_t)$.

2.3   Competitive equilibrium (or decentralising the planner outcome)

In most applications, you will want to study economies where decisions are made by economic agents in a decentralised way, rather than planned economies. We therefore study a decentralised, rational expectations competitive equilibrium of the baseline RBC model. There are many ways we could decentralise the model economy described above, and here we choose the simplest one: a sequential competitive equilibrium in which households and firms interact each period as specified below in markets.

Households own the stock of capital (and hence own the firms since all capital is physical capital) and have to decide: how much capital to accumulate, how much to consume and how much to work in any given period. Let superscript s on any variable denote the households' counterpart to the aggregate one: e.g. K_{t+1}^s is households' stock of capital next period, etc. In order to avoid confusion, let us use bold letters for aggregate values, e.g. aggregate capital stock is $\mathbf{K}_{t+1}$. Households earn a wage rate W_t from working in firms and a rental rate from renting capital to firms each period R_t^K; they take both these prices[2] as given (remember this is a purely frictionless economy), where these prices are functions of the aggregate state of the economy: $W(A_t, \mathbf{K}_t); R^K(A_t, \mathbf{K}_t)$. In a rational expectations equilibrium, agents will forecast these prices, and will have to know the functional forms $W()$ and $R^K()$. They also have to know the laws of motion for A_t and $\mathbf{K}_t$. The households will solve:

$$\max E_t \sum_{i=0}^{\infty} \beta^i U\left(C_{t+i}, L_{t+i}^s\right) \tag{2.8}$$

subject to a budget constraint. The latter is given by:

$$C_t + I_t^s = W_t L_t^s + R_t^K K_t^s + \pi_t \tag{2.9}$$

The left hand-side specifies how much the household spends on consumption and investment respectively. The right-hand side specifies that households' resources

---

[2]Do not get confused by this terminology: R_t^K is NOT the **price** of capital in this economy, but the **return** on capital, net of depreciation. The price of capital is fixed to 1, since the same good is used for consumption and investment, and investment is reversible (you can transform the capital good back into the consumption good freely). See the relevant section on Asset Pricing in Chapter 3. Investment adjustment costs would change this feature (see the part taught by Professor Muellbauer).

come from labor income, capital income (from renting capital to firms) and profit income (if any). Since the law of motion for capital is still $K_{t+1}^s = (1-\delta)K_t^s + I_t^s$ we can write this as:

$$C_t + K_{t+1}^s = W_t L_t^s + \left(1 + R_t^K - \delta\right) K_t^s + \pi_t, \tag{2.10}$$

where we could further define the net real interest rate of this economy as $R_t \equiv R_t^K - \delta$, rental rate net of depreciation. Using the same technique as before to solve this problem, the decision rules of the household $C_t\left(A_t, K_t^s, \mathbf{K}_t\right); K_{t+1}^s\left(A_t, K_t^s, \mathbf{K}_t\right); L_t\left(A_t, K_t^s, \mathbf{K}_t\right)$ are a solution to the optimality conditions (together with 2.10 and transversality):

$$K_{t+1} \quad : \quad U_C\left(C_t, L_t^s\right) = \beta E_t\left\{U_C\left(C_{t+1}, L_{t+1}^s\right)\left[R_{t+1}^K + 1 - \delta\right]\right\} \tag{2.11}$$
$$L_t \quad : \quad -U_L\left(C_t, L_t^s\right) = U_C\left(C_t, L_t^s\right) W_t \tag{2.12}$$

Firms choose how much labor to hire and capital to rent in the spot markets from the household in order to produce the consumption good of this economy. Let a d superscript stand for 'value of variable from firms' standpoint'. Firms are perfectly competitive and choose K_t^d and L_t^d to solve $\max \pi_t$ each period, i.e. maximize profits

$$\pi_t = A_t F\left(K_t^d, L_t^d\right) - \left(W_t L_t^d + R_t^K K_t^d\right), \tag{2.13}$$

where the first term denotes the firms' sales and the term in brackets is the total cost of producing. Optimization leads to:

$$A_t F_L\left(K_t^d, L_t^d\right) \quad = \quad W_t, \tag{2.14}$$
$$A_t F_K\left(K_t^d, L_t^d\right) \quad = \quad R_t^K.$$

Since the production function exhibits constant returns to scale, profits will always be zero - just replace these factor prices in the expression for profits and apply Euler's theorem.

Exercise 1 *Tricky(-ish): can you tell how many firms produce in this economy?*

Market clearing. Households and firms meet in spot markets every period and equilibrium requires that all these markets clear. 'Counting' the markets properly, ensuring their clearing, and understanding how this works is an essential part in practical modelling - and may not be trivial in large models. In this simple economy there are three markets: for labor, capital, and for the consumption good (output). Importantly, when stating the equilibrium conditions for an economy with n markets, you only need to specify market clearing conditions for $n-1$ markets. Walras' Law ensures that then the nth market will also be in equilibrium. In our case, we limit ourselves to factor markets:

$$L_t^d \quad = \quad L_t^s$$
$$K_{t+1}^d \quad = \quad K_{t+1}^s \left(= \mathbf{K}_{t+1}\right).$$

Note that I wrote the capital market clearing condition at $t + 1$ -this may be helpful for any market clearing concerning state variables in practical implementation. Finally, consistence of individual and aggregate decisions requires that the law of motion for capital conjectured by households has to coincide in equilibrium with the aggregate law of motion: $K^s_{t+1}(A_t, \mathbf{K}_t, \mathbf{K}_t) = \mathbf{K}_{t+1}(A_t, \mathbf{K}_t)$.

Exercise 2 *Prove that Walras' Law holds in this economy, i.e. that $Y_t = C_t + I_t$.*

Exercise 3 *4AM: Apply Dynamic Programming (Bellman Principle) techniques that you have learned both to the centralized and the decentralized economies and show that the solution you get is equivalent to the solution in these notes.*

2.4   Welfare theorems

You may recall from Microeconomics (or if you haven't seen this, you will see it in the Micro course next term) that the Pareto optimum (planner equilibrium) and competitive equilibrium coincide under certain conditions on preferences, technology, etc. The following exercise asks you to show that this is the case in our model:

Exercise 4 *Show heuristically that the planner and competitive equilibria coincide (hint: show that first-order conditions coincide).*

Note that this result applies to this very simple, frictionless economy. In most applications the planner and the competitive equilibria will be different: the competitive equilibrium will be sub-optimal due to the presence of externalities, distortionary taxes, trading frictions, etc. I hope we do get to see some examples of this later in the course.

When welfare theorems do apply, however, the big advantage is that we can move freely between competitive and planner equilibrium, and the latter is usually i. unique and ii. easy to calculate (solution to a concave programming problem). Otherwise, existence and uniqueness of a competitive equilibrium may not be trivial to establish.

2.5   Functional forms

In order to simplify analytics, I will now introduce functional forms. We already assumed that the production function is homogenous of degree one (exhibits constant returns to scale). Let us assume that it is of the Cobb-Douglas form, consistent with growth facts:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha},$$

where α is the 'capital share' - if capital is being paid its marginal product, it earns an α share of output. Note that marginal products of capital and labor

respectively are (equal to rental rate and wage):

$$R_t^K = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \frac{Y_t}{K_t}; W_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) \frac{Y_t}{L_t}$$

You immediately see that the capital and labor shares in total output are constant and equal to the respective exponents i the production function.

I will also specialize preferences to take the form:

$$U(C_t, L_t) = \ln C_t - v(L_t), \tag{2.15}$$

where $v()$ is the disutility of labor and is continuously differentiable, increasing and convex. This additively separable utility function is consistent with balanced growth and has some other desirable properties spelled out below. There do exist non-separable utility functions that are consistent with balanced-growth that you may want to use in your applications - see the original article by King, Plosser and Rebelo (1988) dealing with these issues.

2.6 Steady-state and conditions on preferences

We will exploit the welfare theorems in the remainder and focus on the Planner equilibrium in solving the model. You should note, however, that all the techniques described here can be equally applied to the decentralized equilibrium.

To start with, we want to ensure that our model has a unique *non-stochastic steady-state* that is consistent with some 'growth stylised facts' reviewed before, concerning some ratios being constant (see the part taught by Dr. Roland Meeks on Growth). In the non-stochastic steady-state, all variables X_t are constant $X_{t+1} = X_t$ and technology is normalized to 1, $A_{t+1} = A_t = 1$. (note that we abstract from growth: this can be incorporated by assuming that $A_{t+1} = (1+g) A_t$ where g would be the exogenous rate of growth). Moreover, we can drop the expectations operator.

The Euler equation 2.6 evaluated at the steady state yields: $\beta^{-1} = F_K(K, L) + 1 - \delta$. Since the marginal product of capital depends on the capital-labor ratio, it follows directly that the latter is also constant in steady-state: $\alpha \left(\frac{K}{L}\right)^{\alpha-1} = \beta^{-1} - 1 + \delta$. Since the marginal product of labor (real wage) also depends only on the capital-labor ratio, this will also be constant and can be written as a function of deep parameters. Using the definition of the real interest rate we find: $R = \beta^{-1} - 1$. Capital accumulation evaluated at steady state yields the investment-to-capital ratio $\frac{I}{K} = \delta$ - in steady state, investment merely replaces depreciating capital.

Finally, an important remark on the properties of hours worked in steady-state is in order (this confuses surprisingly many people, please do not be among them!!!). Since we observe in post-war data that there is a long-run trend in wages, but no such trend in hours, we want steady-state hours worked to be independent of the wage. Moreover, more generally, we want preferences to be consistent with constant hours for a straightforward reason - per-capita hours

are simply bounded above by the time endowment, so they cannot grow (can you work more than 24 hours?). It turns out that the utility function we have chosen does yield constant steady-state hours. Using the functional form of the utility function to evaluate the intratemporal optimality condition we have:

$$v_L(L) = \frac{W}{C}.$$

Assume for simplicity (this is in no way necessary) that $C = WL$, you see that this becomes $Lv_L(L) = 1$ and hours are independent of the wage or any other potentially trending component. Again, there do exist more general (non-separable) preferences exhibiting this property, but this is enough to make our point.

Rather than trying to find constant steady-state ratios, etc., as I have done above, let's try to **solve for the steady-state explicitly**. To do that, we evaluate the equilibrium conditions at the steady state and use the assumed functional forms for F and U. We assume that technology is constant and equal to A (we do not normalize $A = 1$ as previously). Also, since the steady-state real interest rate and the discount factor are related one-to-one by $R = \beta^{-1} - 1$ I will treat R as a parameter rather than β (just for analytical convenience). From the Euler equation in steady-state we obtain consumption as a function of labor:

$$K = \left(\frac{R+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$$

Substituting this into the reduced constraint 2.5 we have consumption as a function of labor:

$$
\begin{aligned}
C &= A\left(\frac{R+\delta}{\alpha A}\right)^{\frac{\alpha}{\alpha-1}} L - \delta\left(\frac{R+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} L \\
&= \left(\frac{R+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}\left[\frac{R+\delta}{\alpha} - \delta\right] L
\end{aligned}
$$

Substituting both in the intratemporal optimality condition:

$$v_L(L)L = \frac{(1-\alpha)(R+\delta)}{R+\delta-\delta\alpha} = \frac{1}{1 + R\left[(1-\alpha)(R+\delta)\right]^{-1}},$$

which after assuming a functional form for $v_L(L)$ can be solved for L, allowing thence to solve for all other variables. Note, consistent with the intuition above, that hours are independent of the level of technology. Steady-state hours do, however, depend on preferences. Consider for example a standard functional form for $v(L) = \chi\frac{L^{1+\varphi}}{1+\varphi}$, leading to $v_L(L) = \chi L^{\varphi}$. Substituting this in the expression above we have

$$L = \left[\frac{1}{\chi}\frac{1}{1 + R\left[(1-\alpha)(R+\delta)\right]^{-1}}\right]^{\frac{1}{1+\varphi}} \qquad (2.16)$$

Intuitively, the more the agent dislikes work (the higher χ), the less she works in steady-state.

2.7 Loglinearisation

The model we have described consists of a system of non-linear stochastic difference equations. Finding closed-form solutions for these is impossible, unless we assume that there is full depreciation and log utility in consumption (see the Chapter on RBC in David Romer's textbook for this special case). In general, we need to resort to approximation techniques - and there are many that people have used (see the article by Cooley and Prescott for a non-exhaustive review). Arguably the most widely used technique relies on taking a first-order approximation to the equilibrium conditions around the non-stochastic steady-state and studying the behaviour of endogenous variables in response to small stochastic perturbations to the exogenous process. This is the approach we will follow here. This is an instance of the *implicit function theorem* that you must have seen in Maths: calculating the effects of changes in some 'parameters' (here, the stochastic shocks) on the solution to some system of equation for the variables (here, the optimal decision rules for endogenous variables that are implicitly defined by the system of equilibrium conditions).

Let small-case letters denote percentage deviations of the upper-case variable form its steady-state value, or equivalently log-deviations. e.g. for any variable X_t we have

$$x_t \equiv \ln \frac{X_t}{X} \simeq \frac{X_t - X}{X},$$

where the last approximation follows from $\ln (1 + a) \simeq a$.

2.7.1 Intermezzo on loglinearisation

There are many ways you can loglinearize an equation and you should always try to do it in two different ways to minimize the probability of mistakes. In order to loglinearize a (any) system of equations, here is the only one theorem you need to know. Suppose we have a nonlinear equation relating two differentiable functions $G(\mathbf{X}_t)$ and $H(\mathbf{Z}_t)$ defined over vectors of variables $X_t = \left(X_t^1, X_t^2, ..., X_t^n \right); Z_t = \left(Z_t^1, ..., Z_t^m \right):$

$$G(\mathbf{X}_t) = H(\mathbf{Z}_t)$$

This equation can be approximated to a first order around the steady-state values X and Z (of course, equation also holds in steady state $G(\mathbf{X}) = H(\mathbf{Z})$) using the Taylor expansion by:

$$\nabla G(\mathbf{X})(\mathbf{X}_t - \mathbf{X}) \simeq \nabla H(\mathbf{Z})(\mathbf{Z}_t - \mathbf{Z}),$$

where $\nabla G(\mathbf{X})$ is the gradient of G with respect to X, i.e. the row vector stacking the partial derivatives evaluated at steady state $\left\{ G_i() = \frac{\partial G}{\partial X^i} \right\}_{i=1}^n$. This expression contains absolute deviations of each variable from its steady-state value, $X_t^i - X^i$ (whereas what we are after are log-deviations). We can derive the log-linearized version of this by multiplying and dividing each term in

the summation by the steady-state value to get for each term $\frac{X_t^i - X^i}{X^i} X^i \simeq X^i x_t^i$, obtaining:

$$\sum_{i=1}^{n} G_i\left(\mathbf{X}\right) X^i x_t^i \simeq \sum_{i=1}^{m} H_i\left(\mathbf{Z}\right) Z^i z_t^i$$

This has the advantage of being very general -indeed pretty much any equation can be written in this form.

In many cases, however, you can make use of the much simpler tricks:

$$
\begin{aligned}
X_t &= X\left(1 + x_t\right) \\
X_t Z_t &= XZ\left(1 + x_t + z_t\right) \\
f\left(X_t\right) &= f\left(X\right)\left(1 + \frac{f'\left(X\right) X}{f\left(X\right)} x_t\right),
\end{aligned}
$$

where $\frac{f'(X)X}{f(X)}$ is the elasticity of f with respect to X. For example, the last equation says that the log-deviation of f is approximately equal to the elasticity of f times the log-deviation of its argument.

2.7.2   Loglinearising the planner economy

Let's loglinearize our equations. I will use different ways of loglinearising for each of them, not to confuse you but to give you as many 'tools' as possible. To be sure of your result, you can always apply the most general method I gave you above. Some of them are really easy: e.g. the production function is already log-linear, in the sense that taking logs of it we get a linear equation: $\ln Y_t = \ln A_t + \alpha \ln K_t + (1 - \alpha) \ln L_t$. Evaluate this at steady state and subtract from it the resulting steady-state equation, and you get (note that this holds exactly, it is not an approximation):

$$y_t = a_t + \alpha k_t + (1 - \alpha) l_t \tag{2.17}$$

The capital accumulation equation is not log-linear, but we can loglinearize as follows. Divide through by K_t to get :

$$\frac{K_{t+1}}{K_t} = (1 - \delta) + \frac{I_t}{K_t}$$

Applying the second 'trick' above you get:

$$\frac{K}{K}\left(1 + k_{t+1} - k_t\right) = 1 - \delta + \frac{I}{K}\left(1 + i_t - k_t\right),$$

Simplifying:

$$k_{t+1} - k_t = \frac{I}{K}\left(i_t - k_t\right),$$

and substituting the investment to capital SS ratio:

$$k_{t+1} = (1 - \delta) k_t + \delta i_t \tag{2.18}$$

The Euler equation is (having substituted the functional form of the utility function):

$$1 = \beta E_t \left\{ \frac{C_t}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_{t+1}} - \delta + 1 \right) \right\}$$

This is a bit trickier because it involves expectations. However, due to our focus on first-order approximations we implicitly assume certainty equivalence, so the expectation of a non-linear function of a random variable will be equal (to first order) to the function of the expectation of that variable. Having noted this, let's write the perfect-foresight version of the equation[3]:

$$\frac{C_{t+1}}{C_t} = \beta \left(\alpha \frac{Y_{t+1}}{K_{t+1}} - \delta + 1 \right)$$

and apply the second trick again:

$$\frac{C}{C} \left(1 + c_{t+1} - c_t \right) = \beta \left(\alpha \frac{Y}{K} \left(1 + y_{t+1} - k_{t+1} \right) - \delta + 1 \right).$$

The constant terms drop out again, so:

$$c_{t+1} - c_t = \beta \alpha \frac{Y}{K} \left(y_{t+1} - k_{t+1} \right)$$

Recalling that in steady state we have $\frac{Y}{K} = \frac{R+\delta}{\alpha}$ and taking expectations (remember capital is a state variable, $E_t k_{t+1} = k_{t+1}$), the loglinearised Euler equation is:

$$E_t c_{t+1} - c_t = \frac{R + \delta}{1 + R} \left(E_t y_{t+1} - k_{t+1} \right) \tag{2.19}$$

This equation captures intertemporal substitution in consumption: when marginal product of capital is expected to be high, expected consumption growth is high (consumption today falls as the planner saves to augment the capital stock). Note that we have normalized the elasticity of intertemporal substitution (the curvature of the utility function) to unity implicitly by assuming a log utility function. Further discussion of this follows when studying the decentralized economy.

Loglinearisation of the intratemporal optimality condition $v_L(L_t) = (1 - \alpha) \frac{Y_t}{L_t} \frac{1}{C_t}$ yields (applying the 'third trick'):

$$v_L(L) \left[1 + \varphi l_t \right] = (1 - \alpha) \frac{Y}{L} \frac{1}{C} \left(1 + y_t - l_t - c_t \right),$$

where $\varphi \equiv v_{LL} L / v_L$ is the elasticity of the marginal disutility of work to variations in hours worked. A more useful interpretation of this parameter can be found in the loglinearisation of the competitive economy. Simplifying we get:

$$(1 + \varphi) l_t = y_t - c_t, \tag{2.20}$$

---

[3]See the next section for a loglinearisation of the Euler equation without working with the perfect-foresight version.

The economy resource constraint is loglinearised as follows. Apply the first trick to get

$$Y\left(1+y_t\right) = C\left(1+c_t\right) + I\left(1+i_t\right)$$

Divide through by Y and use that this also holds in steady-state, i.e. $Y = C+I$, to get:

$$y_t = \frac{C}{Y}c_t + \frac{I}{Y}i_t$$

We have not yet calculated the steady-state ratios that appear here (consumption to output and investment to output). What I want to emphasize is that often the steady-state ratios you calculate are informed by the loglinearisation. We have found the ratio of investment to capital and the ratio of output to capital previously, and the share of investment to output is just the ratio of the two. The share of consumption in output is then easily found:

$$
\begin{aligned}
\frac{I}{Y} &= \frac{I}{K}\frac{K}{Y} = \alpha\frac{\delta}{R+\delta}; \frac{C}{Y} = 1 - \frac{I}{Y}, \text{ so:}\\
y_t &= \left(1 - \alpha\frac{\delta}{R+\delta}\right)c_t + \alpha\frac{\delta}{R+\delta}i_t
\end{aligned}
\tag{2.21}
$$

A boring but necessary part follows now. We need to 'count' equations and endogenous variables and ensure that the numbers square - i.e. we have as many endogenous variables as equations[4]. We have 5 variables we want to solve for: y, c, i, k, l and 5 equations: 2.17,2.19, 2.18, 2.20,2.21.

Exercise 5 *To make sure you get familiarity with this, substitute out investment i and output y from the above system and get a system in three equations and three unknowns: c, k, l. Now try to loglinearize directly the nonlinear 2.6,2.7 and 2.5 and make sure you get exactly the same result (of course, after using the functional forms for F and U that we have assumed).*

Finally, we need to specify the dynamics of technology - since this is the forcing exogenous stochastic process. It is standard in the literature to assume that A_t follows an AR(1) process in logs, i.e.:

$$a_t = \rho a_{t-1} + \varepsilon_t,$$

where ε_t is white noise. We will return to issues of measurement of a subsequently.

---

[4]You may think this is trivial, but I can assure you that you will not laugh when you try to build your own models. You can bet that in first instance you will always end up with at east one equation or variable too many (or too few...). Of course, you can do the 'counting' after you have derived the fully non-linear equilibrium, but make sure that when you loglinearize you will use the same conditions.

2.7.3 Loglinearising the competitive economy

For completion, let's loglinearize the equilibrium conditions of the competitive economy. To make our life easier, let's use the market clearing conditions and substitute demand and supply for actual aggregate quantities (e.g. K^s and K^d replaced by K, etc.). The production function and capital accumulation concern the environment, i.e. are primitives of the model. Therefore, they are identical to the planner equilibrium above 2.17, 2.18.

The Euler equation is (having substituted also the definition of real interest rate):

$$1 = \beta E_t \left\{ \frac{C_t}{C_{t+1}} \left(R_{t+1} + 1 \right) \right\}$$

As before, certainty equivalence makes it easy to get the loglinearised Euler equation. Note that since the interest rate is already a rate (so it is in percentage points), we need not take its log-deviations: specifically, we define $r_{t+1} \equiv R_{t+1} - R$. To see this, take logs of the perfect-foresight version of the Euler equation (i.e. dropping the expectation operator) to get:

$$0 = \ln \beta + \ln C_t - \ln C_{t+1} + \ln \left(1 + R_{t+1} \right).$$

Add and subtract $\ln C$ and use that $\beta = \left(1 + R \right)^{-1}$ to get $c_{t+1} - c_t = \ln \left(1 + R_{t+1} \right) - \ln \left(1 + R \right) \simeq R_{t+1} - R \equiv r_{t+1}$ and take expectations (again, we can do this due to certainty equivalence) to get the loglinearised Euler equation:

$$E_t c_{t+1} - c_t = E_t r_{t+1} \tag{2.22}$$

You can also get this equation by assuming that real interest rates and future consumption are lognormal and homoskedastic[5]. The Euler equation in logs becomes (again, I (ab)use the equality sign for $\ln \left(R_{t+1} + 1 \right) = R_{t+1}$):

$$
\begin{aligned}
- \ln C_t \; &= \; \ln \beta + \ln E_t \left\{ C_{t+1}^{-1} \left(R_{t+1} + 1 \right) \right\} \\
&= \; \ln \beta + E_t \ln \left\{ C_{t+1}^{-1} \left(R_{t+1} + 1 \right) \right\} + \frac{1}{2} var_t \left(\ln \left\{ C_{t+1}^{-1} \left(R_{t+1} + 1 \right) \right\} \right) = \\
&= \; \ln \beta - E_t \ln C_{t+1} + E_t R_{t+1} + \frac{1}{2} var_t \left(\ln C_{t+1} \right) + \frac{1}{2} var_t \left(R_{t+1} \right) - cov_t \left(R_{t+1} \ln C_{t+1} \right)
\end{aligned}
$$

Under homoskedasticity the conditional second moments are constant and we can drop their time subscript. Hence, evaluating this equation at steady-state and subtracting the result from the dynamic equation we get 2.22 (constants, including second moments, drop out).

Equation 2.22 captures intertemporal substitution in consumption: when real interest rates are expected to be high, expected consumption growth is high (consumption today falls as the household saves). Note that we have normalized the elasticity of intertemporal substitution (the curvature of the utility function) to unity implicitly by assuming a log utility function. In general, the effect of interest rates on consumption will depend on this parameter.

---

[5]If X and Y are jointly lognormal, $\ln E \left[XY \right] = E \left[\ln XY \right] + \frac{1}{2} var \left[\ln XY \right]$.

Combining the definition of the gross interest rate with the equilibrium expression for the rental rate we have

$$1 + R_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta,$$

which, evaluated at steady-state gives $\frac{Y}{K} = \frac{R+\delta}{\alpha}$. A first-order approximation using the first trick on the left-hand side and the second trick on the right-hand side yields:

$$(1 + R) \, r_{t+1} = \alpha \frac{Y}{K} \left(1 + y_{t+1} - k_{t+1}\right) + 1 - \delta$$

Substituting $\frac{Y}{K}$ from the expression just derived we get:

$$r_{t+1} = \frac{R + \delta}{1 + R} \left(y_{t+1} - k_{t+1}\right)$$

Finally, loglinearisation of the intratemporal optimality condition $v_L \left(L_t\right) = W_t / C_t$ yields (applying the 'third trick'):

$$\varphi l_t = w_t - c_t,$$

where $\varphi \equiv v_{LL} / v_L L$ is the elasticity of the marginal disutility of work to variations in hours worked. More importantly, φ is referred to as the inverse elasticity of labor supply l to changes in the wage rate w, keeping fixed consumption c.[6] When $\varphi = 0$, labor supply is infinitely elastic - when demand shifts, the household is ready to work all the extra hours for the given real wage. Also, in this case consumption is independent of non-wage income, and hence of wealth. When $\varphi \to \infty$, labor supply is inelastic, and any labor demand shift generates movement in the real wage, while hours stay fixed.

NOTE: many papers specify the utility function over leisure, $1 - L_t$ rather than hours, by having an utility function of the form: $\ln C_t + h \left(1 - L_t\right)$ where $h\left(\right)$ is a continuously differentiable, increasing and concave function. Make sure you are able to derive the first-order conditions in this case. Notably, the elasticity of labor supply becomes $\varphi L / \left(1 - L\right)$ and hence depends on steady-state hours worked.

The expression for real wage and rental rate are already loglinear, so we have:

$$\begin{aligned} w_t &= y_t - l_t \\ r_t^K &= y_t - k_t \end{aligned}$$

These can be equivalently thought of as 'factor demands' by the firm, for a given level of output. As you would expect, demand is decreasing in the respective factor price.

---

[6] Because utility is separable in consumption and work, φ is also the inverse Frisch elasticity of hours to wage, i.e. the elasticity keeping fixed not consumption, but the marginal utility of consumption (which, when utility is separable, is proportional with consumption). For more general, non-separable preferences, the Frisch elasticity and the elasticity keeping consumption fixed are different objects.

The budget constraint of the household is:

$$\frac{C}{Y}c_t + \frac{I}{Y}i_t = \frac{WL}{Y}\left(w_t + l_t\right) + \frac{R^K K}{Y}\left(r_t^K + k_t\right)$$

You can convince yourselves that this is the same as the economy resource constraint that we found in the planner equilibrium: merely substitute the log-linearised expressions for the real wage and rental rate, and the steady-state shares WL/Y and $R^K K/Y$ to get just y_t on the right-hand side. This makes once more the point that the 'goods market clearing' condition (as it should be called in a competitive equilibrium), or the 'resource constraint', is in fact redundant once we have written down all other equilibrium conditions. Why? Because it is a linear combination of other equilibrium conditions and hence does not have any new information in it! If you -for some strange reason- insist on having this equation when solving the model (i.e. by including it in your computer code), you need to drop one of the following: expressions for factor prices, market clearing conditions, or the household budget constraint itself.

Counting variables and equations again, note that compared to the planner economy we have three more variables $(w, r^K$ and $r)$ and three more equations: two factor prices, and one as the definition of the interest rate. As I hope you already anticipate, after substituting out these three variables we get precisely the same equations as in the planner equilibrium - since the two equilibria are equivalent (something we have shown in the general, non-linear case and carries through for the approximate equilibrium).

2.8 Calibration

By steady-state analysis we found how steady-state variables and ratios (many of which appear in the loglinearised equilibrium) are related to 'deep' parameters, i.e. parameters pertaining to preferences and technology, say $X = S\left(\Theta\right)$, where Θ is the vector of all these Greek letters, in our case $\alpha, \delta, \beta, \varphi$. This mapping is one-to-one. An important step in solving the model is to get numbers for either of these. There are many ways to do this - basically you have to choose whether to treat steady-state variables as observables and solve for 'Greeks' by inverting $S : \Theta = S^{-1}\left(X\right)$ or treat deep parameters as known and solve for steady-state values using $X = S\left(\Theta\right)$.

The first approach is the closest in spirit to what original RBC proponents had in mind: use only observable data on macroeconomic aggregates from the National Income accounts -NIPA- and, using the steady-state relationships between 'Greeks' and steady-state ratios, find the 'Greeks'. This requires a great deal of knowledge of NIPA data and sometimes important choices and a judgement about which macroeconomic aggregates to use (e.g. how to treat consumer durables, profits, what is depreciation, etc.) - different choices imply different values for X and, obviously, different (sometimes very different) values for deep parameters. The '*Users' Manual*' for this approach is the article by Cooley and Prescott.

Example: Let's do this for our model. The discount factor is pinned down by the steady-state condition $R = \beta^{-1} - 1$, hence by merely looking at the average value of the interest rate we have a value for β (you do have to decide which interest rate to use though; King and Rebelo use the average return to capital as given by the average return on Standard&Poor's 500). Alternatively, don't use the interest rate but find the discount factor by using $\beta^{-1} = \alpha\frac{Y}{K}+1-\delta$, which means you first have to find α and δ and then pick β to match the capital-output ratio. In our simple model, the depreciation rate is simply equal to the share of investment to capital, $\delta = \frac{I}{K}$. α is simply found by recognizing that it is the share of capital income in total income: $\alpha = \frac{R^K K}{Y}$ (this sounds far simpler than it is - getting the right measure for 'capital income' is very tricky -see Cooley and Prescott). [IMPORTANT: Be careful to transform all rates - interest, discount, depreciation, etc.- such that you have quarterly values! Also, make sure to use per-capita values for the aggregates since the model economy is per-capita].

Arguably the most difficult parameter to choose is the elasticity of labor supply - or the elasticity of intertemporal substitution in labor supply. This is when even the most 'hard-core' RBC theorist has to give up - or use a 'cheap fix'. This fix consists of using special functional forms for preferences such that this elasticity is not parameterized, but fixed. Within the preference class we work with here, this is achieved by assuming either that $v(L) = \chi \ln L$, which effectively yields a unit elasticity of hours to wages $\varphi \equiv v_{LL}L/v_L = 1$ or that $v(L) = \chi L$, which effectively leads to an *infinitely* elastic labor supply since $\varphi \equiv v_{LL}L/v_L = 0$. In the general case, however, one needs to pick a value for this parameter[7]. Usually, one needs to resort to microeconomic studies estimating this elasticities, although it is not clear at all whether this parameter bears any relationship to what microeconometricians actually estimate. Finally, the relative weight the agent places on labor in the utility function can be inferred once all other parameters have been calculated by assuming a value for steady-state hours worked L and using the expression 2.16.

The other approach, which is best described as 'parameterization', simply picks values for the 'Greeks' from 'micro evidence' (never use the word 'calibration' if you use this approach in your own work, especially if some 'Minnesotta-bred' colleague is in the audience). This approach is being abused in the literature, especially when using very large models with many parameters, and you are better off avoiding it whenever possible.

2.9   'Solving' the model

Our model is finally a system of expectational difference equations. The three equations are (I am just re-stating the loglinearised equilibrium conditions for k, c, l, so I have eliminated i and y by eliminating the equations with a star from the system:

---

[7]See Prescott (1986) for a way of getting an 'empirical elasticity' based only on aggregate data, by using both household and establishment data on hours worked.

$$
\begin{cases}
E_t c_{t+1} - c_t = \frac{R+\delta}{1+R}\left(E_t y_{t+1} - k_{t+1}\right) \\
k_{t+1} = (1-\delta)\,k_t + \delta i_t \\
*i_t = \frac{R+\delta}{\alpha\delta} y_t + \left(1 - \frac{R+\delta}{\alpha\delta}\right) c_t \\
*y_t = a_t + \alpha k_t + (1-\alpha) l_t \\
(1+\varphi)\,l_t = y_t - c_t,
\end{cases} \quad):
$$

$$
k_{t+1} = (1+R)\,k_t + \frac{R+\delta}{\alpha} a_t + \frac{R+\delta}{\alpha}(1-\alpha) l_t - \left(\frac{R+\delta}{\alpha} - \delta\right) c_t \qquad (2.23)
$$

$$
E_t c_{t+1} - c_t = \frac{R+\delta}{1+R}(1-\alpha)\left(E_t l_{t+1} - k_{t+1}\right) + \frac{R+\delta}{1+R} E_t a_{t+1} \qquad (2.24)
$$

$$
l_t = \frac{1}{\alpha+\varphi} a_t + \frac{\alpha}{\alpha+\varphi} k_t - \frac{1}{\alpha+\varphi} c_t, \qquad (2.25)
$$

There are many methods to solve this, and you have seen at least a couple of them with Dr. Meeks[8]. Since most models are larger than this, you usually do need to use the computer. What I want to show you here is how you can solve the simple example 'by hand', make you understand that the solution principle is quite general, and therefore that the same technique is applied when using the computer.

Let's assume that labor is inelastic, $\varphi \to \infty$ for the moment. This allows me to solve this model analitically and be really transparent about what is happening in the 'black box' that many of you may feel computer codes for solving systems of linear expectational equations are. We will return to elastic labor when solving the model numerically. The reasons things get simple is that the equation for hours merely becomes:

$$
l_t = 0,
$$

and the system in standard form is (I have substituted k_{t+1} in the first equation using the second in order to have the endogenous variables on the right hand side appear only at time t):

$$
\begin{aligned}
E_t c_{t+1} &= \left(1 + \left(\frac{R+\delta}{\alpha} - \delta\right)\frac{(R+\delta)(1-\alpha)}{1+R}\right) c_t - \qquad (2.26) \\
&\quad - (R+\delta)(1-\alpha) k_t - \frac{R+\delta}{\alpha}\frac{(R+\delta)(1-\alpha)}{1+R} a_t + \frac{R+\delta}{1+R} E_t a_{t+1} \\
k_{t+1} &= (1+R)\,k_t - \left(\frac{R+\delta}{\alpha} - \delta\right) c_t + \frac{R+\delta}{\alpha} a_t \qquad (2.27)
\end{aligned}
$$

In matrix form (recall that $E_t k_{t+1} = k_{t+1}$), letting x_t be the vector of endogenous variables $\begin{bmatrix} c_t & k_t \end{bmatrix}'$:

$$
E_t x_{t+1} = \Gamma x_t + \Psi a_t,
$$

---

[8]See Campbel (1994) for an analytical solution relying on the 'undetermined coeficients' method.

$$\Gamma = \begin{bmatrix} 1 + \left(\frac{R+\delta}{\alpha} - \delta\right)\frac{(R+\delta)(1-\alpha)}{1+R} & -(R+\delta)(1-\alpha) \\ -\left(\frac{R+\delta}{\alpha} - \delta\right) & 1+R \end{bmatrix}; \qquad (2.28)$$

$$\Psi = \begin{bmatrix} -\frac{R+\delta}{\alpha}\frac{(R+\delta)(1-\alpha)}{1+R} + \frac{R+\delta}{1+R}M \\ \frac{R+\delta}{\alpha} \end{bmatrix}, \qquad (2.29)$$

where M is a general operator - we postulate that the expectation of future technology is this linear function of current technology - for example if the shock is AR(1), $M = \rho$. But this solution method works even if shocks are not AR(1) (e.g. if the shock is AR of higher order, M will also contain lag operators).

So how do we solve this? If you are tempted to say: this is a vector autoregression, so iterate backwards (or use lag operators) to get $x_t = \sum_{i=0}^{\infty}\Gamma^i\Psi u_{t-i}$, you should really read carefully the next Section since this is plainly WRONG.

NOTE: many solution procedures write the system in a slightly different way, expressing current variables as a function of their future expected values and shocks:

$$x_t = \Omega_x E_t x_{t+1} + \Omega_a a_t. \qquad (2.30)$$

This representation is equivalent to the previous one once you recognize $\Omega_x = \Gamma^{-1}; \Omega_a = -\Gamma^{-1}\Psi$.

2.9.1   Solving forward and backward: (local) stability, indeterminacy and equilibrium uniqueness

As a general rule, please remember always that control variables should be solved 'forward' and state (predetermined) variables 'backward'. There is no mysticism involved in this. Control variables are decided upon by the agent looking into the future, maximizing the expected value of the objective function: the future distribution of shocks will hence matter, and we have no initial value from which to start. State or predetermined variables have already been decided upon at time t; indeed, they summarise the whole history of the economy, we have initial values for them and the whole past distribution of shocks will matter.

Suppose s_t is a state variable for which we have the equation, where u is an exogenous shock:

$$s_{t+1} = \lambda_s s_t + u_t.$$

This can be easily solved 'backward' to yield $s_{t+1} = \sum_{i=0}^{\infty}\left(\lambda_s\right)^i u_{t-i}$ if the stability condition $\lambda_s < 1$ is met[9]. If this condition is not met, the equation is unstable.

Suppose x is a control variable and you have the following equation dictating its dynamics:

$$E_t y_{t+1} = \lambda_y y_t + u_t$$

---

[9] Remember that you can solve this equation either by backward iteration or by lag operators: $s_{t+1} = \frac{1}{1-\lambda_s L}u_t$, and recall that $\frac{1}{1-\lambda_s L} = 1 + \lambda_s L + \left(\lambda_s L\right)^2 + ... + \left(\lambda_s L\right)^i + ...$ The powers of λ_s die away if the stability condition is met, otherwise thie whole thing explodes and the equation is unstable.

The way to solve this is to 'iterate forward'[10] (or use the forward operator F, $Fy_t = y_{t+1}$) after writing:

$$y_t = (\lambda_y)^{-1} E_t y_{t+1} - (\lambda_y)^{-1} u_t$$

to get

$$y_t = -E_t \sum_{i=0}^{\infty} (\lambda_y)^{-i-1} u_{t+i}$$

Clearly, you can do this if and only if $\lambda_y > 1$, which is the opposite of what you need for a 'backward' equation.

These were simple univariate examples, but the same principle applies when you have multivariate models. It is a general result due to Blanchard and Kahn (Econometrica) that: in a system of n linear expectational difference equations, if m variables are predetermined, or state variables (and the rest $n-m$ are not, i.e. are 'control' variables), there exists a unique solution if and only if exactly m roots (eigenvalues) of the transition matrix of that system are inside the unit circle. If 'too few' roots are inside the circle, we have 'instability': no stable equilibrium exists, pretty much as in our example above where $\lambda_s > 1$. We are simply unable to solve some of the 'backward' equations backward. If 'too many' roots are inside the unit circle, we have 'equilibrium indeterminacy': we are unable to solve some forward equations forward. For an excellent treatment of these issues (and not only) I warmly recommend the book by Roger Farmer: 'The macroeconomics of self-fulfilling prophecies' published at MIT Press.

Let's try to understand this better by returning to our simple bivariate example. What is so wrong with solving the whole system 'backwards' was that consumption is a control, forward-looking variable. This is not only a technical point, it is economic intuition: at the core of this model (and most models analysing business cycles) lies the permanent income hypothesis; consumption depends on the present discounted value of future income, i.e. on lifetime resources. Capital, on the other hand, is a state variable: it summarizes the history of the economy, i.e. al l the past choices regarding consumption versus investment. So we need to solve one equation forward and one backward.

The two equations, as they appear now, are not independent and cannot be solved separately. However, we can 'uncouple' them by applying a result from linear algebra that uses the eigenvalue decomposition of Γ. Consistent with our intuition above (and with the Blanchard-Kahn result), we need one eigenvalue of Γ to be inside and one outside the unit circle. Let's first see whether this is the case. You can either solve for the eigenvalues by brute force (not recommended) or show this more elegantly, e.g. as follows. First, notice that the determinant of Γ is $\det \Gamma = 1 + R > 1$. Since the determinant is the product of the eigenvalues (remember?), one of the eigenvalues will always be outside the unit circle; hence, we will never have 'too few' explosive roots - the model will not be indeterminate.

---

[10]Solve this e.g. by iterating forward: $x_t = (\lambda_x)^{-1} E_t x_{t+1} - (\lambda_x)^{-1} u_t = (\lambda_x)^{-2} E_t x_{t+2} - (\lambda_x)^{-2} E_t u_{t+1} - (\lambda_x)^{-1} u_t$,

and so forth (using the law of iterated expectations).

We still need to prove that the model will not have 'too many' explosive roots, i.e. that a locally stable solution exists.

The characteristic polynomial of Γ, which has as its roots the eigenvalues $\lambda_{1,2}$, is

$$J\left(\lambda\right) = \lambda^2 - \text{trace}\left(\Gamma\right)\lambda + \det\Gamma,$$

where the trace is

$$\text{trace}\left(\Gamma\right) = 2 + R + \frac{\left(R+\delta\right)\left(1-\alpha\right)}{1+R}\left(\frac{R+\delta}{\alpha}-\delta\right)$$

The condition for existence of an unique RE equilibrium implies: $J\left(-1\right)J\left(1\right) < 0$. Since $J\left(1\right) = 1-\text{trace}(\Gamma) + \det\Gamma$ and $J\left(-1\right) = 1+\text{trace}(\Gamma) + \det\Gamma$, we see immediately that $J\left(1\right) = -\frac{(R+\delta)(1-\alpha)}{1+R}\left(\frac{R+\delta}{\alpha}-\delta\right) < 0$ and $J\left(-1\right) = 4 + 2R + \frac{(R+\delta)(1-\alpha)}{1+R}\left(\frac{R+\delta}{\alpha}-\delta\right) > 0$. QED. Since $J\left(0\right) > 0$, both roots are positive (there are no oscillatory dynamics).

We find the roots by solving

$$J\left(\lambda\right) = 0 \rightarrow \lambda_{\pm} = \frac{\text{trace}\left(\Gamma\right) \pm \sqrt{\left(\text{trace}\left(\Gamma\right)\right)^2 - 4\det\left(\Gamma\right)}}{2}.$$

Note that the smaller root $\lambda_{-} \in \left(0,1\right)$ is stable and the larger one λ_{+} is unstable.

We know from linear algebra that we can decompose our non-singular square matrix Γ as:

$$\Gamma = P\Lambda P^{-1},$$

Λ is a diagonal matrix with the eigenvalues λ_{+}, λ_{-} as entries and P is a matrix stacking the eigenvectors corresponding to these eigenvalues[11]. Replace this decomposition in our system:

$$E_t x_{t+1} = P\Lambda P^{-1}x_t + \Psi a_t,$$

pre-multiply by P^{-1} and define the new variables $z_t \equiv P^{-1}x_t$ to get:

$$E_t z_{t+1} = \Lambda z_t + P^{-1}\Psi a_t$$

Now these equations ARE uncoupled and we can solve them separately. The first one is forward-looking and has an 'explosive' root $\lambda_{+} > 1$, as it should (denote the first element of z by z^c to remind ourselves it comes from consumption):

$$z_t^c = \lambda_{+}^{-1}E_t z_{t+1}^c - \lambda_{+}^{-1}\left[P^{-1}\Psi\right]_{1.}a_t,$$

where for any matrix $G, \left[G\right]_{i.}$ denotes its ith row. The solution is:

$$z_t^c = -\sum_{i=0}^{\infty}\left(\lambda_{+}\right)^{-i-1}\left[P^{-1}\Psi\right]_{1.}E_t a_{t+i}$$

---

[11] Recall that e.g. the first eigenvector (first column of P), call it p_{+} corresponding to λ_{+}, is found from: $\Gamma p_{+} = \lambda_{+}p_{+}$

Now you can use whatever process you want for a_t (however, recall that due to our loglinearisation technique we restrict attention to 'small' shocks). For example for our AR(1) process, $E_t a_{t+i} = \rho^i a_t$, so: $z_t^c = -\left(\lambda_+\right)^{-1}\left[P^{-1}\Psi\right]_{1.} a_t \sum_{i=0}^{\infty} \left(\frac{\rho}{\lambda_+}\right)^{-i}$ and since $\rho < 1 < \lambda_+$, we have:

$$z_t^c = -\frac{1}{\lambda_+ - \rho}\left[P^{-1}\Psi\right]_{1.} a_t$$

The second equation is backward-looking:

$$z_{t+1}^k = \lambda_- z_t^k + \left[P^{-1}\Psi\right]_{2.} a_t,$$

and can be solved in a standard way (the moving average representation can be easily found but is not particularly informative). To find the paths of consumption and capital you merely need to calculate

$$x_t \equiv \left[\begin{array}{c} c_t \\ k_t \end{array}\right] = P z_t = P \left[\begin{array}{c} z_t^c \\ z_t^k \end{array}\right]$$

To find the paths of investment and output you merely use the production function in this case (recall that $l_t = 0$): $y_t = a_t + \alpha k_t$ and the resource constraint.

Exercise 6 *In order to make sure that you understand this solution method, try to solve the model written in the 'forward' form 2.30 and show you get precisely the same solution.*

2.9.2   Elastic labor

Turning to our more general case with $\varphi < \infty$ and eliminating hours, we can still express the model as a two-equation system:

$$k_{t+1} = \left(1 + R + \frac{(R+\delta)(1-\alpha)}{\alpha+\varphi}\right) k_t + \frac{R+\delta}{\alpha}\frac{1+\varphi}{\alpha+\varphi} a_t - \left(\frac{R+\delta}{\alpha}\frac{1+\varphi}{\alpha+\varphi} - \delta\right) c_t \tag{2.31}$$

$$\left(1 + \frac{1-\alpha}{\alpha+\varphi}\frac{R+\delta}{1+R}\right) E_t c_{t+1} = c_t - \frac{R+\delta}{1+R}\frac{\varphi(1-\alpha)}{\alpha+\varphi} k_{t+1} + \frac{R+\delta}{1+R}\frac{1+\varphi}{\alpha+\varphi} E_t a_{t+1} =$$

Let $\gamma = \frac{R+\delta}{1+R}\frac{1-\alpha}{\alpha+\varphi}; \chi = \frac{R+\delta}{\alpha}\frac{1+\varphi}{\alpha+\varphi}$ and the model becomes:

$$k_{t+1} = (1+\gamma)(1+R) k_t + \chi a_t - (\chi - \delta) c_t \tag{2.32}$$

$$E_t c_{t+1} = \frac{1 + \varphi\gamma(\chi - \delta)}{1+\gamma} c_t - \varphi\gamma(1+R) k_t - \frac{\varphi\gamma\chi}{1+\gamma} a_t + \frac{\chi}{1+\gamma} E_t a_{t+1}$$

This system nests the inelastic-labor case (check this! note that $\varphi \to \infty$ implies $\gamma \to 0$; $\varphi\gamma \to \frac{R+\delta}{1+R}(1-\alpha)$ and $\chi \to \frac{R+\delta}{\alpha}$) and can be solved as before.

2.9.3   Discussion of solution in general case.

The solution method described above relies on the Jordan decomposition of
the transition matrix. In more complicated model this won't work, since the
transition matrix may be singular. Solution methods are readily available for
such cases relying on the Generalised Schur decomposition, a generalization of
the Jordan decomposition, but in most instances they require numerical analysis
(i.e. using the computer)[12]. I strongly recommend to those interested in doing
macro the article 'Computing sunspot equilibria in linear rational expectation
models', by Lubik and Schorfheide, Journal of Economic Dynamics and Control,
2004.

Whatever the solution method you use in general in large models, the so-
lution under equilibrium determinacy is typically a recursive equation of the
form:

$$x_t = M_x x_{t-1} + M_e e_t \tag{2.33}$$

for the vector of variables x (note: x includes also the exogenous processes, such
as a above) and white-noise shocks e.

2.10   Welfare analysis - a primer

Another advantage of using a microfounded model is that we can talk mean-
ingfully about welfare. The welfare of the representative agent in our economy
is summarized by the value function $V(K_t, A_t)$. From the Bellman equation
evaluated at the optimum (i.e. where we already recognised that we are along
the optimal path and dropped the 'max'[13]):

$$V(K_t, A_t) = U(C_t) + \beta E_t V(K_{t+1}, A_{t+1})$$

For ease of interpretation, we take a monotonic transformation of the value func-
tion and try to summarize welfare in a variable that is measured in consumption
units, defining the new variable V_t implicitly from:

$$U(V_t) = V(K_t, A_t)$$

Therefore:

$$U(V_t) = U(C_t) + \beta E_t U(V_{t+1}) \tag{2.34}$$

---

[12]For examples of the use of these methods, see the Matlab codes and explanations/examples
provided by Roland Meeks in the computational classes, using the solution method of Paul
Klein and Ben McCallum.

[13]Of course, the Bellman equation without imposing optimality is

$$V(K_t, A_t) = \max\left[U(C_t) + \beta E_t V(K_{t+1}, A_{t+1})\right],$$

but by focusing on the optimal path for consumption (and implicitly for future period's capital
stock) we can drop `max`.

In steady state we have $(1 - \beta) U (V) = U (C)$ and since U is bijective $(1 - \beta) V = C$. A log-linear approximation to 2.34 gives (using the 'third trick'):

$$U (V) \left(1 + \frac{U' (V) V}{U (V)} v_t \right) = U (C) \left(1 + \frac{U' (C) C}{U (C)} c_t \right) + \beta U (V) \left(1 + \frac{U' (V) V}{U (V)} E_t v_{t+1} \right) \rightarrow$$

$$\frac{U' (V) V}{U (V)} v_t = (1 - \beta) \frac{U' (C) C}{U (C)} c_t + \beta \frac{U' (V) V}{U (V)} E_t v_{t+1}$$

Assuming that the elasticity of utility with respect to its argument $U' (X) X/U (X)$ is independent of the level of its argument X (which holds for most utility functions you will use, e.g. for CRRA, log, etc.) we get:

$$v_t = (1 - \beta) c_t + \beta E_t v_{t+1}.$$

Using this equation, the path of v_t can be simulated, as for any other variable, to assess the first-order effect of shocks on the welfare of the representative agent. Further intuition can be gained by solving the equation forward to obtain:

$$v_t = (1 - \beta) E_t \sum_{i=0}^{\infty} \beta^i c_{t+i} \tag{2.35}$$

Therefore, the first-order effect on the representative agent's welfare of any shock that makes consumption deviate from its long-run trend (steady-state value) is measured by the expected present discounted value of these deviations scaled by $1 - \beta$. Note that $1 - \beta$ is a very small number since β is close to 1.

Exercise 7 *What is the welfare cost of fluctuations in this economy? Hint: ask the question in a slightly different way: 'what is the value of eliminating business cycles'? Justify your answer in one sentence.*

2.11 Evaluating the model's performance

Having solved for all endogenous variables as a function of the exogenous driving force allows us to evaluate the model by comparing its predictions to the data. That is, we want to compute moments of our *theoretical variables* (that are, in the model, log-deviations from steady-state, or from a balanced-growth path) and compare them with moments of *data variables*, which are also in log-deviations from a trend component. To do that, we need to discuss two issues: i. how do we measure the technology shock; ii. how do we compute relevant moments.

2.11.1 Measurement of technology

What is the technology shock? How do we measure it? We have a theory that puts this shock at the heart of business cycle fluctuations. So why not get 'data' on this. Any attempt to connect to data sources such as Datastream and

search for something similar is useless - so don't even try. Also, you may read the FT and/or the Economist regularly (probably you should) - but never read about anything resembling this shock, that's supposed to generate the bulk of observed macroeconomic fluctuations[14]. What can we do about it? (i.e.: what have people done in the past?).

The answer that people came up with is something you have already seen in the Growth lectures - extract the stochastic component of productivity from the Solow residual, i.e. the difference between changes in output and changes in measured inputs. The details of the method vary to a large extent across studies, but in essence the method is as follows. Taking logs of our production function we have:

$$\ln A_t = \ln Y_t - \alpha \ln K_t - (1 - \alpha) \ln L_t \qquad (2.36)$$

You can use information on quarterly total measured output, hours worked (either from establishment or household data) and capital stock, together with the 'estimate' of α (see the 'calibration' part) to calculate $\ln A_t$. This is a less trivial problem than it seems, because of measurement issues. For example, no universally accepted measure of the capital stock exists, technology may contain a time trend in the data, etc. I give you two prominent examples from the literature of how people handled some of these issues.

1. Cooley and Prescott (1995) take first differences of 2.36 to get (note that $\ln A_t - \ln A_{t-1} = a_t - a_{t-1}$)

$$a_t - a_{t-1} = (\ln Y_t - \ln Y_{t-1}) - \alpha (\ln K_t - \ln K_{t-1}) - (1 - \alpha) (\ln L_t - \ln L_{t-1}) \qquad (2.37)$$

They assume quarterly variations in the capital stock to be zero ($\ln K_t - \ln K_{t-1} = 0$), since this series is reported only annually and any method of interpolating a quarterly series would be arbitrary and give 'noise' variability to both output and technology. They use real measured GNP data for Y_t, and find the shocks to be well-described by an AR(1) as we assumed above, with $\rho = 0.95$ and $\sigma_\varepsilon = 0.007$.

2. King and Rebelo (1999)'s way of measuring the stochastic component in technology differs in two ways. First, they work with a production function that includes labor-augmenting technological progress H_t that grows exogenously (you have seen this in the growth lectures) to get a modified version of 2.36:

$$\ln SR_t = \ln Y_t - \alpha \ln K_t - (1 - \alpha) \ln L_t \qquad (2.38)$$
$$\text{where } \ln SR_t = \ln A_t + (1 - \alpha) \ln H_t \qquad (2.39)$$

Second, they do use a quarterly series for capital found (following Stock and Watson's paper in the Handbook, same Volume) by the 'permanent inventory method', i.e. generated from the investment series using the capital accumulation equation.

---

[14] Although, to be honest, this has changed recently - see Martin Wolf's article in Wednesday 9th of November's FT on productivity in the UK.

Using the empirical measure found from 2.36 and the deterministic process for H : $\ln H_t = \ln H_{t-1} + \ln g$, where g is the exogenous growth rate, you can estimate a process for $\ln A_t$. Simply fit a linear trend to $\ln SR_t$ to find g and use the residuals to estimate $\rho = 0.979$ and $\sigma_\varepsilon = 0.0072$. Eliminating the trend is consistent with the model being expressed as log-deviations from the steady-state, and hence being stationary (we could have introduced labor-augmenting technological progress from the outset, case in which we would have needed to loglinearize around the balanced-growth path rather than a constant steady state - see King and Rebelo for such a model).

2.12 Impulse responses and intuition

This section presents some impulse response analysis and an intuitive discussion. First, let us remember what an impulse response function actually is. Take the simplest AR(1) process that describes our productivity:

$$a_t = \rho a_{t-1} + \varepsilon_t, \tag{2.40}$$

where ε_t is iid $N\left(0, \sigma_\varepsilon^2\right)$. Suppose we are at time t and we start at the steady-state (so $a_{t-1} = 0$), when an unexpected one-time shock ε_t occurs, and let $\varepsilon_t = 1$. The **impact response** of a_t to ε_t is therefore given simply by 1. To find out what happens from time $t+1$ onwards, simply scroll (2.40) forward one period:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1} = \rho^2 a_{t-1} + \rho \varepsilon_t + \varepsilon_{t+1} = \rho, \tag{2.41}$$

where the last equality follows from $a_{t-1} = 0$ (we started at steady state) and $\varepsilon_{t+1} = 0$ (the shock is one-off). Similarly, you find the impulse response of a at horizon $t + j$, a_{t+j} to a unit shock at time t, ε_t as:

$$a_{t+j} = \rho^j, \forall j \geq 0$$

This is the very simple impulse-response function for our very simple AR(1) productivity shock. Note that the impulse response function is given by the coefficients in the moving average representation; take (2.40) and invert it using the lag operator (or do repeated substitution):

$$a_t = \frac{1}{1 - \rho L} \varepsilon_t = \sum_{i=0}^{\infty} \rho^i \varepsilon_{t-i}, \tag{2.42}$$

The response of a_t to a unit shock i periods ago is given by the corresponding coefficient in the MA representation, ρ^i.

Let's now focus on our endogenous variables and try to find their impulse responses to a unit technology shock. Things are as simple as for the productivity process once we recall that the solution of the linear rational expectations system of equations in the most general case can be represented in the $VAR(1)$ form by (2.33) restated here:

$$x_t = M_x x_{t-1} + M_\varepsilon \varepsilon_t,$$

where x_t is a vector comprising all our variables, including a_t. The impact responses are given in the impact vector M_ε since we start at steady-state where $x_{t-1} = 0$. You can invert (2.33) just as before to get:

$$x_t = (I - M_x L)^{-1} M_\varepsilon \varepsilon_t = \sum_{i=0}^{\infty} (M_x)^i M_\varepsilon \varepsilon_{t-i},$$

where I is an identity matrix of appropriate dimensions. So the response to a unit one-off shock occurring i periods ago on today's variables are found in: $(M_x)^i M_\varepsilon$.

2.12.1   The role of labor supply elasticity

Figure 1 plots the response to a unit technology shock under two scenarios regarding labor supply elasticity for our otherwise baseline calibration (these are obtained by running the Matlab codes that I included in the Appendix and will make available to you). Look first at the solid blue line, plotting the inelastic labor case. Technology increases and this increase is persistent. Ceteris paribus, this increases the productivity of both labor and capital, and hence their marginal products. From the standpoint of the households, this increase in both factor prices translates into an increase in the willingness to invest (since labor is inelastic. the labor supply curve is vertical - all the increase in labor demand is accommodated by an increase in the real wage). It also implies that households will consume more. However, note that since the interest rate will be falling, the household finds it optimal to save some of this increase in wealth and postpone consumption - this is why you see a hump-shaped impulse-response for consumption. From period 2 onwards, investment starts adding to the capital stock of the economy (although capital does not react on impact - remember it is a predetermined variable) and output keeps expanding. Note that the maximum response of consumption is reached in the same period where the interest rate cuts the horizontal axis: when the interest rate becomes negative, it is optimal to substitute consumption intertemporally from the future into today.

With elastic labor, the responses change as follows. When productivity increases, firms increase labor demand. You can see this by looking at the 'labor demand equation' for firms and noting that capital does not respond on impact:

$$LD: \ w_t = a_t + \alpha k_t - \alpha l_t^d.$$

The household is willing to accommodate some of that increase in demand by working more due to an income effect (whereas with inelastic labor, all this increase in demand translates into an increase in wages):

$$LS: \varphi l_t^s = w_t - c_t.$$

Labor market equilibrium ensures that the real wage will hence increase by less than in the inelastic-labor case. However, the increase in hours leads to a larger increase in the marginal product of capital - therefore, it is optimal to

invest more, thereby augmenting the capital stock even more (and ensuring a further expansion in labor demand from time $t+1$ onwards). Both the increase in hours worked and capital ensure that the expansion in output is larger. Recall that φ also governs the intertemporal elasticity of substitution in labor supply (you get this equation by substituting for consumption from the labor supply equation into the Euler equation):

$$\varphi \left(l_t^s - E_t l_{t+1}^s \right) = (w_t - E_t w_{t+1}) + E_t r_{t+1}$$

This says that *ceteris paribus*, if I expect the real wage to be higher tomorrow than it is today, I want to postpone some work for tomorrow (and the more so, the lower is φ, i.e. the higher is the elasticity). This intertemporal substitution effect usually works in the opposite direction of the income effect if the real wage is expected to be increasing for some period (as it is in our case - see the red dashed line). However, the net effect on hours worked is positive (and hence the income effect dominates) since wage growth is expected to be negative for most of the adjustment path. Moreover, since the expected real interest rate is positive at least in the first quarters, there is an intertemporal substitution effect of the interest rate that says you should give up some leisure (work more) today. All these effects disappear when $\varphi \to \infty$.

FIGURE 1: Responses to unit technology shock, comparing labor supply elasticities
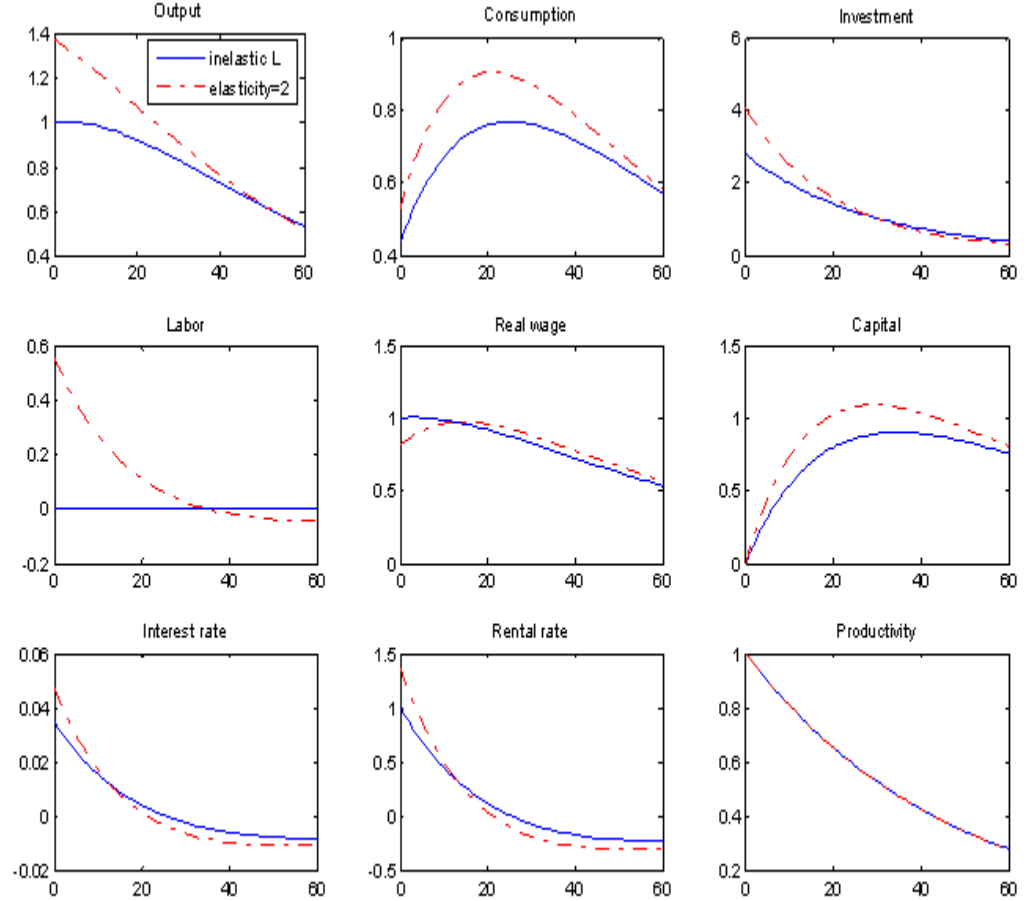

Figure 2 brings this to an extreme, plotting (blue solid line) the case whereby labor supply is infinitely elastic, $\varphi = 0$ (the 'indivisible labor' case). The effects described previously are amplified even further. Note that consumption will track the real wage, and hours adjust fully in order to ensure this optimality condition is met.

Very importantly, note that although the labor supply curve is horizontal when $\varphi = 0$, the real wage still moves! This is due to intertemporal substitution in consumption: on impact, the agent consumes some of the increase in productivity, saves some (the interest rate is high today) and also works as many hours as demanded by the firm. The real wage that clears the labor market is $w_t = c_t$ and is also equal to the marginal product of labor.

FIGURE 2: Responses to unit technology shock, comparing labor supply elasticities
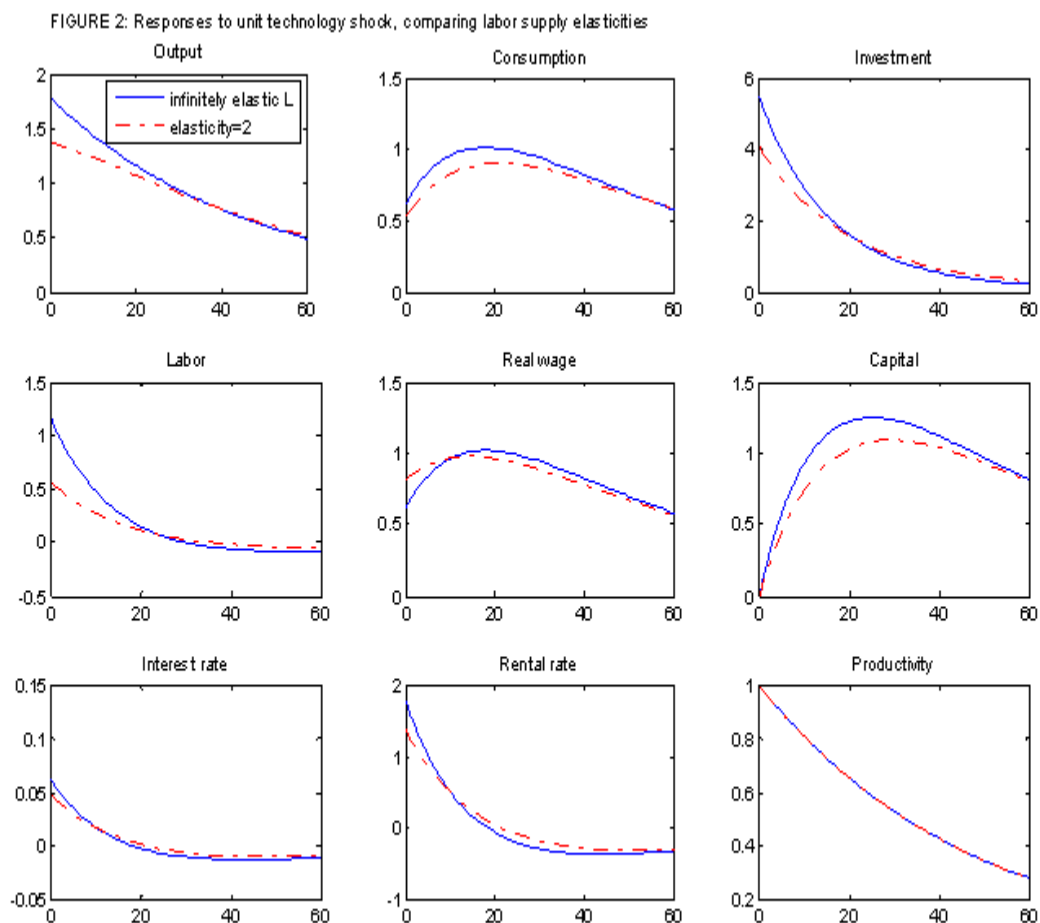
2.12.2 The role of shock persistence

Figure 3 emphasizes the role of persistence by plotting, together with the baseline calibration (red dashed line; notably, labor elasticity is 2), a case whereby the shock is one-off, with zero persistence (blue solid line). Since the marginal product of labor increases, the household again find it optimal to work more hours today: the real wage is very high today as compared to all future periods (remember, the household knows that this shocks is temporary). This increase in hours adds to the direct effect of the increase of productivity to obtain an even larger increase in output. This increase in output ought to be allocated between consumption and investment. Since the interest rate is high today compared to all future periods, it is optimal to save and invest most of the increase in output (postponing some of the gains for consumption in future

periods), and to consume only a small fraction today. Investment increases on impact by a large amount - roughly four times the increase in output. From period 2 onwards, there is no productivity increase. The household finds itself with a higher capital stock (due to previous investment), which she will now optimally consume - and hence disinvest; this is optimal since the interest rate is now low relative to future periods. Consumption and leisure are both normal goods, and the household wants to enjoy more of both (the relative quantities being dictated by the elasticity of labor supply): therefore, hours worked also fall below their steady-state level.

These transitional dynamics emphasize a point first made by Cogley and Nason (1995)- that the benchmark RBC model lacks a strong internal propagation mechanism, or features too little endogenous persistence. Periods of high output are not systematically followed by periods of similarly high output in response to purely transitory shocks. This has led most studies to focus on models in which persistence is inherited from the exogenous process (such as in the responses with red dashed lines); others have focused on enhancing the internal propagation mechanism.

FIGURE 3: Responses to unit technology shock, the role of shock persistence
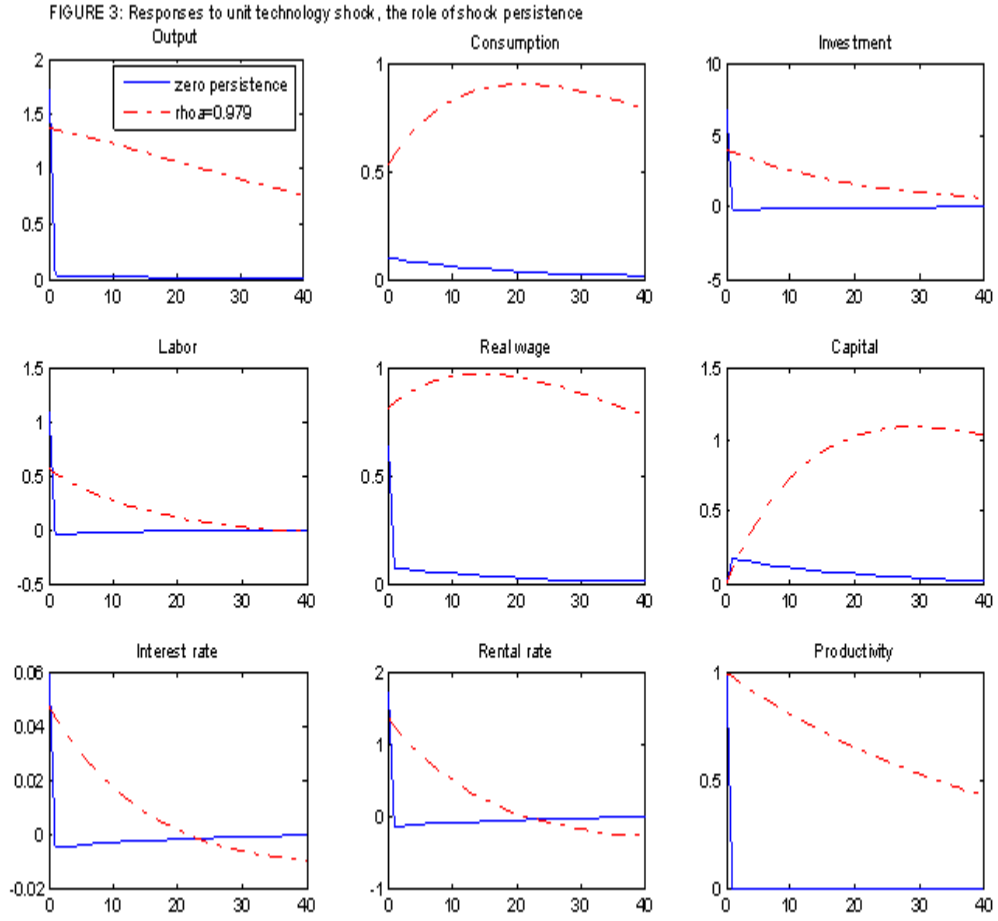
Finally, it is worth to consider the case whereby changes in technology are permanent - i.e. there is a unit root in the technology process. We compare this with our benchmark case in Figure 4. There are two main differences. The first (the easier one) is that permanent (as opposed to temporary, albeit very persistent) changes in technology have permanent effects on output, consumption, investment, real wage and capital (not hours!!!). You can calculate these effects analitically by taking the derivative of the steady-state variables we have calculated in section 2.5 with respect to A.

Secondly, there are important differences concerning transitional dynamics. If technology is permanently higher there are *wealth effects* that are absent otherwise - the household recognizes that it will be permanently richer. These effects combine with the wage effect: as before, the household understands that wages, despite having increased, are lower than in *all* future periods. Therefore,

on impact it will choose to undertake more leisure and work less than in the 'persistent but temporary shock' case. The same effects make the household willing to consume more, and therefore invest less. These choices are consistent with the path of the interest rate, which is monotonically decreasing over time after a positive initial response. Moreover, since interest rates never fall below their steady-state value, consumption is monotonically increasing towards its new steady-state value (it does not 'overshoot'). Investment does overshoot its new steady-state value precisely because the marginal product of capital is high in the first period.

FIGURE 4: Responses to unit technology shock, the role of shock persistence
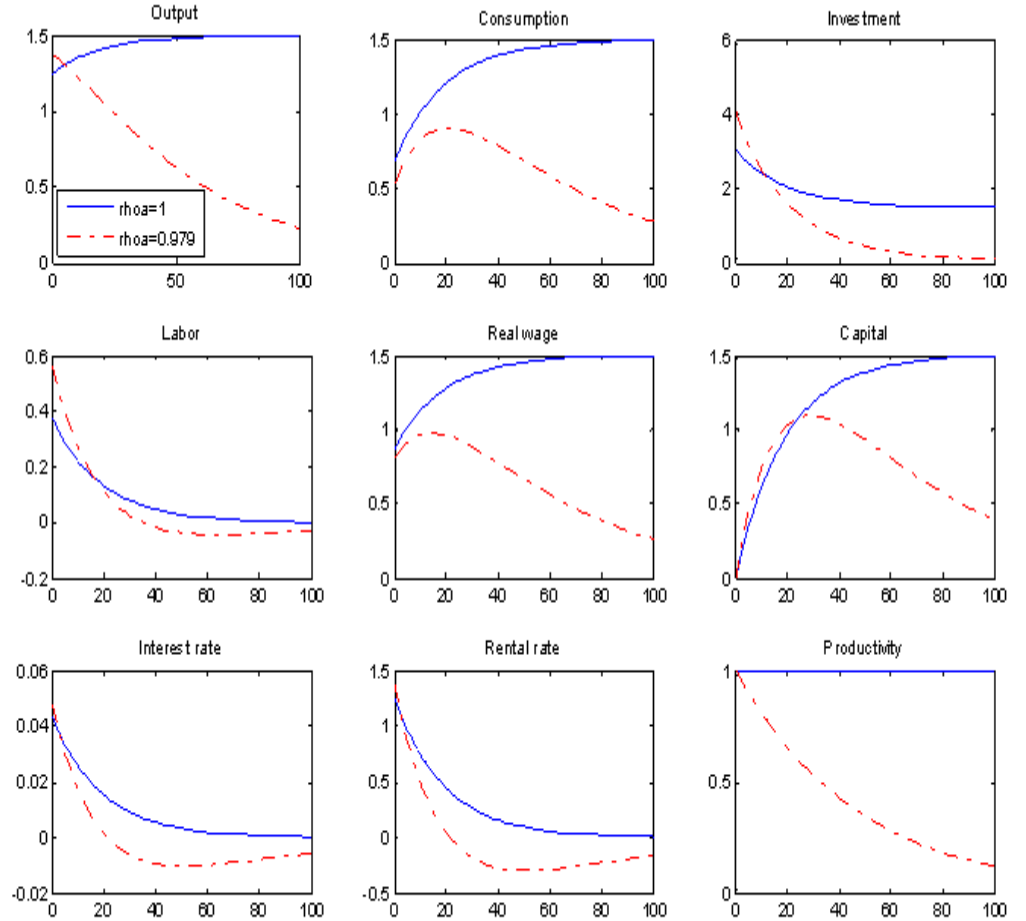
2.13   Second moments

Computation of second moments can be achieved by Monte Carlo simulations (something I won't bother you with) or analitically. Let's use (2.33) to calculate the covariance matrix of x_t, $\Sigma_{xx} = E\left(x_t x_t'\right)$, noting that we know the covariance matrix of the shocks $\Sigma_{ee} = E\left(e_t e_t'\right)$ (in the simple one-shock case, merely the variance of the shock to technology).

Furthermore, since we only consider stationary representations of the economy, such that $\Sigma_{xx} = E\left(x_{t+j} x_{t+j}'\right)$ for any j. Hence, we have

$$\Sigma_{xx} = M_x \Sigma_{xx} M_x' + M_x E\left(x_{t-1} e_t'\right) M_e' + M_e E\left(e_t x_{t-1}'\right) M_x' + M_e \Sigma_{ee} M_e'$$

Remembering that e_t are innovations, and hence orthogonal to x_t, the terms in the middle are zero, so this reduces to:

$$\Sigma_{xx} = M_x \Sigma_{xx} M_x' + M_e \Sigma_{ee} M_e'$$

This is a matrix equation with solution (check you linear algebra books):

$$vec\left(\Sigma_{xx}\right) = \left(I - M_x \otimes M_x'\right) M_e \otimes M_e' vec\left(\Sigma_{ee}\right),$$

where for any matrix $\Sigma_{s\times s}$, $vec\left(\Sigma\right)$ denotes the column vector obtained by stacking its column vectors $\{\Sigma_{.i}\}_{i=1,s}$ on top of eachother $\left(\Sigma_{.1} \vdots \Sigma_{.2} .. \vdots \Sigma_{.i} ... \vdots \Sigma_{.s}\right)'$, I is an identity matrix of dimension $s \times s$ and $\otimes$ is Kronecker product.

Autocovariances $E\left(x_t x_{t-j}'\right)$ (i.e. for leads/lags) are similarly computed:

$$E\left(x_t x_{t-j}'\right) = M_x^j \Sigma_{xx} + M_e M_x^j \Sigma_{ee} M_e'.$$

These formulae can be easily programmed and are used to obtain the numbers you find in the Tables - e.g. the standard deviations and autocorrelation of output, consumption, hours worked, investment, the correlations of each aggregate with output (contemporaneously and at leads and lags), and so on.

Table 2: Moments for Baseline RBC model

Variable x	σ_x	σ_x/σ_y	$E\left[x_t x_{t-1}\right]$	$corr\left(x,y\right)$
y	1.39	1.00	0.72	1.00
c	0.61	0.44	0.79	0.94
i	4.09	2.95	0.71	0.99
l	0.67	0.48	0.71	0.97
Y/L	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Source: King and Rebelo, 1999

Read King and Rebelo, section 4.3

The moments of interest can be divided into two categories:

1. volatilities.

- A first test for the model is the Kydland-Prescott 'variance ratio':

$$\frac{var^{\text{model}}(y)}{var^{\text{data}}(y)} = \left(\frac{1.39}{1.81}\right)^2 = 0.77.$$

- Investment is about three times more volatile than output.

- Consumption is smoother than output in both data and model but too smooth in model compared to data.

- Labor's volatility relative to output is too small compared to the data (mainly because capital is not volatile enough).

2. persistence and correlations.

The model generates persistence, but: (i) this persistence is lower than in the data and (ii) since the exogenous process is very persistent (a point to which we shall return below), it is clear that the model features a very *weak internal propagation mechanism*. I.e., the model generates *little endogenous persistence* (a point first noted by Cogley and Nason (1995, AER)).

The model also generates substantial *co-movement* of macroeconomic aggregates with output (as judged by the contemporaneous correlations), partly consistent with the data. However, there are some discrepancies: the correlations predicted by model for investment, labor, capital and productivity are larger than those found in the data. Moreover, the model generates a highly procyclical real wage (whereas in the data wages are roughly acyclical) and a highly procyclical interest rate (whereas in the data interest rates are countercyclical).

2.14   What have we learned?

A lot. We have built a model that relies on maximization by all agents and rational expectations in order to analyze fluctuations in macroeconomic times series. We learned how to solve this model step-by-step, how to understand the transmission of a technology shock and how to assess its merits by comparing its predictions to the data. You may well be appalled by the insistence on 'technology' shocks as being the main source of fluctuations (indeed, many people would say you should be). But the importance of this framework goes well beyond the focus on technology, as I tried to emphasize in the introduction.

READ King and Rebelo - section 4.5.

2.14.1   Critical parameters

1. Highly persistent -and volatile- technology shock. Note that $E\left(a_t a_{t-j}\right) = \frac{\rho^j}{1-\rho^2} var\left(\varepsilon\right)$. The variance of productivity is hence $\frac{var(\varepsilon)}{1-\rho^2}$, which is increasing in both $var\left(\varepsilon\right)$ and ρ. Moreover, increasing productivity's variance by increasing ρ also implies increasing the persistence of productivity and hence overall persistence.

The crucial role of productivity's persistence can be better understood by conducting the following experiments.

$\rho = 0$ vs. $\rho = 0.979$; $\rho = 0$ vs. $\rho = 1$.

2. Sufficiently elastic labor (either intertemporal or intratemporal - Greenowood, Hercowitz and Huffman) - **section 6.1 of KR.** This implies that work effort is highly responsive to changes in real wages. Therefore, it helps by generating both a lot of movement in hours and little movement in real wages. Assuming a highly elastic labor supply is inconsistent with micro evidence; however, models have been built that reconcile a low micro elasticity with a high macro elasticity. A prominent example is the 'indivisible labor model' of Hansen and Rogerson - read KR, section 6.1 (and the references therein) if you want to know more.

3. Steady-state shares of C and I in Y (I/Y has to be small, otherwise the volatility of I would converge to that of Y).

2.14.2 The Solow residual

Is the Solow residual the right measure for technology shocks? There are three main reasons why the answer is likely to be 'no':

1. The Solow residual can be forecasted using variables that are likely to be orthogonal to productivity: military spending, monetary aggregates, etc.

2. The Solow residual implies a large probability of technological regress (about 0.4 - Burnside, Eichenbaum and Rebelo, 1996).

3. Variable factor utilization (capital utilization and labor hoarding) contaminates the measured Solow residual. Basically, if there is unobserved variation in the utilization of factors of production, the Solow residual erroneously attributes this to 'technology' since it is measured using observed variation in factors of production. You can think of this as an 'endogeneity bias'.

There are two possible ways to correct for variable utilization, i.e. address point 3 above, and doing that is also likely to influence points 1 and 2. First, one can use proxies for unobserved variation and re-compute Solow residuals. Examples of such proxies are: (i) the number of work accidents as a proxy for unobserved work effort (since working harder increases the probability of accidents at least in an industrial setting); (ii) electricity use as a proxy for unobserved variation in capital utilization. The second possibility is to build a model that incorporates unobserved factor variation as an endogeous variable, express this as a function of other endogenous but observable variables and calculate the model-implied productivity series (see next section for an example).

A corrected measure of the Solow residual that takes into account variable utilization (see Burnside, Eichenbaum and Rebelo, 1996) implies that:

1. productivity shocks are much less volatile;

2. the probability of technological regress drops dramatically.

These findings imply that a stronger amplification mechanism than that of the baseline RBC model is needed to explain observed fluctuations. Fortunately, the very same reason that biases the measure of the Solow residual also delivers this extra amplification. An RBC model incorporating variable utilization implies that small shocks to productivity have large aggregate effects.

2.14.3   Enhancing the propagation/amplification mechanism

In the part of the course taught by Professor Muellbauer, you will see two alternative specifications of preferences and technology respectively. In the first, the utility function is not intertemporally separable due to the presence of *habits*: that is, current utility depends on last period's consumption. In the second, investment is subject to *adjustment costs*. Once you will have covered this material, you may want to think of incorporating these features in the baseline RBC model. Both of these features, when introduced in our model, imply an extra channel by which endogenous persistence is generated.

In the remainder, I will intuitively discuss the introduction of **variable capital utilization** that I hinted to above. In the presence of variable utilization, the production function becomes:

$$Y_t = F\left(Z_t K_t, L_t\right) = A_t \left(Z_t K_t\right)^\alpha L_t^{1-\alpha},$$

Z_t being the utilization rate. Using the capital stock more intensively affects the depreciation rate of capital, and the capital accumulation equation becomes:

$$K_{t+1} = \left(1 - \delta\left(Z_t\right)\right) K_t + I_t \tag{2.43}$$

where $\delta\left(Z_t\right)$ satisfies $\delta_Z\left(Z_t\right) > 0$, $\delta_{ZZ}\left(Z_t\right) > 0$: the depreciation rate increases if capital is used more intensively, and does so at an increasing rate. We introduced an extra variable, Z_t, hence we nee done extra equation governing the choice of the utilization rate in order to determine equilibrium. The extra benefit of increasing the utilization rate is given by the extra output that is being created, $F_z\left(.,.\right) = \alpha A_t Z_t^{\alpha-1} K_t^\alpha L_t^{1-\alpha}$. The marginal cost of increasing utilization is given by the higher investment needed to replace capital that is depreciating faster: $\delta_Z\left(Z_t\right) K_t$. The optimal utilization rate is found by equating these two:

$$\begin{aligned}
\alpha A_t Z_t^{\alpha-1} K_t^\alpha L_t^{1-\alpha} &= \delta_Z\left(Z_t\right) K_t \rightarrow \\
\alpha \frac{Y_t}{Z_t} &= \delta_Z\left(Z_t\right) K_t
\end{aligned}$$

Let's loglinearise the production function and the efficiency condition. The production function becomes:

$$y_t = a_t + \alpha k_t + \alpha z_t + \left(1 - \alpha\right) l_t \tag{2.44}$$

The efficiency condition becomes (use the 'third trick' to loglinearise $\delta_Z(Z_t)$, just as we did for the marginal disutility of labor $v_L(L_t)$):

$$
\begin{aligned}
y_t - z_t &= k_t + \frac{\delta_{ZZ}(Z)Z}{\delta_Z(Z)} z_t \rightarrow \\
y_t &= k_t + (1+\xi)z_t,
\end{aligned} \tag{2.45}
$$

where ξ denotes the elasticity of the marginal depreciation rate induced by extra utilization to the utilization rate, $\xi \equiv \delta_{ZZ}(Z)Z/\delta_Z(Z)$. When this elasticity tends to ∞, we are back in the standard model (2.45 implies $z_t = 0$): note that $\xi \rightarrow \infty$ iff $\delta_Z(Z) \rightarrow 0$ which instead implies that the depreciation rate is not affected by utilization.

Substituting the efficiency condition 2.45 into the production function 2.44 and eliminating z_t, we obtain a 'reduced-form' production function:

$$
y_t = \left(1 + \frac{\alpha}{1+\xi-\alpha}\right) a_t + \frac{\alpha\xi}{1+\xi-\alpha} k_t + \left(1 - \frac{\alpha\xi}{1+\xi-\alpha}\right) l_t. \tag{2.46}
$$

Two things can be noted by staring at this expression and comparing it with the benchmark case. First, the partial elasticity of output with respect to technology is increased - and the more so, the lower is ξ, i.e. the more we depart from the benchmark model. This induces extra amplification of technology shocks. Second, the partial elasticity of output with respect to labor is higher (and hence the partial elasticity with respect to capital is lower) than in the benchmark case since $\frac{\alpha\xi}{1+\xi-\alpha} < \alpha$. This tightens the link between output and labor fluctuations and will potentially generate more labor volatility.

Relatedly, variable utilization also affects the cyclicality of wages. Wages are still given my the marginal product of labor, so:

$$
w_t = y_t - l_t = \left(1 + \frac{\alpha}{1+\xi-\alpha}\right) a_t + \frac{\alpha\xi}{1+\xi-\alpha} k_t - \frac{\alpha\xi}{1+\xi-\alpha} l_t.
$$

Since $\frac{\alpha\xi}{1+\xi-\alpha}$ is increasing in ξ, a lower ξ implies a flatter 'labor demand' curve which instead implies that for a given shift of labor supply the wage will react less and hours more than in the benchmark model. However, the shift in labor demand will necessarily be larger under variable utilization, since $\frac{\alpha}{1+\xi-\alpha} > 0$. This is why in the simulation presented in Figure 5 the response of the real wage is higher, at least in the earlier quarters.

2.14.4 The role of variable capital utilization

Finally, Figure 5 shows the effect of variable capacity utilization under the benchmark parameterization. The parameter ξ in the lecture notes has been set to 0.1 in the variable-utilization case (a value borrowed from King and Rebelo), and to a very large value in the 'fixed utilization' case. The figure confirms our discussion in the lecture notes and shows that variable utilization leads to an amplification of a given technology shock. Therefore, smaller shocks are

FIGURE 5: Responses to unit technology shock, the role of variable utilization
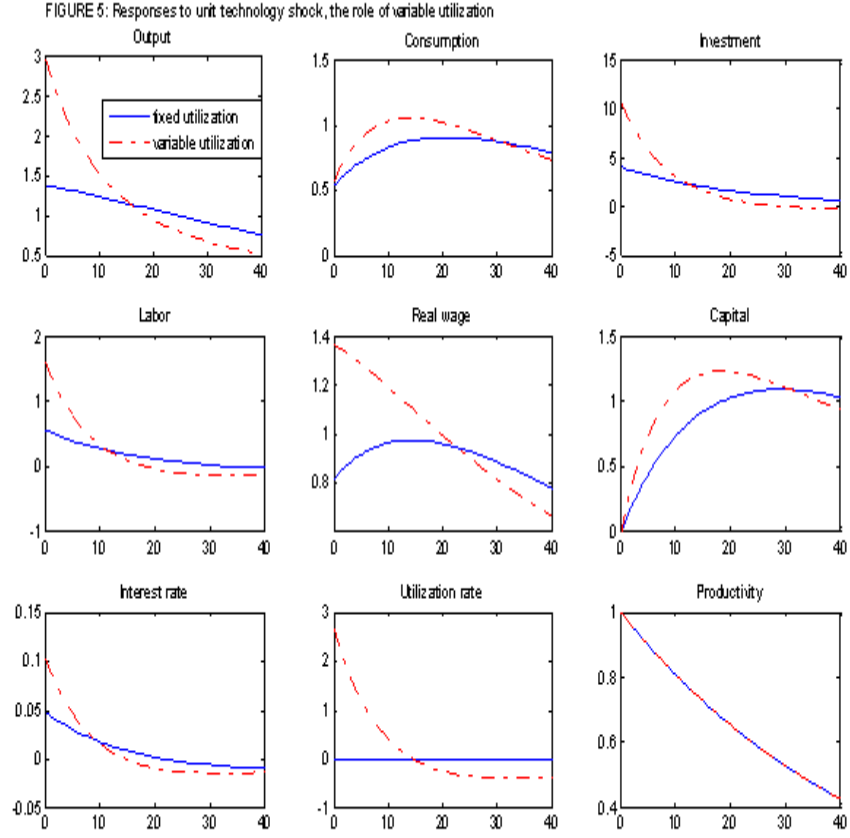
enough to explain observed fluctuations. Note that the figure keeps labor supply elasticity at 2. The amplification induced by variable utilization is increasing with labor supply elasticity - for the infinitely elastic labor case, we would end up with the 'high substitution economy' in King and Rebelo.

The Figure above abstracted from the re-measurement of the Solow residual that was one of the initial reasons to consider this extension in the first place. Following our discussion of the Solow residual above, note the two routes that could be followed in re-measuring productivity using this model. First, one could use 2.44, find a proxy for z_t, say $\tilde{z}_t$, (e.g. electricity use) and compute a_t^* from:

$$a_t^* = y_t - \alpha k_t - \alpha \tilde{z}_t - (1 - \alpha) l_t \tag{2.47}$$

Otherwise, one can use the structure of the model and arrive at the reduced-form

production (hence eliminating the unobservable variable z_t) and compute:

$$a_t^{**} = \left(1 - \frac{\alpha}{1 + \xi}\right) y_t - \frac{\xi}{1 + \xi}\alpha k_t - (1 - \alpha) l_t \tag{2.48}$$

As noted above, after taking variable utilization into account the variance of the computed productivity shocks is generally much lower, but the model is still as able to replicate fluctuations because this very feature induces extra amplification, as demonstrated in Figure 5 and the discussion preceding it.

2.14.5 Where do we go

While the RBC model is successful at explaining some features of the data, it does a pretty poor job at explaining others. This is why research in this area has flourished over the past few decades, and keeps expanding by incorporating various frictions such as imperfect competition, imperfect price or wage adjustment, investment adjustment costs, imperfect labor or financial markets etc. However, what you should take home from this is that (most of) modern macroeconomic literature takes this framework as a starting point in order to examine an incredible variety of issues. To give just one example we will not touch upon at all, modern monetary policy analysis also uses the baseline RBC model as a benchmark and adds imperfect price adjustment in order to examine the effects of nominal disturbances and optimal monetary policy. See Woodford (2003) if you are interested in these issues. For a recent paper on the effects of nominal disturbances and how to account for them in a frictions-rich DSGE model see Christiano, Eichenbaum and Evans (2005).

Chapter 3

Government Spending and Fiscal Policy

In this Chapter I want to familiarize you with the main ideas concerning the effects of government spending shocks in the RBC model, and with related fiscal policy issues. Given the difficulties of technology shocks to account for some of the data features, some authors have naturally looked at other shocks, for examples shocks to government spending. In particular, Christiano and Eichenbaum (AER, 1992) showed that adding this stochastic source of fluctuations helps in resolving an important puzzle, i.e. the discrepancy between the high procyclicality of the real wage implied by the baseline RBC model and the relative acyclicality observed in the data.

However, government spending shocks have other undesirable properties: as will become clearer below, they generally imply a countercyclical consumption, in stark contrast with the data. Consumption falls in response to government spending shocks due to a **negative wealth effect**: government spending absorbs resources and makes the agent feel poorer by the present discounted value of taxes that are used to finance this spending. This makes the agent consume less and work more for a given real wage; the latter effect implies that output increases. Therefore, conditional on government spending shocks, consumption will be countercyclical.

If you want to know more on these issues, I recommend reading Baxter and King (AER, 1993) and Christiano and Eichenbaum (AER, 1992). Some recent developments on these issues can be found in Gali, Lopez-Salido and Valles (2005) and in some of the references therein - in particular, you could check the ones whose author's names start with B (Read these recent papers after you covered sticky prices and monetary policy issues next term).

3.1   Lump-sum taxes, no debt

The main ideas pertaining to fiscal policy analysis in general equilibrium can be understood in the following highly stylised model. We abstract from capital accumulation, for example assuming that the production function is linear in labour:

$$Y_t = A_t L_t \tag{3.1}$$

The real wage is given by profit maximization (this can be regarded as 'labor demand' by firms):

$$W_t = A_t \tag{3.2}$$

We assume that there is an additional source of fluctuations: exogenous movements in government spending G_t. For simplicity, we assume that government spending is pure waste, i.e. it neither contributes to production, nor it affects utility directly: utility is still given by 2.15. Finally, government spending is financed via lump-sum taxes T_t on households. In the simplest case studies here, we assume that the budget is balanced each period, i.e. the government budget constraint reads:

$$G_t = T_t. \tag{3.3a}$$

The budget constraint of the households becomes:

$$C_t = W_t L_t^s - T_t \tag{3.4}$$

Note that there is no intertemporal dimension of the household's maximization problem. The only optimality condition concerns the choice of consumption versus hours worked, which is the same as previously:

$$v_L\left(L_t^s\right) = \frac{1}{C_t} W_t \tag{3.5}$$

As before, this can be regarded as a 'labor supply' schedule for a given level of consumption.

Labor market clearing will again ensure that

$$L_t^s = L_t^d = L_t$$

In the spirit of 'counting equations' and ensuring we have as many equations as new variables, note that we have introduced 2 new variables (exogenous G and endogenous T) and 2 new equations - the balanced-budget rule and an exogenous process for G. As before, by Walras' law the economy resource constraint holds (convince yourself of this, by combining the equilibrium conditions above):

$$Y_t = C_t + G_t, \tag{3.6}$$

stating that the good of this economy is used for either private or public consumption.

Note that there are no endogenous dynamics: all dynamics are extrinsic, coming from dynamics in the exogenous processes for G. Nevertheless, I will

apply the same techniques as in the model with capital (looking at the steady-state, loglinearising, etc.), since in practice you may want to study fiscal policy in more complicated (and hopefully realistic) models. The loglinearised equations are very simple (we substituted the labor market clearing condition $l_t^s = l_t^d = l_t$ already):

$$
\begin{aligned}
y_t &= a_t + l_t \\
w_t &= a_t \\
g_t &= t_t \\
c_t &= \frac{WL}{C}\left(w_t + l_t\right) - \frac{T}{C}t_t \\
\varphi l_t &= w_t - c_t
\end{aligned}
$$

The steady-state shares needed in the loglinearised budget constraint are easily found once we recognise that $WL = Y$ and $T = G$, where G is given exogenously. We parameterize $G_Y \equiv G/Y$ to average US government spending (about 0.2) and observe that $\frac{WL}{C} = \frac{Y}{C} = \frac{1}{1-G_Y}$ and $\frac{T}{C} = \frac{G}{C} = G_Y \times \frac{Y}{C} = \frac{G_Y}{1-G_Y}$.

The steady state is found by merely dropping time subscripts. Combining all the equations above, we have:

$$
\begin{aligned}
v_L\left(L\right) &= \frac{1}{WL - G}W \rightarrow \\
L - \frac{G}{A} &= v_L^{-1}\left(L\right)
\end{aligned}
$$

This equation implicitly defines steady-state hours worked as a function of exogenous forces. Note that due to the presence of government spending and taxation, hours depend on the level of technology. Hours are generally increasing in the level of government spending, since increased taxation introduces and extra income effect, on top of the two effects that, as we saw for this utility specification, cancel out: income and substitution effects due to wage variations. Nevertheless, hours could be constant (i.e. not growing) in steady state even if A grew if we assume that G grows at the same rate as A. Once we found L, we can easily find the steady-state values for all other variables.

3.1.1 The labor market

In most models of fiscal policy (and of business cycles in general), the labor market is crucial for understanding the propagation of shocks and the workings of the model. I try to illustrate this in our simple model. Labor demand in loglinearised form is horizontal in the (l, w) space:

$$
LD : w_t = a_t. \tag{3.7}
$$

Labor demand shifts here only because of exogenous shifts in technology: an increase in productivity a makes firm want to hire more labor at the prevailing real wage.

Labor supply, for a given level of consumption, is:

$$LS : \varphi l_t^s = w_t - c_t.$$

This is an upward-sloping schedule in the (l, w) space that shifts due to endogenous movements in consumption.

Recall first what happens if a positive technology shock hits in the baseline RBC model with no capital and no government spending (let $G = 0$ too) - this is the same as the previous discussion where we now set $\alpha = 0$. Labor demand shifts upward and the real wage increases. Because wage increases, the household feels richer and is willing to work less at a given real wage - so the LS schedule shifts leftward. In equilibrium hours do not move and all the increase in productivity translates into an increase in real wage. This can be seen analitically by (we have imposed labor market clearing and since there is no government spending at all we have $c_t = y_t = a_t + l_t$)

$$\varphi l_t = w_t - c_t = w_t - a_t - l_t \rightarrow w_t - a_t = (1 + \varphi) l_t$$

Combining this with labor demand $(w_t = a_t)$ we obtain $l_t = 0$ and $w_t = a_t$. This is an extreme version of one finding of RBC models that is at odds with the data, where we see the opposite: a lot of movement in hours and little movement in wages.

Take an increase in government spending now, assuming that technology does not move, $a_t = 0$ - which also makes the real wage constant. Let's first work out the effects analitically. First substitute the budget constraint and the balanced-budget rule in the labor supply equation:

$$
\begin{aligned}
\varphi l_t &= w_t - c_t \rightarrow \\
\frac{-G_Y}{1 - G_Y} w_t &= \left(\varphi + \frac{1}{1 - G_Y} \right) l_t - \frac{G_Y}{1 - G_Y} g_t.
\end{aligned}
$$

Substituting the labor demand schedule we get:

$$
\begin{aligned}
l_t &= \frac{G_Y}{\varphi (1 - G_Y) + 1} g_t \\
w_t &= 0
\end{aligned}
$$

Geometrically, the effects of a g shock are easily analysed: the labor demand schedule is unaffected, and labor supply shifts rightward as the household feels poorer and is willing to work more at a given real wage (this is the negative wealth effect on hours). Moving along the labor demand, this implies that hours increase and wages are constant. If labor demand were downward sloping (as in the more general model with capital) as opposed to horizontal, we would get *a fall in real wage* and a (somewhat smaller) increase in hours worked and hence in output.

This is the sense in which government spending shocks make the RBC model fare better in what concerns real wage fluctuations. Since the conditional response of the real wage to technology is highly procyclical, appending one more

shock that implies a (conditionally upon the new shock) countercyclical real wage means that the unconditional (or more precisely, conditional upon both shocks) response of the real wage will be less procyclical, or even acyclical.

However, a shock to g also has unattractive implications. While, as noted above, it implies an increase in hours worked (and in output, since without technology $y_t = l_t$) it also generates a fall in private consumption:

$$c_t = -\varphi \frac{G_Y}{\varphi \left(1 - G_Y \right) + 1} g_t.$$

Intuitively, when the household is taxed it feels poorer and has two margins to adjust: labor and consumption. The relative size of the adjustment along these two margins is governed by the elasticity of labor supply. When labor is inelastic (φ very high), most adjustment takes place by reducing consumption. When labor is elastic (φ very low) the response of consumption becomes negligible and most adjustment takes place through increasing hours worked. This finding (the *negative correlation* between consumption on one hand and hours worked and output on the other) is at odds with the data.

3.2 Short detour on asset pricing

The purpose of this section is merely to introduce some very basic ideas about pricing assets. Suppose that a consumer can invest in some arbitrary asset that costs her P_t today and leads to the following payoff X_{t+1} tomorrow (for example, it can be re-sold tomorrow on the market at price P_{t+1} and also pays some 'dividend' D_{t+1}, such that $X_{t+1} = P_{t+1} + D_{t+1}$). Let us assume that the number of assets that the consumer demands is N_t^d. The consumer budget constraint (where we still assume the consumer receives some labor income $W_t L_t$) is:

$$P_t N_{t+1}^d + C_t = X_t N_t^d + W_t L_t. \tag{3.8}$$

Note that $P_t N_{t+1}^d$ is the amount of assets purchased during period t and carried over to period $t+1$, while $X_t N_t^d$ is the total payoff of assets brought into period t. Therefore, N_t^d is a **state variable.**

You can maximize utility with respect to the number of demanded assets subject to this dynamic constraint using the techniques you learned. Assuming that there is an interior, strictly non-zero demand for these assets (i.e. there are no borrowing or liquidity constraints, no transaction costs, and no short-sale restrictions), you obtain the Euler equation for assets, which is the **fundamental asset pricing formula**:

$$P_t = E_t \left[\Lambda_{t,t+1} X_{t+1} \right], \tag{3.9}$$

$$\text{where } \Lambda_{t,t+1} = \beta \frac{U_C \left(C_{t+1} \right)}{U_C \left(C_t \right)}. \tag{3.10}$$

$\Lambda_{t,t+1}$ is the **stochastic discount factor,** or **pricing kernel**, or **marginal rate of substitution**. It governs the rate at which the consumer is willing to

substitute between consumption at t and at $t+1$. More generally, we will let the stochastic discount factor between time t and $t+j$ be given by

$$\Lambda_{t,t+j} = \beta^j \frac{U_C\left(C_{t+j}\right)}{U_C\left(C_t\right)} = \Lambda_{t,t+1} \times \Lambda_{t+1,t+2} \times ... \times \Lambda_{t+j-1,t+j}$$

Note that the above derivation has not imposed any market structure, any nature of assets (they can be anything) and any particular structure for uncertainty. Indeed, in many cases (i.e. for many types of assets and/or market structures), once you impose equilibrium (and hence market clearing) you will find: $N_{t+1}^d = N_t^d$, and you can normalize this to 1 - but we should not take this route here. Let us now look at some specific assets.

Example 8 *For stock (shares) the payoff is given by the market price plus the dividend: $X_{t+1} = P_{t+1} + D_{t+1}$. The pricing formula becomes:*

$$P_t = E_t\left[\Lambda_{t,t+1}\left(P_{t+1} + D_{t+1}\right)\right]$$

*You can iterate this forward (use the law of iterated expectations also) to obtain the **price of a share** as a discounted present value of future dividends:*

$$P_t = \lim_{T\to\infty} E_t \Lambda_{t,T} P_T + E_t \sum_{j=1}^{\infty} \Lambda_{t,t+j} D_{t+j} = E_t \sum_{j=1}^{\infty} \Lambda_{t,t+j} D_{t+j},$$

where the second equality has used transversality. This is esentially a stripped-down version of Lucas's 'tree model' in 'Asset prices in an exchange economy', 1978, Econometrica.

Example 9 *We can **define net returns of any asset** $1 + R_{t+1}^A$ as payoffs of an asset with price 1, i.e. if I invest one unit of consumption or currency today, how many units of consumption or currency do I get tomorrow. So*

$$1 + R_{t+1}^A \equiv \frac{X_{t+1}}{P_t},$$

which substituted into the fundamental pricing equation:

$$1 = E_t\left[\Lambda_{t,t+1}\left(1 + R_{t+1}^A\right)\right]$$

Example 10 *An important special case occurs for the **risk-free rate** $1 + R_t$, i.e. the return of an asset (a riskless bond) that gives you something tomorrow with certainty:*

$$\begin{aligned} 1 &= E_t\left[\Lambda_{t,t+1}\left(1 + R_t\right)\right] = \left(1 + R_t\right) E_t\left[\Lambda_{t,t+1}\right] \\ &\rightarrow \left(1 + R_t\right)^{-1} = E_t\left[\Lambda_{t,t+1}\right] \end{aligned}$$

Alternatively, you can think of an asset whose payoff tomorrow is 1 with certainty, and whose price by the above formula is $\left(1 + R_t\right)^{-1}$. Such an asset is called a discount bond. More generally, you can define the riskless rate by this formula even when a riskless asset is not being traded or does not exist.

Example 11 *We can even model physical capital as an asset! Think of an asset with price 1 whose 'dividend' is $R_t^K - \delta$, and hence the payoff tomorrow is $1 + R_{t+1}^K - \delta$. The price is 1 since the very same good is used for consumption and investment, and the decision to invest is reversible (i.e. you can transform capital back into the consumption good freely). Because the price is 1, $R_t^K - \delta$ is also the net return of this asset. The quantity (the equivalent of N^d) of the asset is evidently K^s. You will see models in which the price of capital (related to 'Tobin's q') is allowed to vary when you study models with investment adjustment costs.*

3.3   Government debt and Ricardian equivalence

We now re-introduce intertemporal features in the fiscal policy model by assuming that the government can issue bonds that the household purchases for saving purposes. For simplicity, we assume that all bonds are riskless, one-period discount real bonds paying one unit of the consumption good at the end of the period. The budget constraint of the household becomes:

$$(1 + R_t)^{-1} B_{t+1}^d + C_t + T_t = B_t^d + W_t L_t, \tag{3.11}$$

where B_t^d are the bonds the household enters period t with, B_{t+1}^d the bonds that the household decides to purchase during period t at price $(1 + R_t)^{-1}$. Note that B_t is a state variable, whose initial value B_0 is dictated by history.

Exercise 12 *What is the transversality condition in this economy? (Hint: try to rule out 2 scenarios: one in which the household accumulates too many assets by the 'end of time' and one in which the household ends up with debt -a negative B_t^d!- at the 'end of time').*

The Euler equation is, as we found before:

$$
\begin{aligned}
(1 + R_t)^{-1} &= E_t \left[\Lambda_{t,t+1} \right], && (3.12) \\
\text{where } \Lambda_{t,t+1} &= \beta \frac{U_C \left(C_{t+1} \right)}{U_C \left(C_t \right)} = \beta \frac{C_t}{C_{t+1}},
\end{aligned}
$$

is the **stochastic discount factor,** that would be used to price any assets as discussed above. For riskless bonds, this equation states that the price of a discount bond (which is equal to the inverse of its gross return) is the discounted value of the ratio of marginal utilities at two successive points in time.

The government budget constraint dictating debt dynamics is:

$$(1 + R_t)^{-1} B_{t+1}^s + T_t = B_t^s + G_t, \tag{3.13}$$

i.e. total government revenues from taxation and issuance of new bonds equal total spending on goods and on repaying existing debt.

Asset market clearing will imply

$$B_{t+1}^d = B_{t+1}^s \equiv B_{t+1}.$$

Note that we need one more equation dictating the dynamics of tax revenues, i.e. specifying how much of an increase in spending is covered by debt issuance and how much by increased taxes. We specify this arbitrarily for now, for reasons that will become clear:

$$T_t = \Phi_t\left(\mathbf{X}_t\right),$$

where $\mathbf{X}$ could contain any of the variables in the model.

Substituting the asset market clearing condition, the government budget constraint (debt dynamics equation) and the equations from the firm's side that are the same as in the case without debt into the household's budget constraint we obtain the economy resource constraint:

$$Y_t = C_t + G_t. \tag{3.14}$$

Now think about it for a minute. We want to find the equilibrium of this economy, i.e. the decision rules for C, Y, L and prices, namely W and R. We can do this by solving the following 5-equation system:

$$
\begin{aligned}
Y_t &= C_t + G_t. \\
Y_t &= A_t L_t \\
v_L\left(L_t\right) &= \frac{1}{C_t} W_t \\
W_t &= A_t \\
(1 + R_t)^{-1} &= \beta E_t\left[\frac{C_t}{C_{t+1}}\right]
\end{aligned}
\tag{3.15}
$$

Proposition 13 *'Ricardian neutrality'. Debt (or equivalently, the timing of taxation) is irrelevant for the equilibrium allocation 3.15. This can be seen by direct inspection of 3.15.*

Note that the first four equations are identical to the ones used to determine equilibrium in the extreme case without government debt. The fifth equation merely defines the interest rate (price of bonds) of this economy that is consistent with optimal consumption choices. The timing of taxation is completely irrelevant, as is the number of bonds issued. If the government cuts taxes today, the household correctly anticipates that taxes will be increased in the future. It hence increases its bond holdings ('lends') by buying the bonds that the government issues to cover the tax cut. (The opposite holds for a tax increase.). The equilibrium path of the interest rate insures consistency with these choices. Note that this is a very special result, which fails as soon as we depart from our special assumptions (the next exercise makes you think about this issue).

The aspiring macroeconomists amongst you should really read Chapter 13 (and Chapter 10) of Ljungquist and Sargent for a more general exposition of Ricardian equivalence in an asset pricing context. Notably, they allow the government to issue any type of bonds, not just riskless one-period bonds as assumed here.

Exercise 14 *Enumerate three different assumptions under which Ricardian equivalence would fail. Explain why.*

3.3.1 Debt sustainability

Here is where debt does make a difference though. Let us still work under the assumption that the path of taxation is determined somehow arbitrarily and see how far we can get. The budget constraint of the government iterated forward (we iterate forward because $R_t > 0$) gives:

$$B_t^s = (1 + R_t)^{-1} B_{t+1}^s + T_t - G_t \rightarrow$$

$$B_t^s = \lim_{T \to \infty} E_t \Lambda_{t,T} B_T^s + E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} (T_{t+j} - G_{t+j}) \rightarrow$$

$$B_t^s = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} (T_{t+j} - G_{t+j})$$

This is the **intertemporal government budget constraint**; it states that the path of taxes cannot be chosen entirely arbitrarily, but rather to ensure that this constraint holds for any sequence for G_t and any prices. Note that the process for T_{t+j} is the only process that can deliver this adjustment, since G_{t+j}, $\Lambda_{t,t+j}$ and B_t^s are given: the first one exogenously, the second one by equilibrium $\Lambda_{t,t+j} = \prod_{i=0}^{j} (1 + R_{t+i})^{-1}$ and the third one because it is a state predetermined variable. Therefore, the tax process $T_t = \Phi_t (\mathbf{X}_t)$ needs to be chosen so that to satisfy this intertemporal constraint.

4AM: Finally, note that this is a **constraint** only because the bonds are **real.** We have not talked about nominal models, but some really smart people believe in the following, which the aspiring macroeconomists may find interesting: **the fiscal theory of the price level, FTPL**. If bonds were 'nominal' (denominated in currency) the equation above is

$$\frac{B_t^N}{P_t} = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} (T_{t+j} - G_{t+j}), \tag{3.16}$$

where P_t is the price level of the economy (i.e. the price of the final good) and B_t^N is the nominal value of bonds (this is the relevant state variable now). Bluntly, FTPL says that if the path of budget surpluses $T_{t+j} - G_{t+j}$ is exogenous, since B_t^s is given the only thing that can ensure adjustment is P_t, the price level of this economy! Therefore, the price level is determined by fiscal policy. In this case, the equation above is NOT a constraint but a valuation equation for the real value of government debt, which can end up being an equation that determines the price level. This case is labelled 'non-Ricardian' (see Woodford 2003, e.g.) for very good reasons: the path of taxation ends up influencing the price level and hence all variables in the economy. If instead the surplus is determined to ensure that (3.16) holds for any price level P_t fiscal policy is labelled 'Ricardian' and then the price level will not be determined unless we introduce a specific type of monetary policy. We have no time to get into details about this fascinating, counterintuitive and highly controversial topic - see e.g.

John Cochrane's paper 'Money as Stock' (and the references therein) for a nice exposition of these ideas.

3.4   Distortionary taxation and Ricardian equivalence

Suppose that government spending is financed via a tax on labor income at rate $\tilde{\tau}_t$ and there are no lump-sum taxes available. The production function and the real wage equation are unchanged, and so is the Euler equation for the choice of bond holdings. The government budget constraint reads (note that labor income is equal to total income since the production function is homogenous of degree one):

$$(1 + R_t)^{-1} B_{t+1}^s + \tilde{\tau}_t Y_t = B_t^s + G_t.$$

The budget constraint of the households becomes:

$$(1 + R_t)^{-1} B_{t+1}^d + C_t = B_t^d + (1 - \tilde{\tau}_t) W_t L_t, \tag{3.17}$$

The only optimality condition that modifies concerns the choice of consumption versus hours worked, which is now:

$$v_L \left(L_t^s \right) = (1 - \tilde{\tau}_t) W_t \frac{1}{C_t} \tag{3.18}$$

Finally, we need one more equation: we have to specify a rule for the adjustment of tax rates

$$\tilde{\tau}_t = \Phi_t^\tau \left(\mathbf{X}_t \right). \tag{3.19}$$

To see whether Ricardian equivalence holds, we want to find out whether this temporal path of taxes has any effect on the equilibrium allocation. Note that by combining the household and government budget constraints with the bond market clearing equation and recalling that $Y_t = W_t L_t$ we find (or by Walras' law):

$$Y_t = C_t + G_t.$$

Let us list the equilibrium conditions as we did in the lump-sum taxes case in 3.15:

$$
\begin{aligned}
Y_t &= C_t + G_t. \\
Y_t &= A_t L_t \\
v_L \left(L_t \right) &= \frac{1 - \tilde{\tau}_t}{C_t} W_t \\
W_t &= A_t \\
(1 + R_t)^{-1} &= \beta E_t \left[\frac{C_t}{C_{t+1}} \right]
\end{aligned} \tag{3.20}
$$

The path of taxation DOES matter now, and hence Ricardian equivalence fails, since the tax rate affects the intratemporal optimality condition governing

the leisure-consumption trade-off. Therefore, we need the tax rule 3.19 in order to complete equilibrium description in 3.20. Moreover, we need to ensure that the government's intertemporal budget constraint holds:

$$B_t^s = E_t \sum_{j=0}^{\infty} \Lambda_{t,t+j} \left(\tilde{\tau}_t Y_{t+j} - G_{t+j} \right)$$

3.5 The effects of government spending shocks under distortionary taxation

We can understand how exactly does the intertemporal path of taxation matter for the equilibrium allocation by studying, as before, the loglinearized equilibrium. But before we do that, note that we can easily calculate the effects of shocks even without loglinearising in the **balanced-budget case**, i.e. whereby

$$\tilde{\tau}_t Y_t = G_t.$$

Substitute the consumer budget constraint $C_t = (1 - \tilde{\tau}_t) W_t L_t$, into the intratemporal optimality condition to get

$$v_L \left(L_t \right) = \frac{1 - \tilde{\tau}_t}{(1 - \tilde{\tau}_t) W_t L_t} W_t = \frac{1}{L_t}$$

Hours will now be constant (in levels, so both in steady state and outside!), as opposed to the lump-sum taxes case. For our specification for v,:

$$L_t = L = \chi^{-\frac{1}{1+\varphi}}$$

The intuition for this is: if government spending is fully financed via increased tax rates, these rates do nothing else than reduce the after-tax real wage; there is no separate income/wealth effect of taxation (as was the case under lump-sum taxes). But for the assumed log utility function in consumption, income and substitution effects of real wage variations cancel out, so the labor supply is constant. Since there is no intertemporal substitution (no capital and no government debt), hours will not vary at all over time. We will then have (in the absence of technology):

$$Y_t = Y = L, \ W = 1$$

The only variables that do move are consumption and tax rates. Tax rates increase and consumption falls:

$$\tilde{\tau}_t \ = \ \frac{G_t}{Y}$$
$$C_t \ = \ (1 - \tilde{\tau}_t) Y = Y - G_t$$

Let us now consider the general case, whereby **part of the increase in** G **is financed by issuing government debt**. We look at the *loglinearised*

equilibrium conditions in 3.20, together with the loglinearized versions of the tax rate rule and the government's budget constraint. Let's take first the equations in 3.20 (we use the economy resource constraint to replace household's budget constraint and assume technology is constant):

$$
\begin{aligned}
y_t &= (1 - G_Y) c_t + G_Y g_t \\
y_t &= l_t \\
w_t &= 0 \\
\varphi l_t &= w_t - \frac{1}{1 - \tau} \tau_t - c_t \\
c_t &= E_t c_{t+1} - r_t
\end{aligned}
\tag{3.21}
$$

Importantly, I have defined $\tau_t \equiv \tilde{\tau}_t - \tau$, where τ is the steady-state value of $\tilde{\tau}_t$; this is similar to how we defined the interest rate's deviation from steady-state, since the tax rate is also a rate (hence it is already in percentage points!) and we need not take logs of it.

One general point about the distortionary-tax model, and about why the intertemporal path of tax rates influences the equilibrium allocation, should be noted. Substitute the intratemporal optimality condition in the Euler equation, just as we did before, to obtain (since $w_t = 0$):

$$
\varphi \left(l_t - E_t l_{t+1} \right) = \frac{1}{1 - \tau} \left(E_t \tau_{t+1} - \tau_t \right) + r_t
$$

This equation describes *intertemporal substitution in labor supply induced by tax rate variations.* When the tax rate is expected to be higher tomorrow than it is today, the household optimally decides to work more today (because today she is taxed less). This is how taxation at a time-varying rate distorts the labor supply decision of the household.

Moreover, note that, as before, G_Y is something we know (steady-state share of government spending in GDP). We now need to turn to the 'fiscal block'. First, we have one exogenous process dictating the dynamics of g_t, for instance an AR(1) process just as for technology. The remaining two equations are studied in detail in the next subsection.

3.5.1   Debt dynamics and the 'fiscal rule'

Loglinearising the government budget constraint yields (use the tricks, and let $\frac{B}{Y} = B_Y$, $\hat{b}_t \simeq \frac{B_t - B}{B}$):

$$
\beta B_Y \left(\hat{b}_{t+1} - r_t \right) + \tau y_t + \tau_t = B_Y \hat{b}_t + G_Y g_t
\tag{3.22}
$$

I will make the simplifying assumption that budget is balanced in steady-state, i.e.

$$
B_Y = 0, \ \tau = G_Y.
$$

Because B is zero, we can no longer use $\hat{b}_t \simeq \frac{B_t - B}{B}$, so instead we will define $b_t \equiv \frac{B_t - B}{Y} = \frac{B_t}{Y} = B_Y \hat{b}_t$, i.e. debt as a share of steady-state GDP. replacing these elements in 3.22 and keeping only b_{t+1} on the left hand side we get:

$$b_{t+1} = \beta^{-1} b_t + \beta^{-1} \tau g_t - \beta^{-1} \left(\tau y_t + \tau_t \right) \tag{3.23}$$

You now see clearly why we need a 'fiscal rule'. If we ignored the last term, debt would be given by a purely autoregressive process with root $\beta^{-1} > 1$, and hence exploding. But by making **total tax revenues** respond to government debt we will be able to avoid this problem. Therefore, let us assume that tax revenues are determined by the following:

$$\tau_t + \tau y_t = \phi_g \tau g_t + \phi_b b_t \tag{3.24}$$

This rule says that tax revenues respond to government spending and government debt. When $\phi_g = 1$ and there is no debt initially, $(b_t = 0)$, this is the balanced-budget rule we had before: tax revenues respond one-to-one to increased spending. When $\phi_g < 1$, there is some deficit financing as not all of the increase in spending is accommodated by taxes. Finally, note that this rule assumes **automatic stabilization:** tax rates do not have to increase proportionally to the increase in spending, if the latter induces an expansion in the tax base, i.e. in y_t. In fact, if this expansion is strong enough the tax rate may even fall - but since y_t is itself an endogenous process we cannot study this separately. Finally, the response to the outstanding level of debt ϕ_b will ensure that debt itself will not explode.

To see this, substitute 3.24 into 3.23 to get:

$$b_{t+1} = \beta^{-1} \left(1 - \phi_b \right) b_t + \beta^{-1} \tau \left(1 - \phi_g \right) g_t \tag{3.25}$$

This equation is uncoupled with the rest and we can solve it backwards if and only if:

$$\beta^{-1} \left(1 - \phi_b \right) < 1 \rightarrow \phi_b > 1 - \beta.$$

Doing so (do it!!!), you obtain current debt as the present value of all past government spending.

3.5.2 Crowding-in of private consumption

Equilibrium is fully described by 3.21, together with 3.24 and 3.25. We are now ready to study the responses of all variables to a government spending shock. To begin with, since the real wage is constant by assumption, $w_t = 0$, the after-tax real wage changes only because the tax rate changes:

$$w_t - \frac{1}{1 - \tau} \tau_t = -\frac{1}{1 - \tau} \tau_t.$$

The intratemporal optimality condition becomes:

$$\varphi l_t = -\frac{1}{1 - \tau} \tau_t - c_t.$$

Substitute the production function $y_t = l_t$ and the resource constraint $y_t = (1 - \tau) c_t + \tau g_t$ where we used $G_Y = \tau$ to get:

$$[1 + \varphi (1 - \tau)] c_t = -\frac{1}{1 - \tau} \tau_t - \varphi \tau g_t \qquad (3.26)$$

You already see from this equation that consumption may indeed increase in response to government spending shocks if and only if the response of taxes is such that:

$$\tau_t < -\varphi \tau (1 - \tau) g_t.$$

That is, the tax rate has to fall endogenously 'enough', and the more so, the more inelastic is labor supply.

We can derive an exact parametric condition for the crowding-in of consumption by substituting the tax rule into 3.26:

$$[1 + \varphi (1 - \tau)] c_t = -\frac{1}{1 - \tau} \left[\phi_g \tau g_t + \phi_b b_t - \tau y_t \right] - \varphi \tau g_t$$

and substituting the resource constraint for y_t :

$$[1 + \varphi (1 - \tau)] c_t = -\frac{1}{1 - \tau} \left[\phi_g \tau g_t + \phi_b b_t - \tau (1 - \tau) c_t - \tau^2 g_t \right] - \varphi \tau g_t$$

Rearranging:

$$(1 + \varphi) (1 - \tau) c_t = \tau \left[\frac{\tau - \phi_g}{1 - \tau} - \varphi \right] g_t - \frac{1}{1 - \tau} \phi_b b_t \qquad (3.27)$$

This equation gives the equilibrium response of private consumption to government spending shocks and last period's debt. Assuming that we start from steady state, and therefore $b_t = 0$ initially, the impact multiplier of government spending on consumption is positive if and only if:

$$\varphi < \frac{\tau - \phi_g}{1 - \tau}$$

There are three interrelated conditions for g to crowd in private consumption: labor has to be elastic enough (φ low), ϕ_g has to be low - enough deficit financing; and τ has to be high enough.

These three features lead to a fall in the tax rate and an increase in consumption for the following intuitive reasons. A highly elastic labor supply ensures that labor reacts as much as possible to changes in tax rates, therefore expanding the 'tax base'; when this effect is strong enough, the tax base expands so much that the tax rate falls. A high degree of deficit financing ensures that the tax rate does not increase by much on impact (just look at the fiscal rule), while a high τ ensures that the automatic stabilization part is significant enough - i.e., for a given increase in y_t tax revenues will increase automatically more when τ is higher, calling for a lower increase in tax rates.

The effect on output and hours is easily found using the resource constraint:

$$y_t \, (= l_t) = \frac{\tau}{1+\varphi} \frac{1-\phi_g}{1-\tau} g_t - \frac{1}{1-\tau} \frac{\phi_b}{1+\varphi} b_t$$

As expected from the above discussion, the multiplier on output and hours is also higher when τ is higher, ϕ_g is lower, and labor is elastic (φ low).

The equilibrium tax rates evolve according to:

$$\tau_t = \tau \left(\phi_g - \frac{\tau}{1+\varphi} \frac{1-\phi_g}{1-\tau} \right) g_t + \phi_b \frac{1+\varphi(1-\tau)}{(1-\tau)(1+\varphi)} b_t.$$

From period $t+1$ onwards, an expansion in consumption is harder to obtain since debt will have increased and needs to be paid back by an increase in taxes. Indeed, at any time $t + j$ we have:

$$(1+\varphi)(1-\tau) c_{t+j} = \tau \left[\frac{\tau - \phi_g}{1-\tau} - \varphi \right] g_{t+j} - \frac{1}{1-\tau} \phi_b b_{t+j} \qquad (3.28)$$

where b_{t+j} is given by 3.25 evaluated at time $t + j$ as the present value of past government spending:

$$
\begin{aligned}
b_{t+j} &= \beta^{-1}(1-\phi_b) b_{t+j-1} + \beta^{-1}\tau(1-\phi_g) g_{t+j} = & (3.29) \\
&= \beta^{-1}\tau(1-\phi_g) \sum_{i=0}^{j} \beta^{-i}(1-\phi_b)^i g_{t+j-i}. & (3.30)
\end{aligned}
$$

This increase in debt will generally lead to a fall in consumption in later periods.

What is (not) satisfactory about this model? To answer this question we need to think of two interrelated issues: 1. are the parameter restrictions making this happen reasonable?; and 2. is the mechanism by which we obtained this result realistic?

Two of the parametric requirements for a positive consumption multiplier have empirical support : much spending is indeed deficit-financed; and there is a lot of automatic stabilization, precisely because much of taxation is linked to the level of activity (is distortionary). But the remaining requirement, as we learned already, does not have enough support - the elasticity of labor supply is not very high. Moreover, the very mechanism at the heart of this model is likely to be unrealistic: empirical estimates reveal that 'tax base' effects are not likely to be relevant; moreover, tax rates do not vary significantly at business cycle frequency. However, on the last point one can argue that the model's τ_t is not a particular tax rate, but the *effective* tax rate on labour (and estimates of effective tax rates are known to be time-varying).

3.6   Back to Asset Pricing. Some extra embarrassment for the frictionless model: The Equity Premium Puzzle

I want to end this set of lectures with one example of a puzzle that has been known for about 20 years, and has not yet been resolved in a satisfactory way (you will review some attempts, as well as other puzzles in asset pricing, with Professor Muellbauer next term). This is a good example of 'good news' for people who want to do Macro/Finance - there's a lot to do yet. This is the 'Equity Premium Puzzle' (the original contribution is due to Mehra and Prescott, 1985).

Empirically, the average return on equity, as judged by the return on S&P500, in the US economy over the past 50 years has been about 8.1% at an annualized rate. The risk-free rate, judged as return on Treasury Bills, has been much lower, at about 0.9%. Therefore, the equity premium is about 7.2%! The size of the premium varies depending on the period under consideration, the definition of returns, etc., but the main idea is always there: stocks give you a much higher return than bonds. How come? Can we reconcile this with our model?

Remember our asset equations for stocks and riskless bonds respectively (I will now work with shares and denote their return $1 + R_{t+1}^S \equiv \frac{P_{t+1}+D_{t+1}}{P_t}$, but remember you could do the same with physical capital):

$$1 = E_t\left[\Lambda_{t,t+1}\left(1 + R_{t+1}^S\right)\right]$$
$$1 = (1 + R_{t+1})E_t\left[\Lambda_{t,t+1}\right]$$

I have put a $t + 1$ index on the return on bonds although it is known at time t, to emphasize that it is being paid at time $t + 1$. To understand the basic idea, let us assume that returns and the stochastic discount factor are jointly lognormal and homoskedastic, just as we assumed when we loglinearised the Euler equation for the RBC model. Remember that for a lognormal variable:

$$\ln E_t X_{t+1} = E_t \ln X_{t+1} + \frac{1}{2}var_t \ln X_{t+1}.$$

If X is also homoskedastic, conditional second moments are equal to unconditional second moments, so we can drop the time subscript on second moments:

$$\ln E_t X_{t+1} = E_t \ln X_{t+1} + \frac{1}{2}var \ln X_{t+1}.$$

Taking logs of the share-price equation we get (as throughout this course, we use $\ln(1 + a) = a$):

$$0 = \ln E_t\left[\Lambda_{t,t+1}\left(1 + R_{t+1}^S\right)\right] = E_t\lambda_{t,t+1} + E_t R_{t+1}^S + \frac{1}{2}\sigma_\lambda^2 + \frac{1}{2}\sigma_S^2 + \sigma_{\lambda S} \quad (3.31)$$

where $\lambda_{t,t+1} = \ln \Lambda_{t,t+1}$, $\sigma_x^2 = var\left[\ln X_{t+1} - E_t \ln X_{t+1}\right]$, i.e. the variance of innovations to the log of variable $X = R_{t+1}^S, \Lambda_{t,t+1}$ and $\sigma_{\lambda S}$ is similarly the

unconditional covariance between the innovations to the logs of the returns and stochastic discount factor.

Let's do the same now for the riskless asset - an asset whose return is known with certainty and uncorrelated with the stochastic discount factor:

$$0 = \ln\left(1 + R_{t+1}\right) E_t\left[\Lambda_{t,t+1}\right] = E_t\lambda_{t,t+1} + R_{t+1} + \frac{1}{2}\sigma_\lambda^2 \qquad (3.32)$$

Now subtract 3.32 from 3.31 to get:

$$\text{Equity Premium } = E_t R_{t+1}^S - R_{t+1} + \frac{1}{2}\sigma_S^2 = -\sigma_{\lambda S} \qquad (3.33)$$

This equation already illustrates the equity premium. The term on the left-hand side it the equity premium, corrected for a measure of risk for stocks, $\frac{1}{2}\sigma_S^2$, that comes simply from Jensen's inequality (do not get confused about this; this term would disappear if we wrote the premium using the expectation of log gross return on shares, so that the left-hand side becomes $E_t\ln\left(1 + R_{t+1}^S\right) - \ln\left(1 + R_{t+1}\right)$).

The right-hand side says that in this model the equity premium is given by the (negative of the) covariance of the stochastic discount factor with the share return. Consumers demand a high premium in order to hold this asset when the return on an asset co-varies negatively with the stochastic discount factor for a simple reason: the asset tends to have low returns and hence decrease the value of wealth precisely when consumers need it more (when the marginal utility of consumption is lower than in the future, i.e. when the stochastic discount factor is high: remember that $\Lambda_{t,t+1} = \beta\frac{U_C(C_{t+1})}{U_C(C_t)}$ is high when marginal utility of consumption today is low compared to the future one).

A simple functional form for the utility function makes this more transparent. Consider the CRRA utility function (and let labor supply be inelastic): $U\left(C\right) = \left(C^{1-\gamma} - 1\right)/\left(1 - \gamma\right)$, where γ parameterizes both the coefficient of relative risk aversion and the (inverse of) the elasticity of intertemporal substitution. (When $\gamma \to 1$ we are back to the $\ln C$ case.) The stochastic discount factor in this case is $\Lambda_{t,t+1} = \beta\left(C_{t+1}/C_t\right)^{-\gamma}$, so (letting small letters denote logs):

$$\lambda_{t,t+1} = \ln\beta - \gamma\Delta c_{t+1}.$$

The equity premium becomes:

$$\text{Equity Premium } = E_t R_{t+1}^S - R_{t+1} + \frac{1}{2}\sigma_S^2 = \gamma\sigma_{CS}, \qquad (3.34)$$

i.e. the product between the relative risk aversion coefficient and the covariance between innovations to consumption and equity returns. Intuitively, a large equity premium is required when (for a given risk aversion) there is a high covariance between returns and consumption, for in this case the asset delivers low returns when consumption is low (when marginal utility of consumption is high).

You can look at this equation in two ways. First you can think of the equity premium itself as the relevant moment to match. You build a general equilibrium model in which returns and consumption are determined endogenously and calculate the 'artificial' covariance between consumption and returns, parameterize risk aversion and try to see for which risk aversion you can match the observed equity premium of around 6.9%. Secondly, you can use observed, i.e. data covariance between consumption and returns and find the implied risk aversion coefficient, then ask yourself whether the number you get is 'reasonable'.

Most studies, independently of the approach they use and of the type of data employed, find that a risk aversion of the order of 30 is needed to explain the equity premium. This is much, much higher than any plausible empirical estimate; it also has highly unrealistic implications for the behaviour of individuals. Macro and microeconomists usually think of this number as being not higher than 2 or 3.

A related puzzle came to be known as the 'risk-free rate puzzle' (Weil 1989). Look at he equation for bond returns, substituting for the CRRA utility function:

$$R_{t+1} = -\ln\beta + \gamma\Delta E_t c_{t+1} - \frac{\gamma^2}{2}\sigma_C^2$$

Abstract from the last term. Given positive observed consumption growth, say $\Delta c = \Delta E_t c_{t+1}$, a high risk aversion parameter γ can only be reconciled with low risk-free rates, as we observe in the data, if $\beta \geq 1$. This implies negative time preference, something I am sure most of you will consider implausible. Intuitively, CRRA utility links risk aversion and intertemporal substitution: high risk aversion automatically means low desire to substitute consumption intertemporally. A consumer who is unwilling to substitute intertemporally, when faced with low interest rates and positive consumption growth would like to bring consumption into the present, i.e. to borrow. A low interest rate can only be an equilibrium if the rate of time preference is very low or even negative.

The last term, $-\frac{\gamma^2}{2}\sigma_C^2$, helps because for a high γ, the risk-free rate is brought down (the variance term is always positive). This term comes from a precautionary savings motive: agents want to save to protect themselves from uncertainty related to future consumption variability. This desire to save works against the tendency to borrow.

You will study next term various attempts to deal with these puzzles based on non-separabilities in the utility function, on preferences that disentangle risk aversion from intertemporal substitution, on heterogenous agents, limited asset markets participation, etc.

Bibliography

[1] Baxter, M., and R. G. King (1993): "Fiscal Policy in General Equilibrium," *American Economic Review* 83: 315-334.

[2] Blanchard, O. J. and C. M. Kahn, (1980), "The solution of linear difference models under rational expectations," *Econometrica*, 48, 5, 1305-12

[3] Campbell, J. Y., (1994): "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model," *Journal of Monetary Economics* 33: 463-506.

[4] Christiano, L., and M. Eichenbaum (1992): "Current Real-Business-Cycle Theories and Aggregate Labor-Market Fluctuations," *American Economic Review* 82: 430-450

[5] Christiano, L., M. Eichenbaum and Evans (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy", *Journal of Political Economy*, vol. 113, no. 1, 1-46

[6] Cochrane, J, 2004, 'Money as Stock', Forthcoming in *Journal of Monetary Economics*

[7] Cogley, T., and J. M. Nason (1995): "Output Dynamics in Real-Business-Cycle Models," *American Economic Review* 85: 492-511.

[8] *Cooley, T. F. and E. Prescott, 'Economic growth and business cycles', Chapter 2 in Cooley, T.F. (Ed.), 1995. *Frontiers of Business Cycle Research*. Princeton Univ. Press, Princeton.

[9] Farmer, R. (1999) '*The macroeconomics of self-fulfilling prophecies*', MIT Press, Cambridge, MA.

[10] Galí, J., J. D. López-Salido, and J. Vallés (2002): "Understanding the Effects of Government Spending on Consumption," mimeo, CREI.

[11] *King, R. and Rebelo, S. (2000) 'Resuscitating Real Business Cycles', in Taylor J. and M. Woodford, eds, *Handbook of Macroeconomics*, North-Holland

[12] Kydland, F.E., Prescott, E.C., (1982). Time to build and aggregate fluctuations. *Econometrica* 50, 1345–1370.

[13] Ljungquist, L. and T. Sargent (2004), *Recursive Macroeconomic Theory*, 2nd Edition, MIT Press, Cambridge, MA.

[14] Lubik, T. and Schorfheide F. (2003) "Computing Sunspot Equilibria in Linear Rational Expectations Models" *Journal of Economic Dynamics and Control*, 28(2), pp. 273-285.

[15] Lucas, Jr. R. E., (2005), 'Present at the creation: Reflections on the 2004 Nobel Prize to Finn Kydland and Edward Prescott', *Review of Economic Dynamics* 8 (2005) 777–779

[16] Lucas, Jr. R. E., (1981), *Studies in Business-Cycle Theory*, MIT Press, Cambridge, MA.

[17] Lucas, Jr. R. E., (1985), *Models of Business Cycles,* Yrjö Jahnsson Lectures, Basil Blackwell, Oxford.

[18] Prescott, E., (1986) 'Theory ahead of business cycle measurement', *Federal Reserve Bank of Minneapolis Quarterly Review* 10.

[19] Stokey, N.; R. Lucas with E. Prescott, (1989) '*Recursive Methods for Macroeconomic Dynamics*', Harvard University Press, Cambridge, MA.

[20] Woodford, M., (2003), *Interest and prices: foundations of a theory of monetary policy*, Princeton University Press, Princeton, NJ.

Appendix A

Refresher: 'solving' a dynamic optimization problem in 2 minutes

Disclaimer: this part just gives you main intuition behind Euler Equations; a mathematician may faint. The main idea behind calculus of variations is: suppose you have to solve an optimisation problem, i.e. have to find some optimal path for your control variable in order to maximise (minimise) an objective function. The idea is that at any point (period) along that optimal path, a variation in that period's control (action) would reduce the overall payoff. In differential terms, it means that differentiating your objective function along the optimal path with respect to your state variable, you should get ZERO. Again, following the path is always better than deviating (game theory inclined people might have recognised the idea behind some equilibrium concept there). If you know the other two main approaches to dynamic optimisation (Pontryagin and Bellman) you may think of a way to relate the three.

A.1 Euler Equations in the Deterministic Case

Consider the problem:

$$\sup \sum_{t=0}^{\infty} u_t\left(X_t, X_{t+1}\right) \quad s.t. \quad X_{t+1} \in \Gamma_t\left(X_t\right) \tag{A.1}$$

Note that the functional form of the utility function is time-variant, encompassing discounting, i.e. $u_t\left(.\right) = \beta^t u\left(.\right)$ and the constraint correspondence is also time-variant. For simplicity, suppose you have to choose directly tomorrow's state X_{t+1}- you can always substitute out for your control variable (see example below). We will have some additional constraints but I introduce these later, so you see when they are needed.

Now suppose you have an **optimal solution (path)**:

$$\exists \{X_t^*\}_1 = \{X_0^*, X_1^*, ..., X_{t-1}^*, X_t^*, X_{t+1}^*, ...\} \tag{A.2}$$

To see the necessary first order conditions look at a **feasible** one-period deviation from the conjectured optimal path $\{X_0^*, X_1^*, ..., X_{t-1}^*, X, X_{t+1}^*, ...\}$. Feasible means it satisfies the constraint correspondence $\forall t$, more specifically $X \in \Gamma_{t-1}(X_{t-1}^*), X_{t+1}^* \in \Gamma_t(X)$ - rest is trivial, as an optimal path is by definition feasible.

Now by optimality of A.2, the following holds $\forall X$ feasible (if you do not see this directly, note that all other terms in the intertemporal objective function $\sum_{t=0}^{\infty} u_t(X_t, X_{t+1})$ are the same as we consider a one-shot variation):

$$u_{t-1}(X_{t-1}^*, X_t^*) + u_t(X_t^*, X_{t+1}^*) \geq u_{t-1}(X_{t-1}^*, X) + u_t(X, X_{t+1}^*), \forall t \tag{A.3}$$

Now make two additional assumptions to be able to write this in the differentiable case: (i) $u_t(.)$ differentiable; (ii) $\{X_t^*\}_1$ interior solution, i.e. $\forall t$, $X_t^*, X_{t+1}^* \in \text{Int}(\Gamma_t)$. Now from A.3 you see that the derivative of the right hand side (RHS) with respect to X has to be zero, hence we get (where $D_i u(.)$ denotes u's derivative or gradient with respect to its ith argument):

$$D_2 u_{t-1}(X_{t-1}^*, X_t^*) + D_1 u_t(X_t^*, X_{t+1}^*) = 0, \forall t \tag{A.4}$$

This is our friend the Euler Equation. Our best friend will be A.4 in the discounting case, which reads:

$$D_2 u(X_{t-1}^*, X_t^*) + \beta D_1 u(X_t^*, X_{t+1}^*) = 0, \forall t \tag{A.5}$$

Note that this gives you finally a second-order difference equation $X_{t+1}^* = F(X_{t-1}^*, X_t^*)$ from which you get an explicit form for the optimal path - note this holds for all times t. Typically in the models we consider during this course we do not have to solve for these explicitly, as the models do not have a closed-form solution anyway overall, but you should maybe read on.

The bad news is that such a (family of) difference equation(s) has a multiplicity of solutions. But the good news is something you might have forgot: we have boundary conditions to help us choose amongst these. Do not get too excited, though: at first glance we just have the initial condition, an initial value for X_t, and you might know that for a second order equation we need two conditions to pick up the unique path. Hence up to now we pinned down a one-parameter family of paths. The other condition that helps us here is the transversality condition at infinity - keep in mind this is a sufficient, not necessary condition. See the example below for an illustration.

Example 15 *Cake-eating. Suppose a consumer has a fixed endowment X_0 and the only thing he can do with it is either eat it or save it for future consumption. His consumption in period t is what he has today minus what he saves to eat from tomorrow onwards. Hence the constraint $X_{t+1} \in \Gamma_t(X_t)$ is now $X_t \geq X_{t+1} \geq 0$*

$$C_t = X_t - X_{t+1} \rightarrow u(C_t) = u(X_t - X_{t+1}) \tag{A.6}$$

Consider there is discounting at β. This is then a particular case of the problem considered before: the Euler Equation, by A.5 is

$$
\begin{aligned}
\frac{d}{dX_t} u\left(X_{t-1} - X_t\right) + \frac{d}{dX_t} \beta u\left(X_t - X_{t+1}\right) &= 0 \forall t \rightarrow \\
-u'\left(X_{t-1} - X_t\right) + \beta u'\left(X_t - X_{t+1}\right) &= 0 \rightarrow \\
\forall t, \quad u'\left(C_{t-1}\right) &= \beta u'\left(C_t\right)
\end{aligned}
$$

When utility is logarithmic, as is considered many times in our applications we get $C_t = \beta C_{t-1} \forall t$. Do not be mislead: this is indeed not a second-order but a first-order difference equation, but it is expressed in the control, not the state. Replacing back consumption as function of the state you get a second-order equation. We have no initial value for consumption, we just know the initial value of the cake!!! Hence we still need a transversality condition, which in this case is: $\sum_{t=0}^{\infty} C_t = X_0$. Then you can trivially solve the difference equation in consumption and find the optimal path explicitly. In terms of the state, the solution is:

$$
X_t^* = X_0 \beta^t \quad \forall t
$$

This makes sense: you eat a constant fraction of the cake each period, your total consumption over the infinite horizon never hits the limit (there is always some pie around) and more impatient you are faster you eat (faster the pie shrinks).

Note two last things: in the way the problem has been set up I used sup and not max: sometimes a maximum is not achieved, make sure to check this in your applications. Secondly, when you have a minimisation problem you can obviously treat it as maximising the negative of the original objective function.

A.2   Euler Equations in the Stochastic Case

Let me now be very sloppy and merely say what happens in the **stochastic** case informally, as doing it formally would imply I should introduce other concepts like some measure theory, Lebesgue integrals, Markovian operators, etc - if you are really keen on this see the book by Stokey, Lucas with Prescott[1]. Generally, our equation for the discounting case A.5 would modify like

$$
D_2 u\left(X_{t-1}^*, X_t^*, \theta_t\right) + \beta E_{t-1}\left[D_1 u\left(X_t^*, X_{t+1}^*, \theta_t\right)\right] = 0 \tag{A.7}
$$

where θ is a general stochastic shock, capturing the state of the world. Then E_{t-1} is the expectation operator at $t-1$, and how you define this is part of the long story. Now your optimal path will tell you what to do at any given time, AND for any given state of the world: it will be a **state-contingent plan**.

---

[1] If you read Italian, I warmly recommend: Montrucchio, L. and Cugno, F. (2000), *'Scelte intertemporali: teoria e modelli'*, Carocci Editore, Roma.

For example, if you have to eat a stochastic cake (i.e. the size of your cake shrinks or expands each period stochastically by a multiplicative shock z_t which has to satisfy some nice properties), you have the Euler Equation:

$$u'(C_{t-1}) = \beta E_{t-1}[z_t u'(C_t)]$$

Appendix B

Matlab programmes

B.1 Matlab code for the basline, elastic-labor model

The following code solves the baseline RBC model and plots impulse responses. I included a loop over parameters that allows you to plot responses for different parameterizations on the same graph. You need the file `solvek.m` that you already have from Dr. Meeks, from the computational classes. Indeed, the codes here use precisely the same solution method. You also need `dimpulse.m`, but this should already be in the Toolbox. Make sure these are in the same directory as the model file.

Note: *mu* in the code is labor supply elasticity, i.e. φ^{-1}.

clear all;

%Number of Variables nx is the number of variables in the whole system, nz is the number of non-predetermined variables

nx = 11; nz = 2; nu = nz;

%/**/

% @ Parameters of the model;@

%/**/

%Deep Parameters

loop=1; %loop over parameters; here loop over L elasticity and A persistence

for loop =1:2

if loop==1

mu=0; %labor elasticity

phia=1;

else

mu=0; %labor elasticity

phia=0.979;

end

r=0.01;

delta=0.025;

```
alfa=1/3;
%STEADY STATE RATIOS
si=(alfa*delta)/(r+delta);
sc=1-si;
%THE Equilibrium Conditions
A = zeros(nx,nx); B = zeros(nx,nx); C = zeros(nx,nu);
phi = zeros(nu,nu); %exogenous shock processes
cpos = 1; %c(t)
hpos = 2; %l(t)
ypos = 3; %y(t)
ipos = 4; %i(t)
rpos = 5; %r(t)
rkpos = 6; %rk(t)
wpos = 7; %w(t)
ec1pos=8;
er1pos=9;
ek1pos=10;
%Predetermined Variables
nk =1;
kpos = 11; %K(t)
%Shocks
apos = 1;
gpos = 2;
%Definiton of variables
%Equation 1: Euler equation
B(1,cpos) = -1;
%A(1,cpos) = 1;
%A(1,rpos) = -1;
B(1,ec1pos) =1;
B(1,er1pos)=-1;
%@Equation 2: intratemporal optimality@
B(2,hpos)=1;
B(2,wpos) = -mu;
B(2,cpos)= mu;
%@ Equation 5: capital accumulation @
 A(3,kpos)=1;
 B(3,ipos)=delta;
 B(3,kpos)=(1-delta);
%@ Equation 8: production function @
B(4,ypos)=1;
C(4,apos)=-1;
B(4,hpos)=-(1-alfa);
B(4,kpos)=-alfa;
%@ real interest rate definition @
B(5,rpos)=-1;
B(5,rkpos)=(r+delta)/(1+r);
```

```
%@ Equation 10: wage @
B(6,wpos)=1;
B(6,ypos)=-1;
B(6,hpos)=1;
%@ Rental rate @
B(7,rkpos)=1;
B(7,ypos)=-1;
B(7,kpos)=1;
%@ Resource constraint (could replace with HH budget constraint) @
B(8,ypos)=1;
B(8,ipos)=-si;
B(8,cpos)=-sc;
%Define E(t)cs(t+1)
A(9,cpos) = 1;
B(9,ec1pos) = 1;
%Define E(t)r(t+1)
A(10,rpos) = 1;
B(10,er1pos) = 1;
%Define E(t)k(t+1)
A(11,kpos) = 1;
B(11,ek1pos) = 1;
phi(apos,apos) = phia; %technology process introduced here
phi(gpos,gpos) = 0;
[m,n,p,q,z22h,s,t,lambda] = solvek(A,B,C,phi,nk);
bigmn = [m n];
bigpq = [p q];
bigp = phi;
bigpsi = eye(nz,nu);
%%%%%%%%%%%%%%%%% IRF analysis %%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Have to specify ires and ishock, the index values for the
% responding variable and the shock
% Using the solution of the model in state space form
% x(t+1) = Ax(t) + Bu(t+1)
% y(t) = Cx(t) + Du(t)
ishock = 1;
npts = 100; % no of points plotted
% Loop over parameters, assign calculated solution for each case to one IRF
if loop ==1
A1 = [p q;zeros(nz,nk) phi];
C1 = [m n];
D1 = zeros(nx-nk,nu);
B1 = [zeros(nk,nu);bigpsi];
%[Y,X]=dimpulse(A,B,C,D,ishock,npts+1);
%this does not work when have capital defined as here
[Y1,X1]=dimpulse(A1,B1,C1,D1,ishock,npts+2);
```

```
YP1=Y1(2:npts+2,:);
XP1=X1(2:npts+2,:);
else
A2 = [p q;zeros(nz,nk) phi];
C2 = [m n];
D2 = zeros(nx-nk,nu);
B2 = [zeros(nk,nu);bigpsi];
%[Y,X]=dimpulse(A,B,C,D,ishock,npts+1);
%this does not work when have capital defined as here
[Y2,X2]=dimpulse(A2,B2,C2,D2,ishock,npts+2);
YP2=Y2(2:npts+2,:);
XP2=X2(2:npts+2,:);
end %ends if statement
%move on after first case:
loop = loop+1;
end %ends 'for' statement
jj=[0:npts];
%i1 = Y(:,ires); % column index is the element of y you want
subplot(3,3,1)
plot(jj,YP1(:,ypos), jj,YP2(:,ypos),'-.r')
title('Output')
%axis([0 20 -.5 .5])
legend('rhoa=1','rhoa=0.979')
text(0,YP1(1,ypos)+0.7,'FIGURE 4: Responses to unit technology shock, the role
of shock persistence')
subplot(3,3,2)
plot(jj,YP1(:,cpos),jj,YP2(:,cpos),'-.r')
title('Consumption')
%axis([0 20 -.25 .25])

subplot(3,3,3)
plot(jj,YP1(:,ipos),jj,YP2(:,ipos),'-.r')
title('Investment')
%axis([0 20 -.5 .5])
subplot(3,3,4)
plot(jj,YP1(:,hpos),jj,YP2(:,hpos),'-.r')
title('Labor')
%axis([0 20 -1.5 1.5])

subplot(3,3,5)
plot(jj,YP1(:,wpos),jj,YP2(:,wpos),'-.r')
title('Real wage')
%axis([0 20 -.5 .5])
subplot(3,3,6)
%plot(jj,YP(:,ek1pos))
plot(jj,XP1(:,1),jj,XP2(:,1),'-.r')
```

```
title('Capital')
%axis([0 20 -1.5 1.5])
subplot(3,3,7)
plot(jj,YP1(:,rpos),jj,YP2(:,rpos),'-.r')
title('Interest rate')
%axis([0 20 -1.5 1.5])

subplot(3,3,8)
plot(jj,YP1(:,rkpos),jj,YP2(:,rkpos),'-.r')
title('Rental rate')
%axis([0 20 -.5 .5])
subplot(3,3,9)
plot(jj,XP1(:,2),jj,XP2(:,2),'-.r')
title('Productivity')
%axis([0 20 -1.5 1.5])
```

B.2   Matlab code for variable utilization model

The following code solves- and plots impulse responses - for the 'variable uti-
lization' model.

```
clear all;
%Number of Variables nx is the number of variables in the whole system, nz is the
number of non-predetermined variables
nx = 12; nz = 2; nu = nz;
%/***************************************************/
% @ Parameters of the model;@
%/***************************************************/
%Deep Parameters
loop=1; %loop over parameters; here loop over L elasticity and A persistence
for loop =1:2
if loop==1
mu=2; %labor elasticity
phia=0.979;
csi=5000; %this insures (almost) consistency with elasdelta=0
elasdelta=0;%this insures deriv. of delta wrt z is zero
delta=0.025;
else
mu=2; %labor elasticity
phia=0.979;
csi=0.1;
elasdelta=1+csi; %this in when delta=Z^(1+csi)/(1+csi)
delta=0.025;
end
r=0.01;
```

```
alfa=1/3;
%STEADY STATE RATIOS
si=(alfa*delta)/(r+delta);
sc=1-si;
%THE Equilibrium Conditions
A = zeros(nx,nx); B = zeros(nx,nx); C = zeros(nx,nu);
phi = zeros(nu,nu); %exogenous shock processes
cpos = 1; %c(t)
hpos = 2; %l(t)
ypos = 3; %y(t)
ipos = 4; %i(t)
rpos = 5; %r(t)
rkpos = 6; %rk(t)
wpos = 7; %w(t)
ec1pos=8;
er1pos=9;
ek1pos=10;
zpos=11;
%Predetermined Variables
nk =1;
kpos = 12; %K(t)
%Shocks
apos = 1;
gpos = 2;
%Definiton of variables
%Equation 1: Euler equation
B(1,cpos) = -1;
%A(1,cpos) = 1;
%A(1,rpos) = -1;
B(1,ec1pos) =1;
B(1,er1pos)=-1;
%@Equation 2: intratemporal optimality@
B(2,hpos)=1;
B(2,wpos) = -mu;
B(2,cpos)= mu;
%@ Equation 5: capital accumulation @
 A(3,kpos)=1;
 B(3,ipos)=delta;
 B(3,kpos)=(1-delta);
 B(3,zpos)=-delta*elasdelta;
%@ Equation 8: production function @
B(4,ypos)=1;
C(4,apos)=-1;
B(4,hpos)=-(1-alfa);
B(4,kpos)=-alfa;
B(4,zpos)=-alfa;
```

```
%@ real interest rate definition @
B(5,rpos)=-1;
B(5,rkpos)=(r+delta)/(1+r);

%@ Equation 10: wage @
B(6,wpos)=1;
B(6,ypos)=-1;
B(6,hpos)=1;
%@ Rental rate @
B(7,rkpos)=1;
B(7,ypos)=-1;
B(7,kpos)=1;
%@ Resource constraint (could replace with HH budget constraint) @
B(8,ypos)=1;
B(8,ipos)=-si;
B(8,cpos)=-sc;
%Define E(t)cs(t+1)
A(9,cpos) = 1;
B(9,ec1pos) = 1;
%Define E(t)r(t+1)
A(10,rpos) = 1;
B(10,er1pos) = 1;
%Define E(t)k(t+1)
A(11,kpos) = 1;
B(11,ek1pos) = 1;
%fficient utilization
B(12,ypos) = 1;
B(12,kpos) = -1;
B(12,zpos) = -(1+csi);
phi(apos,apos) = phia; %technology process introduced here
phi(gpos,gpos) = 0;
[m,n,p,q,z22h,s,t,lambda] = solvek(A,B,C,phi,nk);
bigmn = [m n];
bigpq = [p q];
bigp = phi;
bigpsi = eye(nz,nu);
%%%%%%%%%%%%%%%%%% IRF analysis %%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Have to specify ires and ishock, the index values for the
% responding variable and the shock
% Using the solution of the model in state space form
% x(t+1) = Ax(t) + Bu(t+1)
% y(t) = Cx(t) + Du(t)
ishock = 1;
npts = 40; % no of points plotted
% Loop over parameters, assign calculated solution for each case to one IRF
if loop ==1
```

```
A1 = [p q;zeros(nz,nk) phi];
C1 = [m n];
D1 = zeros(nx-nk,nu);
B1 = [zeros(nk,nu);bigpsi];
%[Y,X]=dimpulse(A,B,C,D,ishock,npts+1);
%this does not work when have capital defined as here
[Y1,X1]=dimpulse(A1,B1,C1,D1,ishock,npts+2);
YP1=Y1(2:npts+2,:);
XP1=X1(2:npts+2,:);
else
A2 = [p q;zeros(nz,nk) phi];
C2 = [m n];
D2 = zeros(nx-nk,nu);
B2 = [zeros(nk,nu);bigpsi];
%[Y,X]=dimpulse(A,B,C,D,ishock,npts+1);
%this does not work when have capital defined as here
[Y2,X2]=dimpulse(A2,B2,C2,D2,ishock,npts+2);
YP2=Y2(2:npts+2,:);
XP2=X2(2:npts+2,:);
end %ends if statement
%move on after first case:
loop = loop+1;
end %ends 'for' statement
jj=[0:npts];
%i1 = Y(:,ires); % column index is the element of y you want
subplot(3,3,1)
plot(jj,YP1(:,ypos), jj,YP2(:,ypos),'-.r')
title('Output')
%axis([0 20 -.5 .5])
legend('fixed utilization','variable utilization')
text(0,YP2(1,ypos)+0.6,'FIGURE 5: Responses to unit technology shock, the role
of variable utilization')
subplot(3,3,2)
plot(jj,YP1(:,cpos),jj,YP2(:,cpos),'-.r')
title('Consumption')
%axis([0 20 -.25 .25])

subplot(3,3,3)
plot(jj,YP1(:,ipos),jj,YP2(:,ipos),'-.r')
title('Investment')
%axis([0 20 -.5 .5])
subplot(3,3,4)
plot(jj,YP1(:,hpos),jj,YP2(:,hpos),'-.r')
title('Labor')
%axis([0 20 -1.5 1.5])
```

```
subplot(3,3,5)
plot(jj,YP1(:,wpos),jj,YP2(:,wpos),'-.r')
title('Real wage')
%axis([0 20 -.5 .5])
subplot(3,3,6)
%plot(jj,YP(:,ek1pos))
plot(jj,XP1(:,1),jj,XP2(:,1),'-.r')
title('Capital')
%axis([0 20 -1.5 1.5])
subplot(3,3,7)
plot(jj,YP1(:,rpos),jj,YP2(:,rpos),'-.r')
title('Interest rate')
%axis([0 20 -1.5 1.5])

subplot(3,3,8)
plot(jj,YP1(:,zpos),jj,YP2(:,zpos),'-.r')
title('Utilization rate')
%axis([0 20 -.5 .5])
subplot(3,3,9)
plot(jj,XP1(:,2),jj,XP2(:,2),'-.r')
title('Productivity')
%axis([0 20 -1.5 1.5])
```