1 Entry Queues and Attrition*

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1.1 Abstract

An oligopoly game is symmetric if the profit levels of the participating firms are symmetric in the strategies adopted by those firms. This means that if the firms exchange strategies they exactly exchange profits. A symmetric example is introduced in which the second entrant (the follower player in a Stackelberg leader-follower game) makes the greater profit. This leads to a classic attrition game in which each firm waits for the other to set capacity first. The more impatient firm enters first. The timing decisions under common knowledge are solved. With three or more players more complicated possibilities arise, such as second entry advantage over first and third entry pay-off.

1.1.1 The Duopoly Case: An Example

Consider the duopoly game in which x_i $(i = 1, 2)$ is the output of the *ith* producer, both producers have marginal production costs equal to zero and the inverse demand function is:

$$[p_0 + x_1 + x_2]^{-\alpha} \tag{1}$$

with $\alpha > 2$. Notice that the game is symmetrical, in the sense that the value of the game is a symmetrical function of the quantities produced by the two players.

To obtain the Nash solution, let producer 1 maximize profit given x_2:

$$x_1 [p_0 + x_1 + x_2]^{-\alpha} \tag{2}$$

which gives:

$$[p_0 + x_1 + x_2]^{-\alpha} - \alpha x_1 [p_0 + x_1 + x_2]^{-\alpha-1} = 0 \tag{3}$$

$$1 - \alpha \frac{x_1}{p_0 + x_1 + x_2} = 0 \tag{4}$$

$$x_1 = \frac{p_0 + x_2}{\alpha - 1} \tag{5}$$

The symmetric Nash solution is:

$$x_1 = x_2 = \frac{p_0}{\alpha - 2} \tag{6}$$

and the value of this (Nash,Nash) solution to each producer is:

$$\frac{p_0}{\alpha - 2} \cdot \left[\frac{\alpha}{\alpha - 2} p_0\right]^{-\alpha} \tag{7}$$

A first-entrant is a Stackelberg leader who maximizes:

$$x_1 \left[p_0 + x_1 + \frac{p_0 + x_1}{\alpha - 1}\right]^{-\alpha}$$

$$= x_1 \left[\frac{\alpha}{\alpha - 1} p_0 + \frac{\alpha}{\alpha - 1} x_1\right]^{-\alpha} = \left(\frac{\alpha}{\alpha - 1}\right)^{-\alpha} x_1 \left[p_0 + x_1\right]^{-\alpha} \tag{8}$$

The first-entrant's profit is proportional to that of a monopolist, so the first-entrant produces the same output as a monopolist. Maximizing (8) gives:

$$\left[p_0 + x_1\right]^{-\alpha} - \alpha x_1 \left[p_0 + x_1\right]^{-\alpha - 1} = 0 \tag{9}$$

$$\frac{x_1}{p_0 + x_1} = \frac{1}{\alpha} \tag{10}$$

$$x_1 = \frac{p_0}{\alpha - 1} \tag{11}$$

Then the optimal (Nash) response to this quantity, x_2 is:

$$x_2 = \frac{p_0 + x_1}{\alpha - 1} = \frac{\alpha p_0}{(\alpha - 1)^2} \tag{12}$$

Suppose that $\alpha = 3$. Then, from (11) and (12):

$$x_1 = \frac{1}{2} p_0 \text{ and } x_2 = \frac{3}{4} p_0 \tag{13}$$

We may notice that:

- There is late-entry advantage because the second entrant produces more than his Stackelberg leader, while both sellers obtain the same price.
- The combined outputs of the two producers under Stackelberg leadership equal $\frac{5}{4} p_0$ which is less than combined Nash outputs, $2 p_0$, although greater than the monopoly output, $\frac{1}{2} p_0$.

1.2 A Game of Attrition

The method of solving the game of attrition is the same as that shown in Bliss-Nalebuff (1984). We assume a solution of the form $T(r)$, where $T(r)$ is the time that an agent with discount rate r will wait before entering the market. To derive the form of $T(r)$ we employ the condition that a firm may chose to act as if it were any type, when it will find it optimal to be its own type. Thus let a firm of type $\overline{r}$ choose r to maximize:

$$V_S \int_r^\infty h(\rho)e^{-\overline{r}T(\rho)}d\rho + V_F H(r)e^{-\overline{r}T(r)} \tag{14}$$

where $H(r)$ is the cumulative probability distribution of r values, the probability that any particular value of r is less than or equal to the argument of $H(\cdot)$. Also, $h(r) = \frac{dH(r)}{dr}$.

The maximization of (14) requires:

$$-V_S h(r)e^{-\overline{r}T(r)} - V_F H(r)e^{-\overline{r}T(r)}\overline{r}\frac{dT}{dr} + V_F h(r)e^{-\overline{r}T(r)} = 0 \tag{15}$$

when $r = \overline{r}$. This implies:

$$-V_S h(r) - V_F H(r) r\frac{dT}{dr} + V_F h(r) = 0 \tag{16}$$

$$\frac{dT}{dr} = \frac{V_F - V_S}{V_F}\frac{h(r)}{rH(r)} \tag{17}$$

Integrating the non-linear differential equation (17) gives the entry queue for any particular duopoly case. This is the same as saying that the solution to (17) provides a mapping from the r values of the two firms to their entry times. Entry of both will then occur at the minimum of the two times, with the more impatient of the two firms entering at the smaller first-mover capacity level.

Remark 1 *Inspection of (17) shows that the entry queue depends only upon the ratio* $\frac{V_S}{V_F}$ *and the cumulative distribution function* $H(r)$*. For this reason the analysis is good for any model in which there is second-entrant advantage, for whatever reason. For instance, suppose that the second-entrant advantage has nothing to do with capacity levels, but arises because the second-entrant can design a product after inspecting the specification of the first-entrant's product. Then equation (17) applies and, given the ratio* $\frac{V_S}{V_F}$*, the entry queue can be determined.*

Example 1 *The r values are distributed uniformly on $[0, R]$. Then:*

$$H(r) = \frac{r}{R} \tag{18}$$

$$h(r) = \frac{1}{R} \tag{19}$$

$$\frac{dT}{dr} = \frac{V_F - V_S}{V_F}\frac{1}{r^2} \tag{20}$$

Integrating (20) gives:

$$T(r) = \frac{V_S - V_F}{V_F}\frac{1}{r} + a_0 \tag{21}$$

As we should expect, $T(0) = +\infty$; a firm which does not discount waits for an indefinite time for the other firm to move first. We must have $T(R) = 0$; a firm which knows that it has the highest rate of discount possible will enter at once. This condition fixes the constant of integration:

$$a_0 = -\frac{V_S - V_F}{V_F}\frac{1}{R} \tag{22}$$

$$T(r) = \frac{V_S - V_F}{V_F}\left[\frac{1}{r} - \frac{1}{R}\right] \tag{23}$$

1.3 The Triopoly Case

To see why triopoly is far more complicated than duoply, consider the following points:

- in the duopoly case both firms enter together, although with different capacity levels; for this reason the choice of optimal capacity level is that appropriate for duopolistic competition (whether as leader or follower), and is independent of the solution to the attrition game which gives a date at which entry will occur;
- in the triopoly case entry by one firm is not necessarily followed by immediate entry by another; for this reason the first-entrant enjoys a period of monopoly until another firm enters;
- the knowledge that first-entry is followed by a "honeymoon period" of monopoly dominance encourages earlier entry by the most impatient type of firm;
- if the capacity level at which the first-entrant comes in is different from what a simple Stackelberg calculation would indicate, this can affect the duration of the honeymoon period, and that effect has to be taken into account when computing optimal capacity for first-entry.

Another big difference which is made by having three firms can be explained simply in terms of a general model of sequential entry from which three pay-off levels V_F, V_S and V_T emerge, being respectively the pay-offs from first-entry, second-entry and third-entry. The sequence V_F and V_S must be weakly monotonic, because either $V_F \geq V_S$ or $V_F \leq V_S$. The same is not true of the sequence V_F, V_S and V_T.

With these points in mind, we analyse the duopoly example above, but now with three firms. This solution is for the straight Cournot-Nash game, taking no account of the effect on the honeymoon period noted above. A more precise analysis will follow later.

Now x_i $(i = 1, 2, 3)$ is the output of the *ith* producer, all producers have marginal production costs equal to zero and the inverse demand function is:

$$[p_0 + x_1 + x_2 + x_3]^{-3} \tag{24}$$

Let firm 3 maximize profit given x_1 and x_2:

$$x_3\,[p_0 + x_1 + x_2 + x_3]^{-3} \tag{25}$$

which gives:

$$[p_0 + x_1 + x_2 + x_3]^{-3} - 3x_3\,[p_0 + x_1 + x_2 + x_3]^{-4} = 0 \tag{26}$$

$$\frac{x_3}{p_0 + x_1 + x_2 + x_3} = \frac{1}{3} \tag{27}$$

$$x_3 = \frac{p_0 + x_1 + x_2}{2} \tag{28}$$

Equation (28) gives the reaction function of the third entrant to the capacity levels chosen by the first two entrants. Now, given x_1, the second entrant chooses x_2 to maximize:

$$x_2\left[p_0 + x_1 + x_2 + \frac{p_0 + x_1 + x_2}{2}\right]^{-3} = \left(\frac{3}{2}\right)^{-3} x_2\,[p_0 + x_1 + x_2]^{-3} \tag{29}$$

which gives:

$$[p_0 + x_1 + x_2]^{-3} - 3x_2\,[p_0 + x_1 + x_2]^{-4} = 0 \tag{30}$$

$$\frac{x_2}{p_0 + x_1 + x_2} = \frac{1}{3} \tag{31}$$

$$x_2 = \frac{p_0 + x_1}{2} \tag{32}$$

Using (28) and (32), the profit-maximizing decision of the first firm is to maximize:

$$x_1 \left[p_0 + x_1 + \frac{p_0 + x_1}{2} + \frac{p_0 + x_1 + \frac{p_0+x_1}{2}}{2} \right]^{-3} \tag{33}$$

That is equivalent to the maximization of:

$$x_1 \left[p_0 + x_1 + \frac{1}{2}(p_0 + x_1) + \frac{3}{4}(p_0 + x_1) \right]^{-3} = x_1 \left[\frac{9}{4}(p_0 + x_1) \right]^{-3} \tag{34}$$

That maximization requires:

$$\left[\frac{9}{4}(p_0 + x_1) \right]^{-3} - 3x_1 \left[\frac{9}{4}(p_0 + x_1) \right]^{-4} = 0 \tag{35}$$

From (35):

$$x_1 = 3p_0 \tag{36}$$

$$x_2 = 2p_0 \tag{37}$$

$$x_3 = 3p_0 \tag{38}$$

The remarkable result shown by (36) to (38) is that there is early or late entry advantage, but middle entry disadvantage. We can clarify what will happen as follows. There will be a race to enter first. Let any firm, called 1 without loss of generality, be the winner. Then firms 2 and 3 play a game of attrition exactly as the one shown above. So far all is clear. But now consider the following question. If firm 1 winning the entry race gives it the right to set capacity at any level it chooses, what is its optimal choice? That question is answered in the next section.

1.4 Optimal Capacity with a Honeymoon Period

Does firm 1 above choose $x_1 = 3p_0$ or another value? Its motive for deviating from $3p_0$, if it had one, would be to affect the game of attrition which its entry triggers so as to lengthen the period during which it enjoys a monopoly of the market. In the present case this does not happen. Why not? Recall from Remark 1 above that the entry queue depends only upon the ratio $\frac{V_S}{V_F}$. Equations (28) and (32) together show that given the choice of any value of x_1, the resulting responses for x_2 and x_3 are respectively$\frac{1}{2}(p_0 + x_1)$ and $\frac{3}{4}(p_0 + x_1)$. Therefore, regardless of the value of x_1 chosen, the ratio of the profits earned by the second and third entrants is constant. For this particular case, therefore, the honeymoon period is not an issue. It is there, and it adds to the profit associated with winning the entry race, but it does not induce earlier entry, because all firms try to enter first as fast as they can in any case. And the length of the honeymoon period cannot be affected by the first entrant, as has just been shown.

Example 2 *To show a different possibility, drop the assumption that all producers have zero marginal production costs, and give them all the same marginal production cost* $c > 0$. *Now firm* 3 *maximizes profit given* x_1 *and* x_2*:*

$$(x_3 - c)\left[p_0 + x_1 + x_2 + x_3\right]^{-3} \tag{39}$$

which gives:

$$\left[p_0 + x_1 + x_2 + x_3\right]^{-3} - 3\left(x_3 - c\right)\left[p_0 + x_1 + x_2 + x_3\right]^{-4} = 0 \tag{40}$$

$$\frac{x_3 - c}{p_0 + x_1 + x_2 + x_3} = \frac{1}{3} \tag{41}$$

$$x_3 = \frac{p_0 + x_1 + x_2 + 3c}{2} \tag{42}$$

Next, given x_1*, the second entrant chooses* x_2 *to maximize:*

$$(x_2 - c)\left[p_0 + x_1 + x_2 + \frac{p_0 + x_1 + x_2 + 3c}{2}\right]^{-3}$$

$$= \left(\frac{3}{2}\right)^{-3}(x_2 - c)\left[p_0 + x_1 + x_2 + c\right]^{-3} \tag{43}$$

which gives:

$$\left[p_0 + x_1 + x_2 + c\right]^{-3} - 3\left(x_2 - c\right)\left[p_0 + x_1 + x_2 + c\right]^{-4} = 0 \tag{44}$$

$$\frac{x_2 - c}{p_0 + x_1 + x_2 + c} = \frac{1}{3} \tag{45}$$

$$x_2 = \frac{p_0 + x_1 + 4c}{2} \tag{46}$$

Substituting (46) into (42) gives:

$$x_3 = \frac{3p_0 + 3x_1 + 10c}{4} \tag{47}$$

The ratio of the capacity levels for the second and third entrants is no longer independent of x_1. *As profit is capacity multiplied by price minus marginal cost, relative second and third entrant profit levels are affected by the first entrants choice of capacity.*

1.5 First-Entrant's Capacity Affects the Attrition Game

We have now created a case in which the first-entrant's capacity choice does affect the game of attrition which then ensues between the other two players. This extra level of complexity may represent here even more elaborate cases, with more than three firms, and general demand functions. Such cases may feature endogenous games of attrition, as we have here, but also cases in which the first entry is not the result of a dash to get in, but itself comes out of a game of attrition.

All cases, however complex, are solved in the same way; by backward solution. We work from the last entrant(s) back to the first, computing at each step the optimal capacity and attrition strategies of the firms, given earlier entry and attrition choices. We might now apply these principles to solve the case shown in Example 2. Even if we make things as simple as possible, by assuming $r-$values to be uniformly distributed as in Example 1 above, the computation of a precise solution is horribly complicated. Fortunately we do not require an exact solution. The basic insight is obtained once we have seen whether the consideration of the length of the honetmoon period causes the first-entrant to invest in more or less capacity than would be chosen if the length of the honeymoon period were to be independent of first entrant's capacity. The time before entry for a particular firm with discount rate r is:

$$T(r) = \left[\frac{V_S}{V_F} - 1\right]\left[\frac{1}{r} - \frac{1}{R}\right] \tag{48}$$

The game of attrition will end when the firm with the larger r enters. The cumulative distribution of the max of two values of r is:

$$H\left[r\right]^2 \tag{49}$$

and the density of this max is:

$$2H(r)h(r) \tag{50}$$

Thus the first entrant affects the distribution of waiting times for second and third entrants, in the direction of increasing waiting times only if he increases $\frac{V_S}{V_F}$. From (46) and (47) above,

$$\frac{V_S}{V_F} = \frac{1}{2}\frac{3p_0 + 3x_1 + 10c}{p_0 + x_1 + 4c} \tag{51}$$

Differentiating (51) with respect to x_1 gives:

$$\frac{d\frac{V_S}{V_F}}{dx_1} = \frac{1}{2}\frac{3\left(p_0 + x_1 + 4c\right) - \left(3p_0 + 3x_1 + 10c\right)}{\left(p_0 + x_1 + 4c\right)^2} = \frac{1}{2}\frac{22c}{\left(p_0 + x_1 + 4c\right)^2} > 0 \tag{52}$$

Proposition 3 *If the first entrant sets his capacity level higher than he would choose if the other entrants were to enter immediately in order, he will be compensated by a somewhat longer wait before others enter. For this reason, optimal capacity for the first entrant is higher than an analysis which ignores the variablilty of the honeymoon period would indicate. As both second and third entrants' optimal capacities increase with the first entrant's capacity, total industry capacity is larger when the endogeneity of the honeymoon period is taken into account. Obioulsy this particular conclusion depends upon the functional forms selected for the example.*

1.6 Conclusions

Although the IO literature makes it clear that a late entrant may enjoy an advantage, the case has not received much emphasis. In particular the point that late entrant advantage leads to games of attrition which define entry queues has not received the attention which it deserves. Solving for the structure of an entry queue is most straightforward when the games of attrition are independent of the capacity decisions made by arious firms. Yet all cases can be solved by the same method: backward solution from the decisions of the last entrants to that of the first.

References

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