0.1 Investment and Finance with Bankruptcy Risk

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Abstract

Much theory of the firm and financial economics ignores bankruptcy when modelling the investment decision. Bankruptcy is treated as a random disaster, like an earthquake. When bankruptcy risk is incorporated into a forward-looking investment decision some notable conclusions follow. In simple models the marginal product of capital and the marginal cost of extra debt are equal. With bankruptcy risk a wedge is created between these values. Even with a risk-neutral investor bankruptcy causes a mean-preserving spread of risk to discourage investment and to widen the wedge. Unlike the Modigliani-Miller case an optimal degree of leverage can be computed.

Key words: Investment, Bankruptcy, Risk, Leverage

JEL Classification D21 D81 G32

0.1.1 Introduction

The extensive literature on the economic theory of the firm pays surprisingly little attention to the issue of bankruptcy. The *New Palgrave Dictionary of Economics* has no entry under bankruptcy. The index refers the reader to corporate finance, where the text mainly discusses the Modigliani-Miller theorem (Modigliani 1958) and its various offshoots, most of which assume no bankruptcy. There are many papers on bankruptcy in the management journals, but the economics literature is thin, and often addresses questions tangential to what should be a central issue: how does the risk of bankruptcy affect investment and financing decisions? The recent upsurge of interest in banking has generated numerous papers on risk-taking by bankers, mostly with an emphasis on what the banks do, more than on the scale of their operations as such. Of course scale and scope are not unconnected. Banks expanded in part by embracing new operations, such as assembling junk bonds.

In the economics literature bankruptcy is often associated with the question of leverage (called gearing in the UK), measured by the ratio of debt to the capital of the firm. Altman (1968 and 1977) investigates the extent to which several financial ratios can predict bankruptcy using discriminant analysis. Many papers, including Aghion et als. (1992), take bankruptcy as a given outcome and discuss how it should be treated. There is an extensive legal literature concerned with bankruptcy. This reflects the fact that the law has to deal with complex situations. This includes unwinding what has been done by owners who foresee bankruptcy and are motivated to strip out assets and leave as little as possible for creditors. Unusual in analysing strategy to avoid bankruptcy is Lippman (1982). The objective is maximum probability of survival; an assumption more appropriate to biology than the firm.

Brearley et als. (2010) is a text book used widely in management and finance courses. It is mainly non-mathematical and offers no analysis of how bankruptcy risk influences the investment decision. Hart (1995), the indispensible exposition of the firm, contracts and incentives, is likewise silent on the particular question engaged in the present paper.The modelling of bankruptcy is always problematic. A simplistic view of bankruptcy would say that a firm goes bust when it cannot pay its debts. In fact many firms borrow to pay debts. For example a firm established to exploit a new oil field may borrow heavily, and without difficulty, while its cash flow, revenue minus debt service, is negative.

Mainly in the 1970s the Modigliani-Miller theorem was examined in depth and important extensions were exposed, together with its immovable limitations. From that work some anti-Modigliani-Miller results emerged, meaning that the modelling of an optimal financial structure was proposed. As Hirschleifer (1970, p. 264) puts it:

> "...even within complete capital markets, allowing for considerations such as taxes and bankruptcy penalties would presumably permit the determination of an optimal debt-equity mix for the firm."

Fama and Miller (1972) provides an extensive exposition of what has now become orthodox finance theory. Market actors have full information, including knowledge of the probability distributions of unobserved values. Then different finance policies imply various fragmentations of known probability distributions, all of which sum to the same market value. In this case bankruptcy influences how bad outcomes translate to the value of different fragmentations. A higher level of equity finance implies putting more of the cost of bankruptcy onto the equity fragment, which in turn raises the cost of equity finance. The idea here is that the owners of the firm incur no penalty for bankruptcy beyond the loss of residual profit implied by the bad outcome that causes bankruptcy.

In fact bankruptcy may carry additional penalties to the owners of the firm, such as loss of reputation. Kraus and Litzenberger (1973, p. 911) include the paradoxical claim that:

> "...bankruptcy penalties would not exist in a perfect market."

Fortunately these authors do analyse optimal finance with bankruptcy penalties and taxation. They do not include a variable investment rate which is a central concern of the present paper.

In three powerful papers Joseph Stiglitz (1969, 1972 and 1974) extended and hardened the Modigliani-Miller result. He developed the connection between finance theory and the Arrow-Debreu model of markets with uncertainty. The 1972 paper does everything at a highly general and abstract level. It has bankruptcy, choices of the firm that influence bankruptcy, and an optimal debt-equity ratio. Stiglitz notes that in general financial and real decisions cannot be separated. The argument to follow will replicate some of Stiglitz's findings. By operating at a less abstract level it will turn up some particular results that are potentially of great interest, and that a highly general approach misses.

In particular it will be shown that bankruptcy risk creates a wedge between the marginal product of capital and the marginal cost of borrowing. This wedge is present even when the cost of borrowing fully reflects the risk of default on loan payments, and it is present both in a multi-period dynamic programming model and in a simple two-period model where there is no cost to the borrower from bankruptcy. The wedge has further implications. In particular an increase in risk, defined as a mean-preserving spread of the random shocks to profitability discourages investment in what will be called the *Normal Case*.

0.1.2 When does a Firm Go Bankrupt?

Stiglitz (1972) provides an excellent discussion of the issues involved. In the simply two period models popular in the finance literature the problem does not present itself. In the second period the firm is wound up, is financially sound or not as the case may be, and further borrowing cannot help it. In the enduring-firm model examined below something must be included to prohibit Ponzi-scheme-style borrowing to keep an insolvent firm going indefinitely.

Another complication that can arise in a many-period model is encountered when borrowing by the issue of bonds is treated in the manner that has become established in the finance literature. Once it is accepted that the cost of borrowing to a firm varies with the probability of default on the loan, a firm in a many-period world may end up with a bundle of unlike debt obligations reflecting its situations in many periods in the past. Then computing whether bankruptcy will be unavoidable in any particular case becomes formidably complicated. This last problem is circumvented in the analysis below by assuming that the only form of fixed interest borrowing allowed is short-term one-period borrowing. In this case at the beginning of period t the firm is committed to pay a given monetary sum denoted D_t, which payment will extinguish all previous debt. Now the firm may borrow anew, in part to pay the sum D_t; so the issue of an anti-Ponzi borrowing constraint remains.

If the firm's cash-balance including its current profit is insufficient to pay D_t it may still have positive net-worth, and it makes no sense to assume that a firm with positive net worth is always bankrupt. On the other hand the net worth of a firm includes the value of all its capital, and it would be unrealistic to assume that the total of this can be called upon to avoid bankruptcy. There are two issues here. In the first place, if capital is valued at historical cost this will greatly exceed its break-up (or fire-sale) value. Secondly, even if capital really had a market value equal to its book value, it is unlikely that a firm could continue to trade borrowing fully against its capital. A simple assumption is required here, and we assume that the firm is bankrupt if its net worth is less than a given fraction of its capital, that fraction being a level $\beta < 1$. The parameter β also measures the proportion of value in the firm's capital in the event of bankruptcy, when the firm will be dissolved, and what value can be recovered will be paid to the firm's creditors.

This treatment of bankruptcy is surely far from perfect. In particular in a model where the firm can continue into future periods no static test for solvency can ever be fully convincing. A firm may go through bad times, yet retain potential for profit in the future that implies that it should not be wound up. In other words with perfect foresight, not assumed here, lenders might see that the firm can invest profitably in the future. It is possible that future investments will be profitable enough to make present financial difficulties unimportant. Myers (1977) develops the point that a firm's future investment opportunities, that may or may not be exercised, represent an option that like other options has market value. In principle the option value of the firm's future opportunities should be added to its net worth when solvency is decided. This point is not incorporated into the analysis to follow for the obvious reason that it makes the computations impossibly complicated.

0.1.3 The Model of the Firm

The firm is characterized by a profit function that depends only upon its capital stock K and a variable Π that represents the variability of profitability due to random effects:

$$\Pi_t F\left(K_t\right) \tag{1}$$

Here $F\left(\cdot\right)$ is an increasing concave function of capital K. The value Π is determined by a stationary stochastic process:

$$\Pi_{t+1} = \epsilon_t \Pi_t \tag{2}$$

where the variable epsilon ϵ_t is a non-negative i.i.d. random variable. Here the left-hand tail of the distribution of ϵ_t represents what may be called, with apologies to Shakespeare, the heart-ache and the thousand natural shocks that firms are heir to. The firm finances its capital accumulation by a combination of equity finance and fixed-interest debt. Equity takes the form of allocating at time t a share θ_t of the firm's profit to outsiders.

At time t the firm's earlier borrowing requires it to pay D_t. The number D_t may be negative to include the case of a cash-rich firm. The firm's net profit at t is:

$$\Pi_t F(K_t) - D_t \tag{3}$$

To keep the argument simple it is assumed that the firm does not distribute profit as dividends. The firm's nominal net worth at t is:

$$\Pi_{t-1} F(K_{t-1}) - D_t + K_{t-1} \tag{4}$$

The expression (4) has the firm's capital evaluated at historic cost, although we have seen that the break-up value of the capital will be less than K_{t-1}. This point explains the use of the term nominal net worth.

0.1.4 The Bankruptcy Test

At the beginning of Period t the firm has capital K_t, has to pay D_t from debt obligation undertaken in the previous period. Then ϵ_t is revealed and the firm makes profit:

$$\epsilon_t \Pi_{t-1} F(K_t) \tag{5}$$

The firm is bankrupt if:

$$D_t - \epsilon_t \Pi_{t-1} F(K_t) > \beta K_t \tag{6}$$

Thus the firm is bankrupt if the cash that it has to pay out in period t, not including new borrowing, exceeds the break-up value of its capital.

Now the probability of bankruptcy for period t is:

$$Z\left[\frac{D_t - \beta K_t}{\Pi_{t-1} F(K_t)}\right] \tag{7}$$

where $Z[x]$ is the cumulative probability distribution of ϵ, the probability that ϵ will be less than or equal to x.

Differentiating (7) with respect to K_{t+1} gives:

$$Z_1\left[\frac{D_t - \beta K_t}{\Pi_{t-1} F(K_t)}\right] \frac{\left(\frac{dD_t}{dK_t} - \beta\right)\Pi_{t-1} F(K_t) - (D_t - \beta K_t)\Pi_{t-1} F_1(K_t)}{\Pi_{t-1}^2 F(K_t)^2}$$

$$= Z_1\left[\frac{D_t - \beta K_t}{\Pi_{t-1} F(K_t)}\right] \left\{\frac{\frac{dD_t}{dK_t} - \beta - (D_t - \beta K_t)\frac{F_1(K_t)}{F(K_t)}}{\Pi_{t-1} F(K_t)}\right\} \tag{8}$$

So for the probability of bankruptcy to increase with K_t requires:

$$\frac{dD_t}{dK_t} > \beta + (D_t - \beta K_t)\frac{F_1(K_t)}{F(K_t)} > 0 \tag{9}$$

If the firm is not bankrupt it will pay D_t and continue to trade. If it needs to borrow to pay D_t it can do so at its cost of borrowing to be detailed below. Because a non-bankrupt firm can borrow to pay its debts it does not need to break up capital. If the firm is bankrupt its creditors will receive:

$$\epsilon_t \Pi_{t-1} F(K_t) + \beta K_t \tag{10}$$

From (2) it will be seen that the value (4) is less than D_t. Bankruptcy involves partial default on debt. As (4) is non-negative, the creditors always get something.

0.1.5 The Firm's Budget Constraint

In period t the firm will be due to pay D_t, it will own capital K_t, and it will have made profit $\Pi_t F(K_t)$. Then the firm's demand for loanable funds L_t will satisfy:

$$L_t = (K_{t+1} - K_t) + D_t - \Pi_t F(K_t) - (\theta_{t+1} - \theta_t) V^t \tag{11}$$

where V^t is the value of the firm at t, and θ_t is the proportion at t of the firm owned by outsiders; that is holders of the firm's equity. To borrow the sum L_t the firm will have to promise its bankers to pay the sum D_{t+1}, where the expected value of this promise, given that the firm may not be able to pay in full, denoted EP_t, is equal to L_t.

The expected value of the amount that the firm's bankers will receive in all cases in period $t+1$ satisfies:

$$EP_t = (1+r)L_t = \left\{1 - Z\left[\frac{D_{t+1} - \beta K_{t+1}}{\Pi_t F(K_{t+1})}\right]\right\} D_{t+1}$$

$$+ \int_0^{\frac{D_{t+1} - \beta K_{t+1}}{\Pi_t F(K_{t+1})}} \left[\epsilon_{t+1} \Pi_t F(K_{t+1}) + \beta K_{t+1}\right] z\left[\epsilon_{t+1}\right] d\epsilon_{t+1} \tag{12}$$

where E stands for expected value, and $z[\epsilon]$ is the probability density of ϵ. If the probability of bankruptcy is zero (12) becomes:

$$(1+r)L_t = D_{t+1} \tag{13}$$

which is the budget constraint of the standard capital theory that ignores bankruptcy. As the integrand in (12) is less than D_{t+1} almost everywhere it follows from (13) that:

$$(1+r)L_t < D_{t+1} \tag{14}$$

As is expected, a borrower in this model must commit to pay more in period $t+1$ than would be required were there no bankruptcy risk.

0.1.6 The Objective Function

At any time t the firm aims to maximize its share of the value V^t, which is the expected discounted value of its profits going forward:

$$V^t = E\sum_{1}^{\infty}\left[\Pi_t F\left(K_t\right) - D_t\right]\delta^{t-1} \tag{15}$$

where E indicates mathematical expectation, $\delta < 1$ is a constant discount factor, and the summation (15) is terminated should the firm go bankrupt. That means that should the firm become bankrupt it is shut down and any debt in excess of its assets is written off. In other words the firm's owners have *limited liability.*

It follows that the objective function of the firm's owner(s) is:

$$E\sum_{1}^{\infty}\left(1-\theta_t\right)\left[\Pi_t F\left(K_t\right) - D_t\right]\delta^{t-1} \tag{16}$$

so long as the firm is not bankrupt.

0.1.7 Dynamic Programming

Crucial to the understanding of the investment decision is the realization that the choice of capital level K_{t+1} affects the probability of bankruptcy in period $t+1$. This may seem obvious. Over-ambitious investment commonly contributes to bankruptcy. Notice however that the present model includes an extremely unrealistic assumption. If $K_{t+1} - K_t$ in (11) is negative the firm receives the full value of its capital reduction. In other words, given one period to adjust capital is completely fungible. A high investment financed by borrowing can be unwound next period, and the only unavoidable penalty is one period's interest on the debt. This is plainly unrealistic. If a new restaurant opens, and fails to attract sufficient custom, its owner will certainly not be able to recover all the costs of establishing the restaurant. The model could be rewritten to penalize reductions in capital, but as it stands it is well-designed to facilitate comparison with standard neo-classical models of the firm in which capital is fungible and the gross marginal product of capital is $1 + r$.

Our problem can be attacked using dynamic programming. Let the maximized value of (16) when the firm starts with productivity Π, capital K, debt D, and outsiders' share θ, be $W(\Pi, D, K, \theta)$. How does the value θ enter into the function $W(\Pi, D, K, \theta)$? The temptation might be to write:

$$W(\Pi, D, K, \theta) = (1 - \theta)V \qquad 17$$

where V is defined by equation (16) according to the values of Π, D, and K applying at the time. That would be correct if θ were constant forever from the point in time at which the function $W(\Pi, D, K, \theta)$ is defined. In general however, θ will change over time. If, for example, the distribution of ϵ is such that its mean is far higher than 1, then K will grow on average over time, if only because bankruptcy risk for a given level of K will decline. Part of the increase in K will be equity-financed, when θ will rise over time. This is the same as saying that when a firm grows it is normal for the original owner's equity share to be diluted.

Now the Bellman equation is:

$$\begin{aligned} W(\Pi_t, D_t, K_t, \theta_t) &= (1 - \theta_t)\left\{\Pi_t F(K_t) - D_t\right\} \\ &+\delta E\left[\left\{1 - Z\left[\frac{D_t - \beta K_{t+1}}{\Pi_t F(K_{t+1})}\right]\right\} W[\epsilon_t \Pi_t, D_{t+1}, K_{t+1}, \theta_{t+1}]\right] \end{aligned} \qquad (18)$$

where E indicates mathematical expectation, and D_{t+1} is obtained from the borrowing constraint (12).

Just the interest on debt is sufficient to explain why investment is risky in our model. Imagine that an investment bank approaches our firm. It explains that the firm's gross marginal product of capital is far in excess of $1 + r$, and it offers a large loan to purchase capital and to correct this divergence. Would our firm be wise to accept this offer? If it does so next period may see an awful realization of the random variable ϵ, this variable taking an exceptionally low value. It is true that the firm can unwind its debt at no cost, but with drastically reduced profitability the interest charge by itself may push it into bankruptcy, and this is especially likely if the firm starts with a high level of debt relative to its capital. There is a lesson here for everyone. *It is seldom prudent to borrow as much as the capital market will lend.*

Now K_{t+1} is chosen to maximize the right-hand side of (18). What this implies depends upon how capital accumulation is financed. Here we may take advantage of the Viner envelope theorem to argue that for the marginal addition to K_{t+1}each method of finance (equity or debt) must be equally attractive. Then assuming that only debt finance is used at the margin involves no loss of generality. Then the maximization of the right-hand side of (18) requires:

$$-Z_1\left[\frac{D_{t+1}-\beta K_{t+1}}{\Pi_t F\left(K_{t+1}\right)}\right]\frac{\left(\frac{dD_{t+1}}{dK_{t+1}}-\beta\right)\Pi_t F\left(K_{t+1}\right)-\left(D_{t+1}-\beta K_{t+1}\right)\Pi_t F_1\left(K_{t+1}\right)}{\left[\Pi_t F\left(K_{t+1}\right)\right]^2}$$

$$+\left\{1-Z\left[\frac{D_t-\beta K_{t+1}}{\Pi_t F\left(K_{t+1}\right)}\right]\right\}\frac{E\left\{W_2\frac{dD_{t+1}}{dK_{t+1}}+W_3\right\}}{EW[\cdot]}=0 \qquad (19)$$

where $W[\cdot]$ is $W[\epsilon_t\Pi_t, D_{t+1}, K_{t+1}, \theta_{t+1}]$ and subscripts denote differentiation. Notice that when (18) is differentiated with respect to K_{t+1} it is assumed that D_{t+1} varies at the rate required for (12) to be satisfied.

If the firm has no debt it cannot go bankrupt. Then the first term between curly brackets in (19) is 1. However even with zero debt initially a small increase in K will be debt financed and it may be the case that $Z_1(0)$ is positive, meaning that an increase in debt from zero increases the probability of bankruptcy.

We could differentiate again with respect to K_{t+1}, varying D_{t+1} in line with the variation in K_{t+1}. This would give us a second-order condition for the maximum. The computation is messy and probably ambiguous. So we just assume a regular maximum with the second-order term strictly negative. The import of this assumption will be seen on the proof of Theorem 1 below.

As the Bellman equation is an identity in K_t we can differentiate it with respect to that variable. Any consequent effects of a small change in K_t on D_{t+1} and K_{t+1} can be ignored on account of the Viner envelope property. Therefore:

$$W_3(\Pi_t, D_t, K_t, \theta_t) = (1 - \theta_t)\, \Pi_t F_1\,(K_t) \tag{20}$$

A naive line of thought would conclude that the marginal valuation of debt should always equal the marginal valuation of extra capital. This is the same as the familiar property of simple neoclassical models where the marginal product of capital is equal to the rate of interest. That is not the case with the present model because of bankruptcy risk. Borrowing carries the additional risk that hard times will arrive (ϵ exceptionally small) and debt service will prove to be impossible.

0.1.8 The Optimal Choice of θ_{t+1}

Now given θ_t the value of θ_{t+1} must be chosen to maximize the right-hand side of the Bellman equation (18). A variation in θ_{t+1} must be associated with some other change so that the budget constraint (11) can be satisfied. One possibility, followed here, is to have D_{t+2} vary with θ_{t+1}. This exercise tests directly the optimal finance aspect of the model, as we are looking at the substitution of equity for debt. Then from (11):

$$\frac{dK_{t+1}}{d\theta_{t+1}} = -V^t \tag{21}$$

We can take advantage of the fact that the Bellman equation (18) is an identity in θ_t to obtain:

$$W_4^t(\Pi_t, D_t, K_t, \theta_t) = -\{\Pi_t F(K_t) - rD_t\} \tag{22}$$

Other terms involving changes in K_{t+1} and D_{t+1} vanish because of the Viner envelope property. The equation (22) does not seem to lead anywhere.

0.1.9 Increased Risk, Investment, and the Wedge

An increase in risk could mean many things and could be modelled in more than one way. Here we follow the elegant method pioneered by Rothschild and Stiglitz (1970). These authors used the notion of a *mean-preserving spread* (an MPS) in the distribution of outcomes. An MPS implies more risk unambiguously, and an agent's response to an MPS can be used to define risk-aversion. In this section we look at our firm's response to an MPS affecting the random variable ϵ. At first sight it appears that changes in risk should not affect our firm, as it maximizes expected discounted profit, and would seem to be risk-neutral. We will see that such is not the case, and the explanation once again is bankruptcy-risk.

We focus on a case we call *the normal case*. This is illustrated by Figure 1. The dark curve shows the original density function of the random variable ϵ. The lighter curve is the density of the same variable following an MPS of its distribution. The point on the horizontal axis marked B is the bankruptcy point. It is the limit point of the values of ϵ consistent with solvency. For any value of ϵ lower than OB the firm is bankrupt. What is special about the case illustrated in Figure 1 is not the disposition of the two density curves in relation to each other. This is an inevitable implication of the MPS specification. What is crucial is the location of the point B. In the figure this lies well to the left in the area where the MPS has increased the probability density. There is no absolute requirement that such should be the case. It is reasonable however to assume that the majority of firms will be operated in such a way that the probability of early bankruptcy is low, and that justifies the title given to the normal case.

<u>Theorem 1</u>: *In the Normal Case, and assuming that the second order condition noted above (p.7) to be satisfied, a mean-preserving spread of the Distribution of the values ϵ lowers investment and increases the wedge between the marginal product of capital and the marginal cost of debt.*

<u>Proof</u>: Rearrange equation (19) as follows:

$$E\left[\frac{W_2\frac{dD_{t+1}}{dK_{t+1}}+W_3}{W}\right]=$$

$$\frac{Z_1\left[\frac{D_{t+1}-\beta K_{t+1}}{\Pi_t F(K_{t+1})}\right]}{1-Z\left[\frac{D_t-\beta K_{t+1}}{\Pi_t F(K_{t+1})}\right]}\frac{\left(\frac{dD_{t+1}}{dK_{t+1}}-\beta\right)\Pi_t F\left(K_{t+1}\right)-\left(D_{t+1}-\beta K_{t+1}\right)\Pi_t F_1\left(K_{t+1}\right)}{\left[\Pi_t F\left(K_{t+1}\right)\right]^2} \tag{23}$$

Now let there be an MPS in the distribution represented by $Z(\cdot)$ and that this is the normal case. Consider for a start the first term between the square brackets on the right-hand side of (23). The value Z_1 will rise and the value of $1-Z$ will fall, because bankruptcy becomes more probable following an MPS. In all then the first term between square brackets on the right-hand side will increase. Now an inspection of (23) shows that for this equation to be satisfied either the left-hand side of the equation, which is the wedge in percentage form, must increase; or K_{t+1} must fall to lower the final term in the equation. In assuming that K_{t+1} must fall we are relying on the second-order condition being satisfied mentioned in the theorem. When K_{t+1} falls of the first term of (23) is also affected, but this is immaterial because it is a reaction to the MPS and the consequential change in K_{t+1}. Finally note that if and only if the value of K_{t+1} alters, together with the value of D_{t+1}, can the size of the wedge change. Therefore both K_{t+1} and the wedge must change, as stated by the theorem.□

The import of Theorem 1 deserves emphasis. Our firm maximizes expected profit. Given the level of investment, an MPS in the values of ϵ does not affect expected profit. In that sense our firm is risk-neutral. Yet it does not behave as a risk-neutral agent. Greater risk discourages investment, as it would with an agent maximizing a concave utility function. The reason again is bankruptcy risk.

0.1.10 Leverage and the Incentive to Invest

As can be seen from inspection of equation (7) above, the larger is a firm's debt in relation to its capital, which is to say the higher is a firm's leverage, the more vulnerable is that firm to bankruptcy. How does this translate to the investment decision? The answer to this question is surprisingly complicated. To see why compare two firms A and B, alike in every respect, except that B has higher debt than A. Equation (23) above is satisfied by each of our firms. Firm B has a higher probability of bankruptcy for any level of K_{t+1}. This moves its bankruptcy point up the density function of ϵ towards its mean, and this will normally imply a higher probability density, or at least not a lower density. In all if K_{t+1} is the same in the two cases the first term on the right-hand side of (23) is higher for B than for A. Then it would be convenient if the argument could proceed as with Theorem 1 to show that K_{t+1} must fall and the wedge widen.

Now inspection of equation (23) shows that even for a given value of K_{t+1} the higher level of debt with Firm B is reflected in the final term on the right-hand side of that equation. Given the value of K_{t+1}, higher debt lowers that term. This point throws everything into confusion. What now matters is which effect is more powerful: the increase of the first term on the right-hand side of (23), or the decrease of the final term. This finding is not as counter-intuitive as may at first appear. Recall that the bankruptcy test is provided by the debt to cash-flow ratio, which must not exceed β. The marginal effect on that ratio of a little extra investment is given by the expression:

$$\frac{F\left(K_{t+1}\right)-D_{t+1}F_1\left(K_{t+1}\right)}{F\left(K_{t+1}\right)^2} \tag{24}$$

which is smaller the higher is debt. In other words, while the test ratio is higher the larger is debt or leverage, the marginal effect on the ratio of extra debt-financed investment is smaller the larger is debt.

0.1.11 The Growing Firm

Our firm does not grow smoothly like a space vehicle following a fixed conic-section path. Rather it is buffeted along its route by serially uncorrelated shocks, and its optimal path has to be repeatedly re-calculated. Notwithstanding this point, if the mean of ϵ is well in excess of 1 the firm will grow on average. In any case even if the firm shrinks, the question of how the firm should adjust its investment and financing policies as it does so, the subject of this section, will still arise. As K and D and θ can all be adjusted as required, the issue is how these variables should respond to new realisations of Π, which is to say how, given the previous level of Π, how these variables should respond to a new realisation of ϵ.

Theorem 2: *In the normal case, and assuming again a regular second-order condition, if Π_t increases over time then K_t and firm finance, whether via debt or equity, will also increase over time. Although the rise in Π_t moderates the risk of bankruptcy, the extra investment and debt will offset this effect. Should Π_t decrease in value the same results apply with the expected change of sign. For a growing firm the wedge between the marginal product of capital and the marginal cost of debt increases over time.*

Proof: As equation (23) holds throughout, we can see the effect of a change in Π_t through that equation. The argument proceeds much as the proof of Theorem 1. Assume that the value of Π_t increases. In the normal case, so long as K_{t+1} and D_{t+1} are constant, the first term inside the square brackets on the right-hand side of (23) will decrease when Π_t increases. The second term inside the square brackets on the right-hand side of (23) will also fall as Π_t rises. How is the equality in equation (23) to be preserved in this case? As in Theorem 1 this can only be achieved by an increase in K_{t+1} and a widening of the wedge between the marginal product of capital and the marginal cost of borrowing.□

0.1.12 A Simple Computable Model

In our analysis above we have employed dynamic programming to obtain several useful results. The reader may notice that we have not solved the model in the sense of extracting the Bellman valuation function $W(\Pi, D, K, \theta)$. It did not prove necessary to do that. This is fortunate, because solving for that function, with its four arguments, would be extremely difficult, even if simple functional forms were assumed. Our results depend upon the essentially intertemporal character of the model. When the firm invests or takes on debt it is committed to those levels until the start of the next period, and this is why it faces an unavoidable risk of bankruptcy. To capture some of these features it is possible to work with a model in which everything happens within one period, and this is done in the present section. To make the model work we have to sequence events within the single period, so that we have in effect a two-period model. This is done by having the firm take its investment and financing decisions before it knows the value of the random variable ϵ. Once the value of ϵ is revealed the firm is then wound up. If it is bankrupt according to the difference between debt and value it is counted as worth nothing. Otherwise debt is paid off, capital is sold, and the residual value is divided between insider and outsiders.

The reader may object that this model ignores the continuing value of the firm as a going concern looking forward. The firm owns its profit function and has achieved from random effects a particular level of profitability Π. These have market value. However they are independent of the firm's decisions and can be ignored for that reason when we analyse choice within one or two periods.

The firm has the same profit function as above and the variable Π_t is determined in the same way. For any period under consideration we can absorb the current value of Π into the profit function by normalizing it to 1. For this reason the firm's profit once it becomes known will be:

$$\epsilon_t F(K_t) \tag{25}$$

The firm has no capital or debt initially. It must decide on its investment level and how to finance it before it knows the value of ϵ_t. This choice is subject to the budget constraint:

$$(1-\theta_t) K_t = L_t \tag{26}$$

where again θ_t is the proportion of investment that is financed by share issues. When the firm is wound up its value is just its profit (25), minus the debt repayment to which it has commited itself. Notice that we are not assuming here that the firm recovers its capital stock at the end of the period. So K_t is more investment than a level of capital.

Then (26) says that the firm must borrow that part of its investment that is not financed by the sale of equity to outsiders. To obtain L_t the firm must commit to pay D_t at the end of the period, after ϵ_t has been revealed. Should ϵ_t take a low value the firm may be unable to pay in full. Then the lender may receive part payment, and the level of D_t will take this into account. Now there is no explicit bankruptcy as the firm is wound up at the end of the period in any case. The probability that the firm will be unable to pay D_t in full, or will only just be able to pay is:

$$Z\left[\frac{D_t}{F(K_t)}\right] \tag{27}$$

The equivalent of equation (12) for this model is:

$$(1+r)(1-\theta_t)K_t = \left\{1 - Z\left[\frac{D_t}{F(K_t)}\right]\right\}D_t + F(K_t)\int_0^{\frac{D_t}{F(K_t)}} \epsilon_t z(\epsilon_t)\,d\epsilon_t \tag{28}$$

When the firm can repay D_t the lender is happy. When the firm can pay less than D_t the equity holders get nothing. The level of θ_t affects L_t through equation (26), but this is already incorporated in (28).

Another way of writing equation (28) is useful for the following discussion:

$$(1+r)(1-\theta_t)K_t = D_t\int_{\frac{D_t}{F(K_t)}}^{\infty} z(\epsilon_t)\,d\epsilon_t + F(K_t)\int_0^{\frac{D_t}{F(K_t)}} \epsilon_t z(\epsilon_t)\,d\epsilon_t \tag{29}$$

Inspection of equation (29) reveals a fascinating point. With K_t and θ_t and hence the left-hand side of the equation fixed, consider the implication of increasing D_t. This has two consequences. The value $\frac{D_t}{F(K_t)}$ increases. This by itself lowers the right-hand side of (29), because less of the range of integration covers the full repayment D_t, and more of it covers under-payment. A second consequence of increasing D_t arises because D_t multiplies the first integral in (31). When debt is repaid more is repaid. This increases the right-hand side of (31). *For this reason there is no guarantee that equation (31) given K_t and θ_t will have a unique solution for D_t.*

In fact this last conclusion is fairly familiar. The size of D_t is equivalent to setting the rate of interest to be charged for a loan where there is a risk that the borrower may default. What rate is fair and appropriate? Obviously the higher the risk of default the higher is the fair rate of interest. Yet the higher the rate of interest the higher is the probability of default. Then two or more different interest rates may be equally fair.

The owner of the firm chooses K_t and θ_t to maximize:

$$(1-\theta_t)\int_{\frac{D_t}{F(K_t)}}^{\infty}\left[\epsilon_t F\left(K_t\right)-D_t\right]z(\epsilon_t)d\epsilon_t$$

$$=(1-\theta_t)\left[E^{\frac{D_t}{F(K_t)}}\left(\epsilon_t\right)F\left(K_t\right)-D_t\left\{1-Z\left[\frac{D_t}{F\left(K_t\right)}\right]\right\}\right] \tag{30}$$

where D_t and K_t satisfy (29) above, and $E^{\frac{D_t}{F(K_t)}}\left(\epsilon_t\right)$ is the expected value of ϵ_t conditional on $\epsilon_t \geq \frac{D_t}{F(K_t)}$.

Then the maximization of (31) by choice of K_t requires:

$$(1-\theta_t)\left[E^{\frac{D_t}{F(K_t)}}\left(\epsilon_t\right)F_1\left(K_t\right)-\frac{dD_t}{dK_t}\left\{1-Z\left[\frac{D_t}{F\left(K_t\right)}\right]\right\}\right]$$

$$-(1-\theta_t)D_tZ_1\left[\frac{D_t}{F\left(K_t\right)}\right]\frac{d\frac{D_t}{F(K_t)}}{dK_t}=0 \tag{31}$$

When the integral in (30) is differentiated with respect to the lower limit of the integral the result is zero.

Equation (31) confirms that in this simple model, just as with the dynamic programming model above, we have a wedge between the expected value of the marginal product of capital and the marginal cost of borrowing provided that $\frac{d\frac{D_t}{F(K_t)}}{dK_t}$ is positive.

Lemma 1: $\frac{d\frac{D_t}{F(K_t)}}{dK_t} > 0$.

Proof: Suppose that capital is available to the firm regardless of scale or risk at a constant gross interest charge equal to $1+r$. Then:

$$\frac{D_t}{F(K_t)}=\frac{(1+r)K_t}{F(K_t)} \tag{32}$$

Then:

$$\frac{d\frac{D_t}{F(K_t)}}{dK_t}=(1+r)\frac{F(K_t)-K_tF_1(K_t)}{F(K_t)^2} \tag{33}$$

For $F(K_t)$ concave (33) must be positive. Therefore the ratio of debt to $F(K_t)$ increases with K_t when the cost of borrowing is independent of K_t. It is evident then that if the cost of borrowing increases with K_t, as it will when bankruptcy risk is taken into account, that D_t will increase even more rapidly with K_t. That ensures that the ratio of debt to $F(K_t)$ will increase in general as required.□

With bankruptcy risk involved it is no surprise that, as Hirschleifer indicated, an optimal value for θ_t can be computed. The Modigliani-Miller result does not hold. As we have revealed a wedge between the expected value of the marginal product of capital and the marginal cost of borrowing, just as with the dynamic programming model, it is straightforward to replicate the results obtained in that case. Thus we can show again, using an argument exactly similar to that of Theorem 1, that a mean-preserving spread of the variable ϵ_t will discourage investment.

0.1.13 Specific Functional Forms

If we introduce specific functional forms we can simplify equations (29) and (30). So let:

$$F\left(K_t\right) = \sqrt{K_t} \tag{34}$$

$r = 0.1$, and assume that ϵ_t is uniformly distributed on $[0.1, 2]$. Now:

$$Z(x) = \frac{x - 0.1}{1.9} \tag{35}$$

for $0.1 \leq x \leq 2$.

We retain the standard condition:

$$L_t = (1 - \theta_t)K_t \tag{36}$$

The probability of survival for our firm is:

$$1 - \frac{\frac{D_t}{\sqrt{K_t}} - 0.1}{1.9} = 1 - \frac{D_t - 0.1\sqrt{K_t}}{1.9\sqrt{K_t}} = \frac{1.8\sqrt{K_t} - D_t}{1.9\sqrt{K_t}} \tag{37}$$

Then rearranging (30) and substituting for our assumed forms and values we obtain:

$$1.1\,(1 - \theta_t)\,K_t = \frac{1.8\sqrt{K_t} - D_t}{1.9\sqrt{K_t}} D_t + \sqrt{K_t} \int_{0.1}^{\frac{D_t}{\sqrt{K_t}}} \epsilon_t z\,(\epsilon_t)\,d\epsilon_t \tag{38}$$

Or, equivalently:

$$1.1\,(1 - \theta_t)\,K_t = D_t \int_{\frac{D_t}{\sqrt{K_t}}}^{2} z\,(\epsilon_t)\,d\epsilon_t + \sqrt{K_t} \int_{0.1}^{\frac{D_t}{\sqrt{K_t}}} \epsilon_t z\,(\epsilon_t)\,d\epsilon_t \tag{39}$$

Simplifying (39) gives:

$$(1 - \theta_t) = 0.478 \frac{D_t}{K_t}(2 - \frac{D_t}{\sqrt{K_t}}) + 0.239 \frac{1}{\sqrt{K_t}}(\frac{D_t^2}{K_t} - 0.01) \tag{40}$$

Similarly, embodying our simplifications, the maximand takes the form:

$$.526(1 - \theta_t)\left[2\sqrt{K_t} + \frac{1}{2}\frac{D_t^2}{\sqrt{K_t}} - 2D_t\right] \tag{41}$$

To maximize (41) subject to (40) we need the help of the computer. This gives us the following solution:

$$K_t = 2.605 \quad D_t = 1.436 \quad \theta_t = 0.592 \tag{42}$$

Given these values we can compute the probabilty that our firm does not go bankrupt as only 48 per cent.

This argument may provoke the response that the Modigliani-Miller theorem applies to a firm where the original owners retain part ownership of the firm, even if all new investment is equity financed. It is the cost of capital to the original owners by debt or equity that is compared. In the present model all the firm's capital is debt- or equity-financed. While this is correct there is residual profitability for the owner(s) of the firm, which is the share of profit not assigned to outsiders less the cost of debt finance. It is certainly obvious that with $\theta_t = 1$, 100 per cent equity finance, the maximand (31) takes the value zero for all K_t. Granted that, it remains striking that we do not even have the Modigliani-Miller property close to the maximizing values of K_t and θ_t, which are interior to the space of choice variables.

We have seen that the probability of bankruptcy for our firm is 52 per cent; a huge risk. This is unsurprising when account is taken of the riskiness implied by the uniform distribution (36). The average growth of profitability is only 5 per cent; against which must be set the huge downside risk, with a 25 per cent probability of profit falling at least 42 per cent. Given this it is surprising at first that the firm does not lay off more risk by increasing the level of equity participation in its investment. This would enable it to reduce D_t and to lower the probability of bankruptcy. In this simple model however there is no penalty for bankruptcy beyond the fact that the firm collects no revenue in the bankrupt state. Such considerations as the inability to trade in the future, or loss of reputation, do not arise. For this reason the firm is willing to accept huge risks for the chance to increase expected profit.

The computations just completed only apply to the simple computable model, and to one special case. There is every reason to believe nonetheless that important features of the special example are quite general. Notable among these is the emergence of an optimal level of leverage, a maximizing value for θ_t.

0.1.14 Conclusion

Adding the risk of bankruptcy to a simple model of the firm, with the assumption that the firm is aware of the risk, and takes it into account, leads to new and interesting conclusions. Additional capital investment does not only lead to added cash-flow. It also increases the risk that the firm will be bankrupted should a large negative shock to profit arrive next period. This additional consideration creates a wedge between the marginal product of capital and the marginal cost of borrowing. These two values are equal in standard neoclassical theory but are unequal with bankruptcy risk whenever the firm owes debt. It could be argued that when additional bankruptcy risk is added to the cost of borrowing the classical equality between the marginal product of capital and the cost of borrowing will be seen to apply after all. That point does not subtract from the importance of taking bankruptcy risk into account.

A mean-preserving spread of risk in the case of greatest interest widens the wedge and lowers the level of investment. It might be expected that the gap would be larger and investment lower the higher is the firm's leverage, the ratio of its debt to its capital. Surprisingly this is hard to prove, as has been demonstrated. In the case of a growing firm investment increases with profitability and the wedge rises over time. With bankruptcy risk the Modigliani-Miller result that leverage makes no difference no longer applies, and an optimal level of leverage can be computed. This has been illustrated by a special example from a simple computable model. The same exercise leaves no doubt that the result will be replicated in similar cases, however difficult they may be to solve for explicit values.

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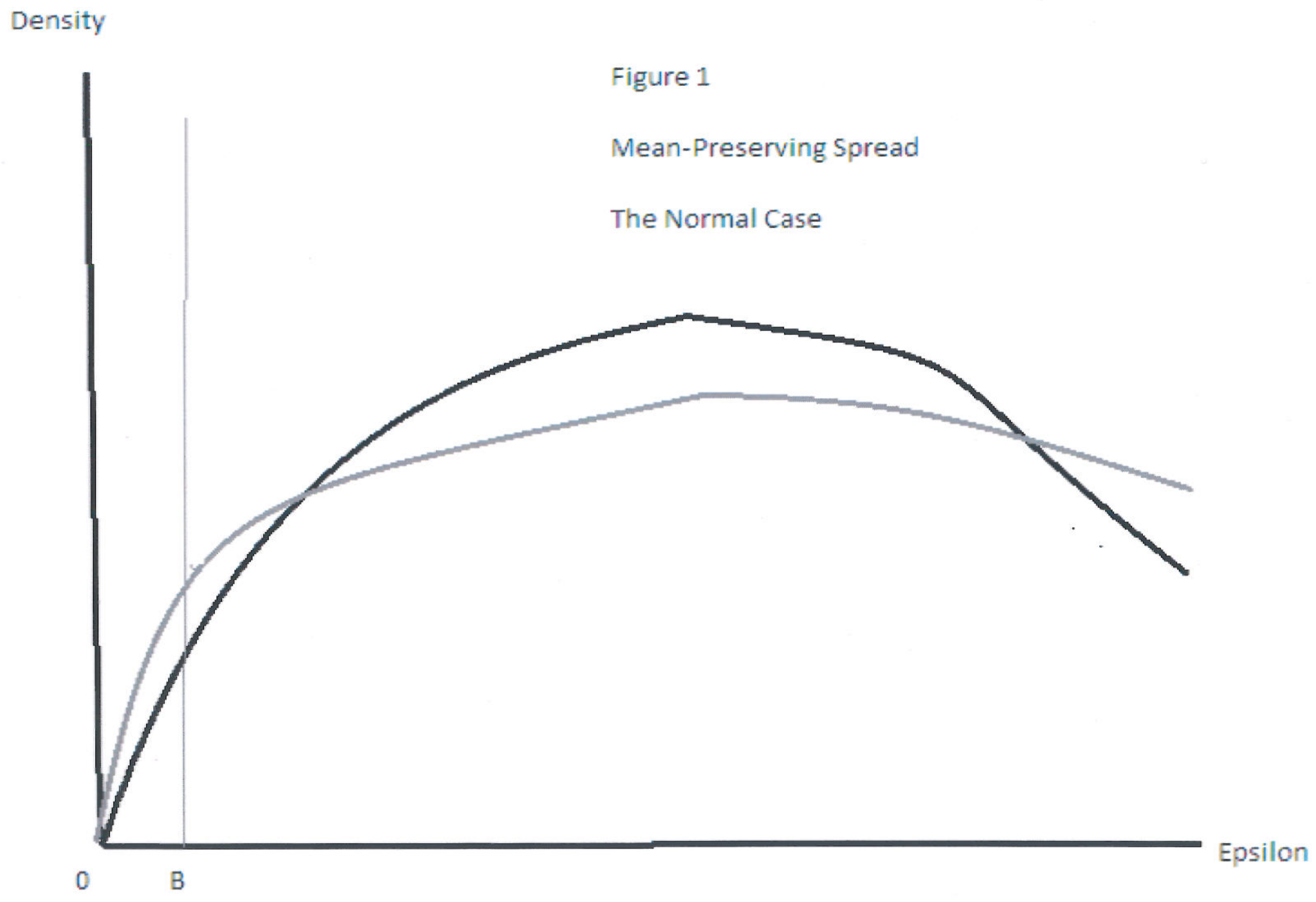
Density
Figure 1
Mean-Preserving Spread
The Normal Case
Epsilon
0
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