1. Koopmans Recursive Preferences and Income Convergence

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Abstract

Stiglitz [12] shows income convergence in a many-agent Solow growth model with integrated capital markets (ICM). The many-agent Ramsey model (MARM) without ICM also gives income convergence. With a MARM, equal discount rates, and ICM, convergence of incomes (as opposed to product per capita) cannot occur. These results depend upon fixed saving propensities (Stiglitz) or separable additive preferences (Ramsey). Non-convergence of incomes is shown when preferences are identical Koopmans separable (KS). Endogenous discount rates may violate KS. A model for that case is developed when, even under favourable assumptions, oscillations or chaotic dynamics may result.

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2. Introduction

What happens when many distinct infinitely-lived agents save? Will their incomes converge together, or will initial wealth differences be replicated indefinitely? How important for deciding your current wealth position is the wealth status of an ancestor 500 years ago? Obviously these questions address issues which are enormously complicated, and the answers to them may in practice be impossible to resolve. Even so, reviewing how these questions are decided within highly simplified and stylized models is certainly of interest. Moreover it turns out that one important determinant of the answer within such models is the extent of capital market integration. And that is likely to be an important issue however much the argument is made complicated and sophisticated in other respects. Another question which will be seen to be central is: how are saving rates determined? Are they given by arbitrary propensities to save, or are they determined by explicit optimization?

In reality agents differ in numerous respects: IQ, risk aversion and time discount rates are all variables which could cause life histories to differ even were agents to start with identical wealth holdings. Although that is true, the particular case in which all agents are fundamentally the same in their earning power, and in their tastes and their discount rates, is of central and abiding concern. If the market system cannot iron out inequalities when all savers are fundamentally the same, it will certainly not do so where there are root differences between agents.

Stiglitz [12] examines a disaggregated neoclassical growth model - a Solow model in fact. Capital ownership is assigned separately to individual agents; the rate of return and the wage rate depend upon marginal products in an aggregate production function; and individuals accumulate capital using a simple proportional saving rule for all incomes. These assumptions define an integrated capital markets model, denoted an ICM model below. Stiglitz's important paper should be cited by every current paper on convergence, and rarely is. It does not show existence for a strict general equilibrium, because agents follow a mechanical - proportional - savings rule, and cannot be shown to be optimizing intertemporally. That said, the finding that all agents converge to holding the same steady state level of capital per head, and thus to the same per capita income, provides an early and striking instance of theoretical income convergence.

Ramsey [11] himself considered a version of his optimal saving problem which will be called here the many-agent Ramsey model (MARM). He looked at steady states, and noted a paradoxical feature of many-consumer steady states. If agents discount future utility and use different constant discount rates, then, in any steady state, all the capital will be owned by agents with the lowest discount rate[1]. One way round having all capital end up owned by one agent type would be to have the discount rate depend on consumption per head. For this to help, however, the discount rate would have to be low for the poor, which is the opposite of what intuition may suggest. On these issues, see more below. The optimal growth problem with many consumers is examined by Lucas and Stokey [10].

To explain why capital does not migrate instantly between different agents to equalise the marginal product of capital, Barro, Mankiw and Sala-i-Martin [4], henceforth (BMS), proposes a radically different argument. Human capital is introduced into the production function. By itself this does not make a great difference, as human capital is simply accumulated optimally to combine with physical capital. These authors, however, add an extra assumption. The accumulation of human capital cannot be financed by borrowing.

The idea is that imperfect capital mobility by impeding output convergence will assist income convergence. That is what happens in the case considered by the authors. This case is quite special. One small low-wealth country converges to a steady state which the rest of the world occupies from the start. The converging country is borrowing-constrained all the way to steady state. In a more general many-agent equilibrium, for the same model with borrowing constraints, with a low wealth country (or countries) having significant weight in the world equilibrium, convergence is far from certain.

Section 2.1 examines many-agent steady states and notes that for a given number of agents they are uncountably infinite. Section 2.2 states that if agents start with initial capital unequally distributed incomes will not converge. It asserts that this result extends to Koopmans recursive preferences. Section 2.3 defines Koopmans recursive preferences and shows that utility functions for finite initial consumption streams (overtures) conditional on given infinite continuations may be extracted. Section 2.4 defines pseudo-Pareto-optimal equilibrium for infinite many agent programmes; defines price systems for such programmes; and shows that a programme which has a price system is pseudo-Pareto-optimal. The section also defines a class of programmes called pseudo planning solutions. Section 2.5, called The Structure of Recursive Equilibrium, identifies pseudo-efficient programmes and planning solutions. Section 2.6 examines the implications of earlier

[1] Barro and Sala-i-Martin [3] pp. 100-101 discusses the implications of differences in discount rates. However these authors do not provide a full discussion of what happens out of steady state when all agents have the same discount rate.

results for convergence. As any market solution must maximize a weighted sum of agents' utilities with the weights being constant over time, the non-convergence of incomes for unlike agents follows. This implies that the constant equal saving proportions of the Stiglitz model can never be Ramsey optimal.

In Section 2.7 it is shown that local β-convergence or local β-divergence are both possible. This is different from a result of Barro and Sala-i-Martin [3] for the simple Ramsey case. That difference arises because these authors make a special, and actually an implausible, assumption concerning the elasticity of marginal utility. Section 2.8 considers endogenous discount rates and proposes a simple dynamic model of their determination. That model is shown to contradict Koopmans separability. It allows no unequal many-agent steady states in which all agents have positive capital holdings. However a simple convergence proof is shown to fail. Oscillations or chaotic dynamics my result. The concluding section 2.9 contains some sceptical remarks concerning rational expectations solutions for this type of model.

2.1. Steady States

It is known that in the MARM equal-discount-rate unequal-income steady states exist. Consider a period model in which each of N agents supplies one unit of labour inelastically each period and agent i maximizes:

$$\sum_{t=1}^{\infty} \left(\frac{1}{1+\delta}\right)^{t-1} U\left[c_t^i\right] \tag{1}$$

where c_t^i denotes consumption at t for the agent of type i. In stationary state factor prices are constant over time, as is capital held by any type of agent. In any equilibrium, steady state or otherwise, no agent wishes to alter consumption so as to transfer it marginally between periods. That requires:

$$\frac{U_1\left[c_t^i\right]}{U_1\left[c_{t+1}^i\right]} = \frac{1+r}{1+\delta} \tag{2}$$

where r is the rate of interest and non-time subscripts denote differentiation. In steady state the left-hand side of (2) is unity, from which it follows that $r = \delta$ is a necessary condition for a steady state.

As all agents have the same discount rate δ, there is only one possible steady state value for aggregate capital, K, and that is the value for which:

$$f_1(K,N) = r = \delta \tag{3}$$

The wage rate, of course, will be the one corresponding to aggregate capital defined by (3). Although only one value of aggregate capital is consistent with steady state, it can be distributed between agents in any way. The divisions of a fixed total between two or more agents is uncountably infinite.

For steady-state to hold, when we vary the capital holding of agents of significant weight, we have to vary the capital holdings of other agents so as to keep total capital in the economy constant. This implies that if one large group of agents decides, for whatever reason, to accumulate more capital, they will drive down the rate of return, and other agents through their optimizing responses will eventually end up holding less capital. In any case, if one starts in steady state with total capital equal to its long-run equilibrium level, there will be no tendency at all for agents' incomes to converge.

2.2. Conditions for non-Convergence of Incomes

So far we have only shown that unequal-income steady states exist. This by itself points up a contrast with the Stiglitz model, as in that case an unequal-income steady state is not an equilibrium[2]. In fact starting from arbitrary unequal initial conditions, and analysing optimal developments in the MARM with ICM, it can be shown that convergence to income equality for otherwise like agents never happens. That result emerges below in a more general framework.

The non-convergence of incomes which we have so far demonstrated, illustrated by unequal-income steady states, depends heavily upon the linear additive Ramsey utility function (1). It is surprising that, with an optimal general equilibrium solution, this function is always inconsistent with Stiglitz's fixed saving coefficient. But such is evidently the case, as different convergence conclusions are implied in the two instances. The Solow-Stiglitz assumption used to be the more fashionable; then a revival of interest in Ramsey's model largely displaced it. Bliss [7] argues that for a theory of the long-run real rate of interest both these approaches suffer from shortcomings. Before reaching a final conclusion on the issues it is interesting

[2] Consider two agents with unequal capital holdings. They earn equal wages by assumption, and save the same share of total income. Capital income is a smaller share of total income for the poorer agent. But capital growth (saving) has the same share. So the poorer agent's capital grows faster, and an unequal distribution is not a steady state.

to know the answer to another question. In the MARM with ICM, how much do non-convergence of income results depend upon linear additive separability?

To provide the answer in advance, such results do depend upon a separability assumption, but it is weaker than Ramsey separability.

Theorem 1. *In a MARM/ICM model non-convergence of incomes follows when agent preferences are Koopmans recursive.*
Proof: Follows discussion and analysis below.

2.3. Utility Functions for Overtures

This section explains Koopmans recursive preference and clarifies how it may be applied when preferences are over infinite consumption streams. A fine treatment of Koopmans recursive preferences is provided by Becker and Boyd [5].

Let $\mathbf{c}$ denote the infinite sequence of consumptions $\{c_1, c_2, ..., c_t, ...\}$, and let S_t be the $t-$shift operator such that:

$$S_t\mathbf{c} = \{c_t, c_{t+1}, ..., c_{s+t}, ...\} \tag{4}$$

Thus $\mathbf{c} = S_1\mathbf{c}$.

Definition 1. *Given an infinite sequence* $\mathbf{c}$, *the T-period overture of* $\mathbf{c}$ *is the finite sequence* $(c_1, c_2, ..., c_T)$.

Obviously:

$$\mathbf{c} = \{c_1, c_2, ..., c_T; S_{T+1}\mathbf{c}\} \tag{5}$$

which means that an infinite consumption sequence is defined by the union of its infinity of overtures.

Lemma 2. *Koopmans recursive utility:*

$$U(\mathbf{c}) = W_1 \left[u\left(c_1\right), U\left(S_2\mathbf{c}\right)\right] \tag{6}$$

implies recursive utility for overtures as:

$$U(\mathbf{c}) = W_t \left[u\left(c_1\right), u\left(c_2\right), .., u\left(c_t\right); U\left(S_{t+1}\mathbf{c}\right)\right] \tag{7}$$

Proof: Is by induction on t. For $t = 1$, (7) is the same as (6). Now suppose that the induction hypothesis is satisfied for $t = s$.

$$U(\mathbf{c}) = W_s\left[u\left(c_1\right), u\left(c_2\right), .., u\left(c_s\right); U\left(S_{s+1}\mathbf{c}\right)\right] \tag{8}$$

Substituting from (7) gives:

$$U(\mathbf{c}) = W_s\left[u\left(c_1\right), u\left(c_2\right), .., u\left(c_s\right); W_1\left[u\left(c_{s+1}\right), U\left(S_{s+1}\mathbf{c}\right)\right]\right] \tag{9}$$

Now the right-hand side of (9) defines W_s, depending as required on:

$$u\left(c_1\right), u\left(c_2\right), .., u\left(c_s\right), u\left(c_{s+1}\right), U\left(S_{s+1}\mathbf{c}\right) \tag{10}$$

The induction proof is complete.

Definition 2. *Koopmans recursive preferences are concave just as simple preferences; i.e.:*

$$U(\lambda\mathbf{c}_1 + (1-\lambda)\,\mathbf{c}_2) \geq \lambda U(\mathbf{c}_1) + (1-\lambda)\,U(\mathbf{c}_2) \tag{11}$$

From the Lemma it follows immediately that recursive preferences define a utility function, and hence a preference relation, over all overtures, given their infinite continuations as defined by the equilibrium. We call a utility function for an overture of length T the *T-period utility function.* Such a utility function is always defined conditional on an infinite consumption continuation, and is only meaningful when so understood.

2.4. Pseudo-Pareto-Optimal Equilibrium

An equilibrium of the Koopmans recursive MARM is defined by infinite sequences. These specify consumption of each agent, $\mathbf{c}^i$ $(i = 1, ..., n)$, and capital owned by each agent, $\mathbf{k}^i$ $(i = 0, 1, ..., n)$; plus capital prices and wage rates:

$$S_0\mathbf{p} = \{p_0, p_1, ..., p_t, ...\} \tag{12}$$

$$S_0\mathbf{w} = \{w_1, w_2, ..., w_t, ...\} \qquad 13$$

Prices (12) and (13) must be such that they lie for each t on the factor-price frontier. That is they must be such that the profit maximization decision of producers is well defined. Where the production function is constant returns this condition is the same as requiring that the wage rate and the implicit rate of profit $\frac{p_t - p_{t+1}}{p_t}$ should equal the marginal products of capital and labour corresponding to the level of capital at that time.

At the end of any period, capital held through the previous period is sold, and capital required to be held through the next period is purchased forward. In period t, the sale is at price p_{t-1}, while the purchase is at price p_t. As prices fall with time, an agent holding a constant unit capital stock makes a return:

$$p_t - p_{t-1} \tag{14}$$

in period t.

Each period one unit of labour is supplied by each agent.

The budget constraint of agent i for period t is:

$$w_t + p_t \cdot k_t^i + q_t^i \geq p_t \cdot c_t^i + p_{t+1} \cdot k_{t+1}^i \tag{15}$$

where q_t^i is the sum of dividends received by agent i at t. Then (15) says that the wage rate plus the sale value of capital held at the end of period $t-1$ plus dividends must be no less than the cost of purchasing consumption for period t plus the cost of buying capital to be held through period t. Notice how prices falling through time implies that if the k values in (14) are constant the net effect is a boost to consumption possibilities. For each T, consumers maximize their T-period utilities subject to fixed starting and finishing capital holdings k_0^i and k_T^i.

Dividends appear in this specification to allow for the possibility that the production function will exhibit diminishing returns to scale (to all factors). In that case firms will make pure profits in equilibrium after they have paid the cost of capital. These profits must be distributed to consumers in the form of dividends.

Summing (15) gives:

$$\sum_{t=1}^{T} \left(w_t + q_t^i \right) + p_1 \cdot k_1^i - p_{T+1} \cdot k_{T+1}^i \geq \sum_{t=1}^{T} p_t \cdot c_t^i \tag{16}$$

Then (16) says that consumers sell their initial capital holding and buy their final capital holding forward, and the net credit due from that transaction is added to discounted wage income plus discounted dividends, to give the total which constrains discounted consumption for the first T periods. Summing (16) over i and denoting the totals by upper case letters gives:

$$\sum_{t=1}^{T} w_t n + Q_t + p_1 \cdot K_1 - p_{T+1} \cdot K_{T+1} \geq \sum_{t=1}^{T} p_t \cdot C_t \tag{17}$$

Value maximization for aggregate producers is the maximization for each T of:

$$\sum_{t=1}^{T} \{p_t \cdot F(K_t, n) - (p_t - p_{t+1}) \cdot K_t - w_t \cdot n\} \tag{18}$$

The expression (18) treats the producer as renting the capital used in each period, with the rental for period, with the rental for period t being $p_t - p_{t+1}$. Notice again that if capital always has positive marginal productivity, then the series (12) must be monotonically decreasing. Now (18) is equivalent to:

$$\sum_{t=1}^{T} \{p_t \cdot F(K_t, n) - w_t \cdot n\} - p_1 K_1 + p_{T+1} \cdot K_{T+1} \tag{19}$$

The maximizations described above are valid for all T. So in the case of both consumers and producers maximization involves an infinite stream of nested maximizations in which, as the length of the plan increases without limit, the final capital holding is moved forward in time to apply to a more and more distant time.

Definition 3. *An n-Agent Ramsey Equilibrium is specified by n Koopmans recursive preference relations; n non-negative initial capital holdings; consumption and capital-ownership histories for each agent; $\mathbf{c}^i$ and $\mathbf{k}^i$; and prices $S_0\mathbf{p}$ and $\mathbf{w}$, such that:*

- Each agent maximizes utility subject to the budget constraints (16) with labour supply 1 in each period;
- Production maximizes (18) or (19), and in each period $L_t = n$; and $K_t = \sum_{i=1}^{n} k_t^i$.

Definition 4. *The N-Agent Ramsey Equilibrium is Pseudo-Pareto-Optimal if each overture is a Pareto-optimal plan subject to the total initial capital and final total capital levels defined by the equilibrium.*

The definition is similar to the definition of pseudo-efficiency for infinite production programmes, see Bliss [6], p.219. Notice that the definition only makes sense because we have established recursive utility functions for overtures, conditional on given infinite continuations. Efficiency of all overtures is required for the strict efficiency of infinite programmes, but does not imply it. For example, a stationary solution to the Solow model with the rate of interest below the rate of growth is pseudo-Pareto-efficient, yet inefficient.

Theorem 3. *The N-Agent Ramsey Equilibrium is Pseudo Pareto Optimal*
Proof: *The proof proceeds exactly as the standard proof of the efficiency of general equilibrium, taking advantage of the fact that recursive utility functions have been defined for overtures, conditional on the infinite continuation of the plan. Suppose that, contrary to the theorem, we have a pseudo-inefficient overture of length S. This implies that there exist feasible sequences, $i = 1, .., n$:*

$$c_1^{'i}, c_2^{'i}, .., c_s^{'i} \tag{20}$$

such that the utility provided to each individual, conditional on the infinite continuation of the said n-Agent Ramsey Equilibrium, is no smaller, and in the case of at least one agent is larger, than that provided by the S-overture to the plan:

$$c_1^i, c_2^i, .., c_s^i \tag{21}$$

Then the agents cannot afford to buy the said consumption sequence, which state of affairs, as all terms on the left-hand side of (16) are constants for agents, implies for the alternative development that:

$$\sum_{t=1}^{S} p_t \cdot C_t < \sum_{t=1}^{S} p_t \cdot C' \qquad 22$$

As these are feasible developments, (22) entails:

$$\sum_{t=1}^{S} (p_t \cdot [F(K_t, n) - K_{t+1} - K_t]) - n \cdot w_t$$

$$< \sum_{t=1}^{S} \left(p_t \cdot \left[F\left(K_t^{'}, n\right) - K_{t+1}^{'} - K_t^{'}\right]\right) - n \cdot w_t \qquad 23$$

which contradicts producer profit maximization.

Definition 5. *Feasible consumption sequences* $\mathbf{c}^i$ *are a pseudo-planning solution if they maximize:*

$$\sum_{i=1}^{n} \alpha^i U^i \left(\mathbf{c}^i\right) \tag{24}$$

for all T*, subject to the fixed consumption continuations after* T *as defined by the solution concerned. The* α *values are non-negative and* $\sum_{i=1}^{n} \alpha^i = 1$*. Denote such a vector of* α *values in the unit simplex is denoted* $\boldsymbol{\alpha}$*.*

2.5. The Structure of Recursive Equilibrium

This paper is not so much concerned with proving existence of equilibrium as with examining the structure of an MARM when it exists. The demonstration of existence is straightforward. The method shown by Negishi [9] lends itself naturally to this model. The key step is contained in the following theorem:

Theorem 4. *An equilibrium of the* $n-$*agent Ramsey Economy is pseudo-efficient iff it is a pseudo-planning solution.*
Proof: *(i) sufficiency: The pseudo-efficiency of a planning solution is immediate. If such a solution were to be pseudo-inefficient, it could immediately be replaced by a better plan with no negative implication for the subsequent history - because final capital is not varied.*
(ii) necessity: Suppose that an equilibrium is pseudo-efficient but is not a planning solution. Let it provide a vector of utilities to consumers $\mathbf{u}*$*. It is easily shown that the vector of utilities provided to consumers is a continuous function of* $\boldsymbol{\alpha}$*. Denote the mapping from* $\boldsymbol{\alpha}$ *to* $\mathbf{u}$ *by* $\mathbf{u} = \mathbf{h}[\alpha]$*. Consider the continuous mapping from* $\boldsymbol{\alpha}$ *in the unit simplex into that same simplex:*

$$\frac{\boldsymbol{\alpha}+ \left[\mathbf{h}\left[\boldsymbol{\alpha}\right] - \mathbf{u}*\right]^{+}}{N\langle \boldsymbol{\alpha}+ \left[\mathbf{h}\left[\boldsymbol{\alpha}\right] - \mathbf{u}*\right]^{+}\rangle} \tag{25}$$

where a superscript $^+$*indicates the non-negative elements of the vector concerned, and* $N\langle\cdot\rangle$ *is a scalar norm by which a vector is divided to reduce the sum of its elements to unity. The Brouwer Fixed Point Theorem guarantees a fixed point of the above mapping. At that point:*

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\alpha}+ \left[\mathbf{h}\left[\boldsymbol{\alpha}\right] - \mathbf{h}\left[\boldsymbol{\alpha}\right]\right]^{+}}{N\langle \boldsymbol{\alpha}+ \left[\mathbf{h}\left[\boldsymbol{\alpha}\right] - \mathbf{u}*\right]^{+}\rangle} \tag{26}$$

Therefore $[\mathbf{h}[\boldsymbol{\alpha}] - \mathbf{u}*]^{+}$ *is proportional to* $\boldsymbol{\alpha}$. *Then Either* $[\mathbf{h}[\boldsymbol{\alpha}] - \mathbf{u}*]^{+} = \mathbf{0}$, *in which case the weights* $\boldsymbol{\alpha}$ *have produced* $\mathbf{u}*$, *which is then a planning solution, contrary to assumption. Or, the weights* $\boldsymbol{\alpha}$ *have produced a solution with utility of all agents higher than* $\mathbf{u}*$, *in which case the supposed equilibrium is certainly not pseudo-efficient. In any case, an equilibrium that is not a planning solution cannot be pseudo-efficient, as required.*

One can build on Theorem 4 to obtain an existence proof for market equilibrium, on the same lines as the method of Negishi [9]. The details are left to the interested reader. Vectors of weights in the unit simplex $\boldsymbol{\alpha}$ map continuously to equilibria of the MARM. Such an equilibrium assigns an initial capital endowment to each agent, the vector $k(\boldsymbol{\alpha})$, being the present value of lifetime consumption less the present value of lifetime wage income. Agents in turn have an actual endowment vector $\mathbf{k}_0$. Then a fixed point of the mapping:

$$\frac{\boldsymbol{\alpha} + [\mathbf{k}_0 - \mathbf{k}[\boldsymbol{\alpha}]]^{+}}{N\langle \boldsymbol{\alpha} + [\mathbf{k}_0 - \mathbf{k}[\boldsymbol{\alpha}]]^{+} \rangle} \tag{27}$$

is a competitive equilibrium.

2.6. Implications for Convergence

Theorem 4 has some rather stark implications for the question of whether income inequality will tend to iron itself out with time due to a natural process of convergence or trickle down. As any market solution must maximize a function of the form of (24), in which note the weights are constant over time, it follows that strict asymptotic convergence for wealth holdings cannot happen. Take two agents who start unequal with regard to capital endowments. Let agent a have less initial wealth than agent b. These agents must have unequal weights in the objective function (24) - a lower weight for agent a. Yet if the agents have unequal weights, their paths cannot converge together. If they did, the planning solution could shift consumption at future times to agent b so as to increase the weighted sum of utilities.

The conclusion comes from a model which incorporates strong simplifications. These might be regarded as particularly favourable to convergence. In particular:

- All agent types have the same tastes
- All supply the same quantity of labour in all periods and earn the same wage

- All have access on exactly equal terms to the same capital market, where they all earn the same rate of return
- All have perfect foresight and there are no stochastic effects in the model to upset convergence

Note the contrast between the non-convergence (of incomes) just noted, and the convergence (of incomes) finding of Stiglitz [12]. The reconciliation of the two conclusions is evident, if somewhat surprising. *Constant savings shares equal for all agents are always inconsistent with full general equilibrium intertemporal optimization when the agents differ in their initial conditons.*

We have been able to rule out strict asymptotic convergence for wealth holdings: the unequal cannot become completely equal, not even in the limit. This is the fundamental non-convergence of incomes result for this type of model. The finding however has little relevance to empirical studies of convergence. These only examine convergence over quite short periods of time. The present theory can throw light on partial convergence. However the findings are ambiguous. Unequal agents may come closer together (meaning here that the ratio of their consumption levels moves closer to unity), or they may move further apart (meaning here that the ratio of their consumption levels moves away from unity). Just as both these outcomes may be observed, so both may occur simultaneously in different areas of the global income distribution across agents. The possibilities are rich. This is not inconsistent with the findings of concrete empirical studies, which similarly seem to suggest a variety of possibilities[3]. Too much should not be made of that fact. The model is highly stylized and unrealistic.

To see what happens to the distribution of consumption over time, consider that the maximization of the objective function:

$$\sum_{i=1}^{n} \alpha^i U^i \left(\mathbf{c}^i\right) \tag{28}$$

is subject to the usual principles of dynamic programming. This implies, in particular that the solution which maximizes (28) also maximizes:

[3]Note that empirical studies usually examine output per head, not income per head, which more closely reflects wealth owned. In fact these two levels do not deviate as much as a perfect capital mobility model might lead one to expect. Feldstein and Horioka [8] provides the original study of this pattern.

$$\sum_{i=1}^{n} \alpha^i U^i \left(S_t \mathbf{c}^i\right) \tag{29}$$

for any value of t subject to the total capital available at time t .

Given that the maximizing solution provides total consumption C_t in period t, then the c_t^i values must maximize:

$$\sum_{i=1}^{N} \alpha^i \cdot W_t \left[u\left(c_t^i\right); U\left(S_{t+1}\mathbf{c}^i\right)\right] \tag{30}$$

subject to all $S_{t+1}\mathbf{c}^i$ fixed and:

$$\sum_{i=1}^{N} c_t^i \leq C_t \tag{31}$$

Now (30) and (31) imply:

$$\alpha^i \cdot W_1 \left[c_t^i; U\left(S_{t+1}\mathbf{c}^i\right)\right] - \eta_t = 0 \tag{32}$$

for all i, where η_t is the value of the Lagrange multiplier on the constraint (31) at t. Taking any two values of i, say j and l, we have, from (32):

$$\frac{W_1\left[c_t^j; U\left(S_{t+1}\mathbf{c}^j\right)\right]}{W_1\left[c_t^l; U\left(S_{t+1}\mathbf{c}^l\right)\right]} = \frac{\alpha^l}{\alpha^j} = A \tag{33}$$

where $\frac{\alpha^l}{\alpha^j}$ is a constant, denoted A. Thus, taking adjacent time periods:

$$\frac{W_1\left[c_t^j; U\left(S_{t+1}\mathbf{c}^j\right)\right]}{W_1\left[c_t^l; U\left(S_{t+1}\mathbf{c}^l\right)\right]} - \frac{W_1\left[c_{t+1}^j; U\left(S_{t+2}\mathbf{c}^j\right)\right]}{W_1\left[c_{t+1}^l; U\left(S_{t+2}\mathbf{c}^l\right)\right]} = 0 \tag{34}$$

Or,

$$\ln W_1\left[c_t^j; U\left(S_{t+1}\mathbf{c}^j\right)\right] - \ln W_1\left[c_{t+1}^j; U\left(S_{t+2}\mathbf{c}^j\right)\right] \qquad 35$$

is equal for all agents.

By the mean value theorem (35) may be written:

$$\frac{-W_{11}\left[c_t^{MVj}; U\left(S_{t+1}\mathbf{c}^j\right)\right]}{W_1\left[c_t^{MVj}; U\left(S_{t+1}\mathbf{c}^j\right)\right]}\left(c_{t+1}^j - c_t^j\right) \tag{36}$$

where the c values with superscripts MV are chosen to satisfy the mean value theorem. Then (36) may be written:

$$\frac{-W_{11}\left[c_t^{MVj}; U\left(S_{t+1}\mathbf{c}^j\right)\right]}{W_1\left[c_t^{MVj}; U\left(S_{t+1}\mathbf{c}^j\right)\right]}\frac{c_t^j}{c_t^{MVj}}\frac{c_{t+1}^j - c_t^j}{c_t^j} \tag{37}$$

Or,

$$\xi_t^{MVj}\frac{c_t^j}{c_t^{MVj}}g_t^j \tag{38}$$

where ξ is the elasticity of current marginal utility conditional on the continuation $S_{t+1}\mathbf{c}^j$, and g is the growth rate of consumption. Then (38) is the same for all agents j.

In general the relationship between the first two terms of (38) and the level of consumption may be anything. That is true even in the standard linear separable case, and is so a fortiori with Koopmans recursive preferences. Therefore it is possible, comparing two different agents with different consumption levels, that the one with the lower consumption will have consumption growing faster or growing slower. In the continuous time version of the MARM the dependence of (38) on mean value theorem computations disappears, and the relative growth rates of consumption for different agents depends entirely upon how the elasticity of conditional marginal utility varies with consumption. That in turn depends upon the third derivative of the utility function.

If ξ varies with the level of consumption the β-convergence of Barro [1] and Barro and Sala-i-Martin [2] and [3] can be undermined. This is no mere curiosum point. It is in fact quite plausible to suppose that the poor may be reluctant to save because the intertemporal substitution of consumption which saving requires may be difficult for the poor particularly, not because they have a high discount rate, but rather because they have a high elasticity of marginal utility.

2.7. Endogenous Discount Rates

Above, pages 2-3, we mentioned the possibility that the utility discount rate might vary with income or consumption. That is a different possibility from Ramsey's suggestion that different individuals might have different utility discount rates. In the latter case discount rates are exogenous but variable among individuals. In the former case discount rates are endogenous in the sense that they depend upon the state of the model.

Becker and Boyd [5] covers several results on asymptotic discounting. With Koopmans separable preferences discount rates may be present, but when they are present they are hidden in the form of the $W_t[\cdot]$ function. For that reason the fact that many natural specifications of the endogenous discount rate idea are inconsisent with Koopmans separability may not be as evident as it should be. To illustrate the point consider the following simple specification for individual lifetime preferences:

$$\text{Lifetime Utility} = \sum_{t=1}^{\infty} \Delta_t U\left[c_t^i\right] \tag{39}$$

$$\Delta_1 = 1 \tag{40}$$

$$\Delta_{t+1} = \Delta_t \phi\left[c_t\right] \tag{41}$$

where $\phi[c_t]$ is monotonic in c. Now an endogenous discount effect can easily be spelt out in the form of the function $\phi[c_t]$. Thus if:

$$\frac{d\phi\left[c_t\right]}{dc_t} > 0 \tag{42}$$

high current consumption causes the agent to discount utility in the next period, and for given following consumptions, utility in all later periods, less than he would with a lower c_t.

At first glance this new specification seems to hold out the possibility of convergence of incomes. Thus the starting point for a non-convergence argument above was to note that unequal-income steady states are perfectly possible in the MARM. In the many-agent version of the model (39)-(41) above unequal-income steady states are again possible. Under the monotonicity assumption, however, they are of one special type. Assume a steady state with two agents, a and b, consuming at different levels. On account of monotonicity these agents must have unequal utility discount rates. We are back to Ramsey's original MARM. Of the two agents a and b one (the one with the higher discount rate) must own no capital. That agent would like to borrow against present wage income to consume more now. Supposing that such a transaction is not permitted, we have an equilibrium steady-state.

As was remarked above, the case favourable to convergence is $\frac{d\phi[c_t]}{dc_t} < 0$, when the poor have lower discount rates than the rich. That rules out the special steady-state inequality where the poor own no capital. Intuition suggests that $\frac{d\phi[c_t]}{dc_t} > 0$

is more plausible. Ignoring plausibility, however, in the light of the powerful non-convergence result shown above, how is income convergence possible even in that case?

Simply, and in short, the model (39)-(41) violates Koopmans separability. This is plain if we rewrite (39) as:

$$\text{Lifetime Utility} = U\left[c_1^i\right] + \phi\left[c_1\right]\sum_{t=2}^{\infty}\frac{\Delta_t}{\Delta_2}U\left[c_t^i\right] \tag{43}$$

which cannot be translated to the form (6), that defines Koopmans recursive preferences.

Given the assumption:

$$\frac{d\phi}{dc} < a < 0 \tag{44}$$

can it be shown that a many agent optimal growth with preferences (39)-(41) converges asympotically to a steady state in which all agents consume at the same level? An argument similar to Theorem 4 above can be applied to the present case to show that an equilibrium solution maximizes:

$$\sum_{i=1}^{N}\beta_i\sum_{t=1}^{\infty}\Delta^{it}U(c^{it}) \tag{45}$$

For each agent $\Delta^{i1} = 1$. As time proceeds the current Δ values for the various agents are altered by their respective consumption histories. However a form like (45) with t starting higher is still maximized. As $\Delta^{it} \neq 1$, the product $\beta_i\Delta^{it}$ plays the same role as β_i in a programme starting at $t = 1$. So we may call $\beta_i\Delta^{it}$ the *effective weight* for agent i at t. If the effective weights of all agents converge to equality, then so will consumption levels. This requires only a standard convexity assumption.

Now consider two agents i and j at time t. Let agent j consume less than agent i at that time t. Then agent j must have the lower effective weight at t. In that case:

$$\frac{\Delta^{j,t+1}}{\Delta^{j,t}} > \frac{\Delta^{i,t+1}}{\Delta^{i,t}} \tag{46}$$

which implies:

$$\frac{\Delta^{j,t+1}}{\Delta^{i,t+1}} > \frac{\Delta^{j,t}}{\Delta^{i,t}} \tag{47}$$

So long as their ranking is preserved, the effective weights of the agents as measured by their ratio move closer together. If that continues to be the case convergence is guaranteed. The effective weights cannot move closer together but asymptote to distinct values because $\frac{d\phi}{dc} < a$ ensures that convergence will be large until the values are extremely close together.

This looks like the shell of a convergence proof, with only the details needed. However any attempt to fill out the argument to make a formal theorem is bound to fail. Equation (47) shows unequal growth rates, with the lower consumption agent's Δ growing faster. As long as the relative ranking of the two agents is unaltered that is enough to give convergence. However (47) is consistent with the lower consumption agent's Δ overtaking the Δ of his better-fed partner. Then their roles are reversed and by next period the relative growth rates of the two parties will be reversed. Now convergence cannot be established and two possibilities emerge:

- Equilibrium oscillations, with two or more agents swopping consumptions from period to period
- Chaotic dynamics, with the equilibrium consumptions of various agents meandering around the steady state

These conclusions come out of the discrete time model which is used in this paper, and which has to be used to examine Koopmans separability. For this reason there is no contradiction with Frank Ramsey's arguments, because he had a continuous time model in mind.

2.8. Conclusions

Although Ramsey theory appears to be consistent with convergence, for incomes a fundamental conservative principle operates. With optimal saving and unified capital markets, and without public or international intervention to redistribute wealth, inequality persists. It does so because it is not optimal for individual agents to remove it by their own saving. They may partially remove it within a convergence club but there is no guarantee of this, and there may indeed be anti–convergence clubs, within which agents' wealth holdings tend to diverge. We have shown that this discouragaing result extends to a MARM model in which agents have Koopmans recursive preferences.

Further analysis indicates that the problem with the original Ramsey model may not be that it is a special case of Koopmans recursive preferences, but rather that it is an instance of Koopmans recursive preferences. The point is that the important issue of endogenous discount rates cannot be handled under the Koopmans separability assumption. Even so, it is unlikely that endogenous preferences can rescue an income convergence result within the MARM, as the type of monotonicity required (lower discount rates for the poor) is not what most would expect.

If these results are thought to be unappealing, there are many ways in which the model may be modified to soften them. For instance:

1. The excessive stability of income distributions in the basic capital model is due to the fact that inter-generational transmission of wealth is perfect, which makes the system extremely conservative. Less perfect wealth transmission would help there.

2. Stochastic shocks may be helpful to realistic modelling. They spread wealth out and avoid unrealistic convergence. To avoid a random walk outcome one needs imperfect transmission.

The type of theory considered in this paper is fundamentally long-run, so that it is not easy to distinguish the realistic from the unrealistic. We know that global income distributions can shift significantly within individual lifetimes, due to macroeconomic developments, policy shifts or technical changes. Yet other studies show a remarkable stability of inequality. Even partial convergence in the models examined will require many generations. Qualitative findings, however, may be suggestive. If there is no tendency to convergence for incomes, as can happen within the model, the shortness of our line of view will not matter.

The BMS example is quite special. It suggests that imperfect capital markets, while tending to slow down output convergence, may also increase the extent of income convergence among a poor subset of agents whose poverty forces then to experience borrowing constraints. Even so, the implication that the incomes of all poor agents will converge together in the limit; to equality with the steady state return to labour plus steady-state human capital; is not a general result. Usually initial income differences between even poor agents will persist after optimal accumulation to infinity - although their extent may be modified.

Finally a sceptical note is in order. We are used as economists to starting - often finishing - with an equilibrium. An equilibrium solution has a solidity and definiteness to it. It is consistent with basic economic postulates, by construction, and we think we know what it means. That is harder to credit with the many agent infinitely-lived equilibrium, the properties of which are examined above. A fixed-point theorem may tell us that it exists, but it never shows how to compute it, or whether it could be computed. To wave hands and talk about an auctioneer is simply to brush the difficulty away. If agents cannot know the full infinite intertemporal history of prices, they will have to act, presumably using rules of thumb. Then the Stiglitz (1969) model may provide a reasonable approximation to what actually happens. In this case the non-convergence result can be turned right around to say that in reality we may see less persistent income inequality than would characterize the full equilibrium optimum, precisely because agents cannot know future prices for certain and cannot compute the equilibrium.

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