

Co-breaking: Recent Advances and a Synopsis of the Literature

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Abstract

The purpose of this paper is twofold. First, it aims to provide a synopsis of a literature on co-breaking which has been developing in several, seemingly disconnected, strands. This includes establishing a consistent terminology, collecting theoretical results, delimiting co-breaking to cointegration and common features, and reviewing recent contributions to co-breaking regressions and the budding analysis of co-breaking rank. The second purpose of the paper is to present new results in the field. In particular, the importance of co-breaking for policy analysis takes centre stage, with special emphasis on impulse-response functions. Moreover, a new test for co-breaking rank is presented, evaluated by means of Monte Carlo experiments and illustrated using UK macroeconomic data.

1 Introduction and a selective historical overview

This section overviews much of the literature over the past three decades on modelling non-stationary time series. It is indicative of various themes and practices that have prevailed, and continue to prevail, in macro-econometrics.

Many macroeconomic time series are generated by non-stationary random processes since various moments of observed data are not time invariant. Examples include upward-trending GDP, the aggregate price level (especially in hyperinflation periods,) or time series affected by political, legislative or technological structural change. Economic processes subject to breaks often experience shifts in their locations, primarily means and trends: we refer to these as deterministic shifts as a shorthand for shifts in the coefficients of deterministic variables. The history of economic forecasting in the UK confirms the existence of major irregularities: see e.g. Burns (1986), Pain and Britton (1992), Wallis (1989), and Stock and Watson (1996). Barrell (2001) illustrates the frequency of breaks by six major episodes of change during the 1990s alone. Importantly, a major source of forecast failure seems to be deterministic shifts, with other structural breaks being much less pernicious: see Hendry (2000c).

There has been considerable progress in econometric modelling of non-stationarities. Two prominent approaches to mapping the data to stationarity involve prior regression on a polynomial time trend, and the taking of first, or higher order, differences. The sequence of papers by Courakis (1978), Hendry and Mizon (1978) and Williams (1978) on econometric modelling practices at the Bank of England provides a snapshot of the debate on the pros and cons of differencing. Also, the paper by Nelson and Plosser (1982) on some long-range US macroeconomic time series initiated a lively debate on whether macroeconomic time series are trend-stationary or difference-stationary, i.e., whether disturbances have

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a transitory or permanent effect on the level of the process. As a result, a number of test procedures were devised that sought to differentiate between a unit root and a deterministic time trend; see Dickey and Fuller (1979, 1981) and Said and Dickey (1984).

Since the mid-1980s, however, three important developments may be noted. First, the deterministic approach no longer subscribes to [the view that](#) a simple time trend [may](#) describe the behaviour of a time series. Instead, more complicated functions of time are used to capture non-stationarities. Examples include the use of ‘blip’ dummy variables to model outlying observations, or ‘step’ dummy variables – possibly multiplied by a linear trend function – to represent shifts: see Box and Tiao (1965, 1975) on intervention analysis; and Gallant and Fuller (1973) on segmented polynomial regression models. In econometrics, a segmented trend model was employed by Rappoport and Reichlin (1989), who revisited the Nelson–Plosser data set and argued that the number of disturbances with a permanent effect on the process of interest may be smaller than the difference-stationarity hypothesis suggests, and could be modelled by means of a few large intermittent changes in the growth rate.

The second major extension of the dichotomy between unit roots and linear time trends was to augment the Dickey–Fuller unit-root tests by deterministic components. Perron (1989, 1990) proposed testing the null of a unit root against the alternative of a linear trend and an intercept, respectively, allowing for a structural break. Applying his tests to the Nelson–Plosser data, Perron (1989) concluded that the unit-root hypothesis may be rejected for most of the time series involved, arguing that a linear time trend which is subject to a few large structural breaks better describes the non-stationarity inherent in many macroeconomic processes. This conclusion was supported by Andreou and Spanos (2003), who re-estimate the models specified by Nelson and Plosser (1982) and Perron (1989), ensuring they are data congruent.

Finally, the third development in the modelling of non-stationarity has been the contribution of cointegration by Granger (1981), according to which a linear combination of unit-root processes may be integrated of a lower order. Engle and Granger (1987) subsequently operationalized cointegration, building on the equilibrium-correction representation of an autoregressive process introduced by Sargan (1964) and popularized by Davidson, Hendry, Srba and Yeo (1978). Establishing a relationship that is stable over time between several stochastic variables, even though its constituents are individually non-stationary, added a new dimension to econometric modelling. The mapping to stationarity of the time series could now be achieved within a system, i.e., without resorting to purely exogenous deterministic factors and without losing information by differencing. The concept was subsequently extended to linear combinations that cancelled other types of stationary or non-stationary ‘common feature’, to use the term coined by Engle and Kozicki (1993). For instance, Kang (1990) considers a deterministic time trend as a putative common feature, while Engle and Kozicki (1993) provide examples of the feature being an ARCH component or a cycle. Indeed, co-breaking as in Hendry (1996) is another member of this last class, which we now consider in more detail.

1.1 Common features

The literature on common features may be broadly classified into two strands. The first concerns the estimation of relationships that cancel the common feature of interest under the assumption that there is a known number of such relationships, usually one. In contrast, the second strand deals with procedures that simultaneously test for the number of common features and estimate the corresponding linear relationships. The following two sub-sections deal with these approaches in turn. In accordance with standard terminology, they will be called the single equation and the system approach to common feature analysis.

1.1.1 Single equation analysis

This approach amounts to what Engle and Kozicki (1993) call a common feature regression. Generally, it proceeds in two steps. In the first step, the relationship between the variables believed to have a feature in common is estimated. The second step then investigates whether or not the first step residuals continue to display the feature of interest. The most prominent example of a common feature regression is the original ‘cointegrating regression’ of Engle and Granger (1987). In a similar development to the extension of the Dickey–Fuller test by deterministic components (nuisance parameters), the residual-based cointegration test has also been augmented by dummy variables or made robust against structural breaks; see, for instance, Gregory and Hansen (1996) and Arranz and Escribano (2000), respectively.

Another example of a common feature regression is the so-called ‘co-trending regression’ suggested by Ogaki and Park (1998) and Kang (1990). In these studies, each of the time series of interest is assumed to be generated by a linear deterministic time trend, which is no longer displayed in a linear combination thereof. This setup was subsequently extended by Chapman and Ogaki (1993) to include structural breaks, where the intercept and linear trend of individual time series may be subject to an arbitrary number of breaks, whose number is reduced in the residuals of the estimated co-trending relationship. Chapman and Ogaki (1993), and subsequent authors, call the cancellation of breaks ‘co-trending’, since trend breaks and the trend itself are varieties on a continuum of deterministic terms. Empirical examples of what might more aptly be called a ‘co-breaking regression’ are provided by Hendry and Mizon (1998) and Clements and Hendry (1999) on the one hand, and Morana (2002) on the other, who use dummy variables and a stochastic break process based on a Markov chain, respectively, to model irregularities in the data, and examine whether they vanish in linear combinations of the series.

1.1.2 System analysis

A more general approach to common feature analysis was taken by Johansen (1988), who suggested a procedure for estimating cointegrating vectors jointly with determining their number in a system of equations (the latter is implicitly assumed known in cointegrating regressions). Johansen (1988) used a vector version of the equilibrium correction model by Davidson *et al.* (1978), in which the number of cointegrating vectors is determined by the rank of a particular coefficient matrix, for a purely stochastic model (not taking deterministic factors into account). Since processes whose levels are solely determined by their initial values cannot describe all macroeconomic time series, Johansen (1994) added a constant and linear trend to the analysis. The distribution of the standard rank test, like the Dickey–Fuller unit-root and Engle–Granger cointegration tests, turned out not to be similar with respect to deterministic terms but Doornik, Hendry and Nielsen (1998) and Nielsen and Rahbek (2000), building on Kiviet and Phillips (1992), argue that similarity may be obtained by augmenting the rank hypothesis to include the intercept or the linear trend parameter, depending on the setting. Tests for cointegrating rank that allow for structural breaks include Inoue (1999), who proposes a rank test when there is a single trend break under the alternative hypothesis, and Johansen, Mosconi and Nielsen (2000) who employ the segmented trend model in Rappoport and Reichlin (1989) to allow for an arbitrary number of trend breaks.

Using the terminology of Engle and Kozicki (1993), the concept of cointegrating rank may be generalized to that of ‘co-feature rank’. Similarly, since many macroeconomic variables appear subject to structural breaks, Hendry (1996) suggests the idea of ‘co-breaking rank’ as the number of linear combinations of the variables that no longer display the breaks. However, while co-breaking regressions are by now firmly established in the literature co-breaking rank analysis is still in its infancy. In particular, there appear to be only three publications that deal with this or related issues. Firstly, Bierens (2000) develops a procedure for testing for ‘nonparametric non-linear co-trending rank’ which he applies to a data set of the Federal funds rate and the CPI inflation rate in the US. Secondly, Krolzig and Toro (2002) formulate a parametric statistical model for the estimation of linear combinations that cancel the deter-

ministic components of a vector autoregressive model. Lastly, Hatanaka and Yamada (2003) suggest an approach that combines the idea of co-trending in the process of interest with cointegration analysis in a segmented trend model.

1.2 Co-breaking

The purpose of this paper is twofold. First, it aims to provide a synopsis of a literature which has been developing in several, seemingly disconnected, strands. This includes establishing a consistent terminology, collecting theoretical results, delimiting co-breaking to cointegration and common features, and reviewing recent contributions to co-breaking regressions and the budding analysis of co-breaking rank. The second purpose of the paper is to present new results in the field. In particular, the importance of co-breaking for policy analysis takes centre stage, with special emphasis on impulse-response functions. Moreover, a new test for co-breaking rank is presented, evaluated by means of Monte Carlo experiments and illustrated using UK macroeconomic data.

The remainder of the paper is structured as follows: Section 2 outlines the theory of co-breaking, section 3 relates co-breaking to empirical econometric research, section 4 reviews the literature on co-breaking regressions before section 5 discusses the analysis of co-breaking rank. Finally, section 6 concludes.

2 The concept of co-breaking

Section 2.1 defines our terminology. In section 2.2, the concept of co-breaking is defined as a linear combination of time-varying parameters which is constant over time. This might be thought impossible in the case of frequent breaks in the series concerned: however, in some important special situations, co-breaking arises naturally. Section 2.3 discusses such cases, dealing with co-breaking in a cointegrated framework.

2.1 Terminology

Central to our analysis are location shifts of random variables, i.e., changes in the unconditional expectations of the non-integrated transformations of the variables. Since unconditional expectations are functions of deterministic variables – variables whose future values are known with certainty but the parameters of which could change – location shifts are represented by changes in the parameters of deterministic terms. To illustrate, consider the simplest location model of a variable y_t :

$$y_t = \alpha + u_t \quad (1)$$

with $\alpha \neq 0$ and where $\{u_t\}$ is $I(0)$ such that $E(u_t) = 0$. A location shift to α^* from T onwards induces:

$$y_{T+h} = \alpha^* + u_{T+h} \equiv \alpha\mu + u_{T+h} \quad (2)$$

where $\mu = \alpha^*/\alpha \neq 1$ is the shifted intercept for $h > 0$. Location shifts may be mimicked by other factors, such as mis-estimating or mis-specifying deterministic components in models. The simulation evidence in Hendry (2000c) confirms their pernicious effects on forecasts, as well as highlights the difficulty of detecting other forms of break, especially those associated with mean-zero changes, which have no effects on unconditional means.

To relate the notion of location shifts to econometric models consider the following illustrative static bivariate system:

$$\begin{aligned} y_t &= \alpha + \beta z_t + v_t \\ z_t &= \mu_z + \varepsilon_{z,t} \end{aligned} \quad \text{with} \quad \begin{pmatrix} v_t \\ \varepsilon_{z,t} \end{pmatrix} \sim \text{IN}_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix} \right] \quad (3)$$

where $(\alpha, \beta, \mu_z, \sigma_v^2, \sigma_z^2) \in \mathbb{R}^3 \times \mathbb{R}_+^2$. The first equation relating y_t to z_t is deemed to be causal, i.e. $\partial y_t / \partial z_t = \beta$, with z_t strongly exogenous for $(\alpha, \beta, \sigma_v^2)$; whereas the second equation for z_t is a policy process. Thus, the system in (3) consists of one structural and one policy equation. However, its specification does not directly entail how changes in any subset of parameters might affect the remainder, so additional assumptions are required. Note that (3) can be solved for its reduced form to yield:

$$\begin{aligned} y_t &= \mu_y + \varepsilon_{y,t} \\ z_t &= \mu_z + \varepsilon_{z,t} \end{aligned} \quad \text{with} \quad \begin{pmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{pmatrix} \sim \text{IN}_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_y^2 & \rho \\ \rho & \sigma_z^2 \end{pmatrix} \right] \quad (4)$$

such that

$$\alpha = \mu_y - \beta \mu_z, \quad (5)$$

$$\beta \rho = \sigma_z^2 \text{ and } \sigma_y^2 = \sigma_v^2 + \beta \rho.$$

Using ∇ to denote a parameter change, we define the following terminology: a *location shift* refers to the reduced form in (4) and is defined as a change in the unconditional mean, e.g., $\nabla \mu_y \neq 0$ or $\nabla \mu_z \neq 0$. Regarding the structural system in (3), a *structural break* occurs when the parameters of the causal model change, here either $\nabla \beta \neq 0$ or $\nabla \alpha \neq 0$ or both. A *policy regime shift*, on the other hand, occurs when the parameters of the policy process change, here either $\nabla \mu_z \neq 0$ or $\nabla \sigma_z^2 \neq 0$ or both. Finally, a change where $\nabla \mu_y \neq 0$ is called a *target shift* when it is the result of a policy regime shift. Thus, both regime and target shifts are special cases of location shifts.

Structural invariance occurs when a policy regime shift does not entail a structural break, i.e., $\nabla \mu_z \neq 0$ or $\nabla \sigma_z^2 \neq 0$ leaves $\nabla \alpha = 0$, $\nabla \beta = 0$ and $\nabla \sigma_v^2 = 0$. In particular, contemporaneous mean *co-breaking* occurs when α is invariant to changes in μ_z because there is a proportionate change in μ_y . To see this, consider (5) where β does not change, such that $\nabla \alpha = 0$ entails that:

$$\nabla \mu_y = \beta \nabla \mu_z. \quad (6)$$

Put differently, co-breaking occurs when a regime shift causes a target shift of proportion β . In general, changes in α may be due to changes in any of μ_z , μ_y and β , since $\nabla \alpha = \nabla \mu_y - \beta \nabla \mu_z - \mu_z \nabla \beta$. It is thus conceivable that a regime shift is exactly off-set by a structural break in β , such that $\beta \nabla \mu_z + \mu_z \nabla \beta = 0$, implying that for $\nabla \alpha = 0$, there need be no shift in the mean of y_t .

Finally, to distinguish parametric changes in the unconditional expectation of a random variable from those in its expectation conditional upon its own history, the term *deterministic shift* will refer exclusively to changes in the intercept of the autoregressive representation of the process. To illustrate, suppose that a scalar random variable w_t follows an autoregressive process of first order, so $w_t = \gamma_0 + \gamma_1 w_{t-1} + \varepsilon_{w,t}$ where $\varepsilon_{w,t} \sim \text{IN} [0, \sigma_w^2]$ and $|\gamma_1| < 1$ such that $\{w_t\}$ is covariance-stationary. Then the distribution of w_t given w_{t-1} is $w_t | w_{t-1} \sim \text{IN} [\gamma_0 + \gamma_1 w_{t-1}, \sigma_w^2]$ and the unconditional distribution is $w_t \sim \text{N} [\mu_w, \sigma_w^2]$, with $\mu_w = (1 - \gamma_1)^{-1} \gamma_0$. While the conditional distribution could change in response to shifts in γ_0 , γ_1 or σ_w^2 , a deterministic shift will denote $\nabla \gamma_0 \neq 0$. This may or may not lead to a location shift $\nabla \mu_w \neq 0$, depending on whether there is a compensating shift in the dynamic parameter γ_1 .

2.2 Contemporaneous mean co-breaking

We now review some established results to clarify what is known about co-breaking, and set the scene for the new developments.

2.2.1 The basic definition

Consider the n -dimensional vector stochastic process $\{\mathbf{x}_t\}$ over $\mathcal{T} = (1, \dots, T)$ and assume \mathbf{x}_t has well-defined unconditional expectations around an initial parameterization $E[\mathbf{x}_0] = \boldsymbol{\rho}_0$. In particular, $\boldsymbol{\rho}_0$

depends only on deterministic variables whose parameters, such as of an intercept and linear time trend, do not change: e.g., $\rho_0 = \rho_c + \rho_\ell t$. The process $\{\mathbf{x}_t\}$ is either an $I(0)$ process, or a derived function from an $I(1)$ process, where the retained combinations of variables have well-defined expectations; examples include first differences and cointegrated combinations.

Following the terminology of Section 2.1, a location shift in $\{\mathbf{x}_t\}$ is said to occur if, for any $t \in \mathcal{T}$,

$$E[\mathbf{x}_t - \rho_0] = \boldsymbol{\rho}_t \neq \mathbf{0}_{(n \times 1)} \quad (7)$$

where $|\boldsymbol{\rho}_t| < \infty$ but otherwise is unrestricted. Put differently, the location shift occurs if the expected value of \mathbf{x}_t deviates from ρ_0 at $t \in \mathcal{T}$.

Co-breaking is defined as the cancellation of location shifts across linear combinations of variables.

Definition 1 *The $(n \times r)$ matrix Φ of rank $r < n$ is said to be contemporaneous mean co-breaking of order r for $\{\mathbf{x}_t\}$ in (7), denoted $CMC(r)$, if $\Phi' \boldsymbol{\rho}_t = \mathbf{0}_{(r \times 1)} \forall t \in \mathcal{T}$.*

It follows from this definition that $\Phi' E[\mathbf{x}_t - \rho_0] = \Phi' \boldsymbol{\rho}_t = \mathbf{0}_{(r \times 1)}$, so that the parameterization of the reduced set of [the \$r\$ so-called co-breaking relationships \$\Phi' \mathbf{x}_t\$ is independent of the location shifts](#).

Contemporaneous mean co-breaking seems unlikely when the $\{\boldsymbol{\rho}_t\}$ can change in any possible way from period to period, since a single matrix Φ is required to annihilate all changes. Nevertheless, there are many cases where co-breaking can occur in principle even though no $\rho_{i,t}$ stays constant. To see this possibility, we next link CMC to a reduced-rank condition. Consider the $(n \times T)$ matrix $\mathbf{P} = (\boldsymbol{\rho}_1 : \boldsymbol{\rho}_2 : \dots : \boldsymbol{\rho}_T)$ where $T > n$. The condition that $\phi' \boldsymbol{\rho}_t = 0 \forall t \in \mathcal{T}$ when $\phi \neq \mathbf{0}_{(n \times 1)}$ can be written as $\phi' \mathbf{P} = \mathbf{0}_{(1 \times T)}$, so that Theorem 2 follows from the equivalence of reduced rank and linear dependence.

Theorem 2 *A necessary and sufficient condition for $\phi' \boldsymbol{\rho}_t = 0 \forall t \in \mathcal{T}$ when $\phi \neq \mathbf{0}_{(n \times 1)}$, i.e. $CMC(1)$, is that:*

$$\text{rk}(\mathbf{P}) < n. \quad (8)$$

This theorem is tantamount to saying that the matrix \mathbf{P} must have reduced row rank for CMC to be feasible. The rank-plus-nullity theorem, see for instance Meyer (2000), then allows the deduction of the following corollary.

Corollary 3 *CMC is at least of order r if there exist r linearly independent vectors satisfying $\phi_i' \boldsymbol{\rho}_t = 0$ such that the $(n \times r)$ matrix $\Phi = (\phi_1 : \dots : \phi_r)$ has rank $r < n$. Then $\Phi' \mathbf{P} = \mathbf{0}_{(r \times T)}$ so $\text{rk}(\mathbf{P}) \leq n - r$, and the nullity of \mathbf{P} , denoted $\text{nul}(\mathbf{P}) \geq r$, determines the order of CMC.*

[Note that the matrix \$\Phi\$ in Corollary 3 is not unique without suitable normalization, since if \$\mathbf{H}\$ is any \$\(r \times r\)\$ non-singular matrix, then \$\Phi' \mathbf{P} = \mathbf{0}_{\(r \times T\)}\$ implies that \$\mathbf{H} \Phi' \mathbf{P} = \mathbf{0}_{\(r \times T\)}\$ as well.](#)

Given that the order of co-breaking is equivalent to a rank condition, the following terminology is introduced.

Definition 4 *The number r of linearly independent vectors ϕ_i , $i = 1, \dots, r$, in $\Phi = (\phi_1 : \dots : \phi_r)$ or, equivalently, the nullity of $\mathbf{P} = (\boldsymbol{\rho}_1 : \boldsymbol{\rho}_2 : \dots : \boldsymbol{\rho}_T)$ is referred to as co-breaking rank.*

This terminology is particularly useful when rank tests, employed to determine the order of CMC from the data, are discussed in Section 5.

Since $CMC(r)$ implies $CMC(r-1)$, and unmodelled remaining breaks entail model mis-specification, it seems natural to seek the maximum degree of CMC. [However, an upper bound on the order of CMC is established in the following theorem.](#)

Corollary 5 *CMC cannot exceed order $n - 1$ if breaks occur (i.e. $\rho_s \neq \mathbf{0}_{(n \times 1)}$ for some s) since when Φ is $(n \times n)$ and non-singular, the equations $\Phi' \mathbf{P} = \mathbf{0}_{(r \times T)}$ would entail $\mathbf{P} = \mathbf{0}_{(n \times T)}$.*

Conversely, for CMC not to be trivial, no $(m \times 1)$ -dimensional sub-vector $\rho_{1,t}$ of ρ_t , $m < n$, must remain constant $\forall t \in \mathcal{T}$ since otherwise an obvious co-breaking vector ϕ would consist of zeros apart from one unity value in the position that corresponds to a constant element in ρ_t .

Corollary 6 *When any $(m \times 1)$, $m < n$, sub-vector $\rho_{1,t}$ of ρ_t is zero $\forall t \in \mathcal{T}$ then m linearly independent $(n \times 1)$ vectors ϕ_i may be found such that $\phi_i' \rho_t = 0$, $i = 1 \dots, m$, and CMC of at least order m occurs.*

Finally, the fact that \mathbf{P} must have reduced row rank may be alternatively illustrated by, first, reformulating the breaks in terms of $(1 \times T)$ -dimensional row vectors $\rho^{(i)'} = (\rho_{i,1} : \dots : \rho_{i,T})$, $i = 1, \dots, n$, so that $\mathbf{P} = (\rho^{(1)} : \dots : \rho^{(n)})'$ and, secondly, noting Corollary 7:

Corollary 7 *CMC cannot occur when the $\rho^{(i)}$ are linearly independent, since then, for all $\phi \neq \mathbf{0}_{(n \times 1)}$, $\mathbf{P}' \phi = \sum_{i=1}^n \phi_i \rho^{(i)} \neq \mathbf{0}_{(T \times 1)}$.*

2.2.2 The reduced-rank restriction

In addition to the instance mentioned in Corollary 6, at least two further cases may be identified that induce co-breaking. These correspond to scenarios in which the reduced row rank of \mathbf{P} , see Theorem 2, is not a restriction. First, an undersized sample induces a reduced rank by default:

Corollary 8 *When $T < n$, then $\text{rk}(\mathbf{P}) \leq T < n$ and there must be CMC of at least order $n - T$.*

Secondly, and relatedly, if ρ_t does not vary in every period $t \in \mathcal{T}$ but in fact only changes $k < n$ times over the course of the sample then at least $n - k$ co-breaking vectors can always be found. To see this, assume that ρ_t has $k < T$ distinct values, leading to the decomposition of \mathbf{P} into

$$\mathbf{P} = \boldsymbol{\theta} \mathbf{D} \quad (9)$$

where $\boldsymbol{\theta}$ is $(n \times k)$ and comprises the $k < T$ distinct columns of \mathbf{P} , while \mathbf{D} is $(k \times T)$ and specifies how the remaining $(T - k)$ columns of \mathbf{P} are generated as linear combinations of $\boldsymbol{\theta}$. In this setting, the reduced row rank of \mathbf{P} translates into an equivalent condition on $\boldsymbol{\theta}$ such that co-breaking occurs if $\boldsymbol{\theta}$ has reduced row rank. This, however, must happen if $\boldsymbol{\theta}$ has fewer columns than rows, i.e., if there are fewer distinct shifts in the sample than variables in the system:

Corollary 9 *If (9) holds and $\text{rk}(\boldsymbol{\theta}) \leq k < n$ then CMC occurs for at least order $n - k$.*

A case in point would be a single simultaneous blip in all elements of \mathbf{x}_t for which $(n - 1)$ co-breaking vector could always be found.

More interesting than the scenario described in Corollary 9 is the decomposition in (9) coupled with the assumption that there are more distinct values in ρ_t than variables in the system, so $n \leq k$. As Engle and Kozicki (1993) point out, it is only in this setting that a reduced rank condition on $\boldsymbol{\theta}$ becomes a restriction, namely $\text{rk}(\boldsymbol{\theta}) < n$. Assuming that $\boldsymbol{\theta}$ is of rank $\text{rk}(\boldsymbol{\theta}) = n - r < k$ with $r > 0$, then it may be decomposed into the product of two full-rank matrices ζ and ν such that:

$$\boldsymbol{\theta} = \zeta \nu' \quad (10)$$

where ζ is of dimension $(n \times (n - r))$ and of rank $\text{rk}(\zeta) = n - r$, while ν is of dimension $(k \times (n - r))$ and of rank $\text{rk}(\nu) = n - r$; see, for instance, Rao (1973, Section 1b). This scenario corresponds closely to the idea that co-breaking involves shifts being related across variables *and* over time, so the shifts are not merely distinct but also linearly dependent. The setting in Corollary 9, however, obfuscates whether CMC occurs because the shifts are ‘truly’ related over time, i.e. $\text{rk}(\theta) < k$, or whether it occurs despite $\text{rk}(\theta) = k$. The assumption that $n \leq k$ avoids the possibility of such ‘spurious’ co-breaking.

Substituting (10) into (9) yields $\mathbf{P} = \zeta \nu' \mathbf{D}$ so the r co-breaking vectors $\Phi = (\phi_1 : \dots : \phi_r)$ are given by the orthogonal complement of ζ , namely the $(n \times r)$ matrix ζ_\perp such that $(\zeta : \zeta_\perp)$ is of full rank n and $\zeta_\perp' \zeta = \mathbf{0}_{(r \times (n-r))}$.

Corollary 10 *If ρ_t has $k < T$, $k \geq n$, distinct values over the sample such that $\mathbf{P} = \theta \mathbf{D}$ as in (9) and $\text{rk}(\theta) = n - r$, with $r > 0$, so that $\theta = \zeta \nu'$ as in (10), then there exists an $(n \times r)$ matrix $\Phi = \zeta_\perp$ such that the r linear combinations $\zeta_\perp' \mathbf{E}[\mathbf{x}_t - \rho_0] = \mathbf{0}_{(r \times 1)}$ are the co-breaking relationships.*

2.2.3 Extensions

Various extensions of the concept of contemporaneous mean co-breaking are conceivable. For instance, co-breaking could be defined for higher moments by direct extension, leading to, e.g., variance co-breaking; or inter-temporal mean co-breaking (IMC) could be defined as the cancellation of deterministic shifts across variables and time periods, see Hendry (1996). Moreover, the class of processes $\{\mathbf{x}_t\}$ in Definition 1 may be extended to conditional functions of past variables. Defining $\mathbf{X}_{t-1}^1 = \sigma(\mathbf{x}_1, \dots, \mathbf{x}_{t-1})$ as the σ -field generated by $\{\mathbf{x}_t\}_{t-1}^1$, a deterministic shift in $\{\mathbf{x}_t | \mathbf{X}_{t-1}^1\}$ occurs if, for any $t \in T$:

$$\mathbf{E}[(\mathbf{x}_t - \pi_0) | \mathbf{X}_{t-1}^1] = \sum_{i=1}^g \Pi_i \mathbf{x}_{t-i} + \pi_t \quad (11)$$

and the conditional mean is time-varying such that $\pi_t \neq \mathbf{0}_{(n \times 1)}$. While changes in the dynamic parameters Π_i are possible, previous research has established that the main impacts on policy variables and forecast biases seem to derive from entailed changes in the series location; see, for instance, Clements and Hendry (1998). Juxtaposing the condition $\Phi' \rho_t = \mathbf{0}_{(r \times 1)}$ in (7) to $\Phi' \pi_t = \mathbf{0}_{(r \times 1)}$ in (11) then leads to the following distinction:

Definition 11 *If there exists an $(n \times r)$ matrix Φ of rank $r < n$ such that $\Phi' \rho_t = \mathbf{0}_{(r \times 1)} \forall t \in T$ in (7), then the transformations $\Phi' \mathbf{E}[\mathbf{x}_t - \rho_0]$ are said to be unconditional co-breaking relationships. If, on the other hand, Φ is such that $\Phi' \pi_t = \mathbf{0}_{(r \times 1)} \forall t \in T$ in (11), then $\Phi' \mathbf{E}[(\mathbf{x}_t - \pi_0) | \mathbf{X}_{t-1}^1]$ are said to be conditional co-breaking relationships.*

It is instructive to decompose the shift vector ρ_t into intercept and trend shifts such that:

$$\mathbf{E}[\mathbf{x}_t - \rho_0] = \rho_t = \rho_{c,t} + \rho_{\ell,t}. \quad (12)$$

Concatenating the values of $\rho_{c,t}$ and those of $\rho_{\ell,t}$ into $(n \times T)$ matrices yields $\mathbf{P}_c = (\rho_{c,1} : \dots : \rho_{c,T})$ and $\mathbf{P}_\ell = (\rho_{\ell,1} : \dots : \rho_{\ell,T})$, respectively. The analysis in Section 2.2.1 now carries over directly to both \mathbf{P}_c and \mathbf{P}_ℓ such that conditions for intercept co-breaking or trend co-breaking are, according to Theorem 2, $\text{rk}(\mathbf{P}_c) < n$, and $\text{rk}(\mathbf{P}_\ell) < n$. The following definition summarizes.

Definition 12 *If there exists an $(n \times r)$ matrix Φ of rank $r < n$ such that $\Phi' \rho_{c,t} = \mathbf{0}_{(r \times 1)}$ or $\Phi' \rho_{\ell,t} = \mathbf{0}_{(r \times 1)}$, $\forall t \in T$ in (12), then the transformations $\Phi' \mathbf{E}[\mathbf{x}_t - \rho_0]$ are said to be intercept, or trend, co-breaking relationships, respectively.*

This definition is easily generalized to other forms of location shifts. To see this, decompose ρ_t into the sum of p deterministic terms, $\rho_t = \sum_{i=1}^p \rho_{i,t}$ or, in full-sample notation, $\mathbf{P} = \sum_{i=1}^p \mathbf{P}_i$. Suppose that each $\rho_{i,t}$ in fact has only $k_i < T$ distinct values θ_i , $i \in \{1, \dots, p\}$, such that, in analogy to (9), $\mathbf{P} = \sum_{i=1}^p \theta_i \mathbf{D}_i$. Further, let \mathcal{I} be any subset of $\{1, \dots, p\}$ and \mathcal{I}_c its complement. Then, concatenate the θ_i into $\theta_{\mathcal{I}}$ and $\theta_{\mathcal{I}_c}$ such that $\theta_{\mathcal{I}}$ is $(n \times k_{\mathcal{I}})$ and $\theta_{\mathcal{I}_c}$ is $(n \times k_{\mathcal{I}_c})$, where $k_{\mathcal{I}} = \sum_{i \in \mathcal{I}} k_i$ and $k_{\mathcal{I}_c} = \sum_{i \in \mathcal{I}_c} k_i$. As a consequence, \mathbf{P} may be written as

$$\mathbf{P} = \theta_{\mathcal{I}} \mathbf{D}_{\mathcal{I}} + \theta_{\mathcal{I}_c} \mathbf{D}_{\mathcal{I}_c} \quad (13)$$

where $\mathbf{D}_{\mathcal{I}}$ and $\mathbf{D}_{\mathcal{I}_c}$ are of dimension $(k_{\mathcal{I}} \times T)$ and $(k_{\mathcal{I}_c} \times T)$, respectively. Imposing the reduced-rank restriction of (10) on $\theta_{\mathcal{I}}$ it is clear that $\Phi = \zeta_{\perp}$ are co-breaking vectors for the deterministic terms in $\mathbf{D}_{\mathcal{I}}$.

Finally, examine more closely the rôle played by the initial parameterization $E[\mathbf{x}_0] = \rho_0$. It was defined above to be a constant-parameter polynomial in t so as to abstract from the ‘predictable’ component of $E[\mathbf{x}_0]$ and, instead, focus attention on deterministic shifts, i.e. changes in the expected value of \mathbf{x}_t at time $t \in \mathcal{T}$. As an example, ρ_0 was defined as $\rho_0 = \rho_c + \rho_{\ell} t$ such that ρ_c is a constant-parameter baseline intercept and ρ_{ℓ} is a constant-parameter baseline trend in the deterministic component. The parameter vector ρ_t would then comprise deterministic shifts in those two components, as in (12). Re-write, then, (7) as:

$$E[\mathbf{x}_t] = \rho_0 + \rho_t = \varrho_t \quad (14)$$

such that the expected value of $E[\mathbf{x}_t]$ is a piecewise polynomial in t , i.e., a generalized spline function. The concepts outlined in section 2.2.1 now apply to (14): in particular, the following is a direct extension of Definition 1:

Definition 13 An $(n \times r)$ matrix Φ such that $\Phi' \varrho_t = \mathbf{0}_{(r \times 1)} \forall t \in \mathcal{T}$ is said to be spline co-breaking for $\{\mathbf{x}_t\}$, with the result that the spline co-breaking relationships have mean zero, so $E[\Phi' \mathbf{x}_t] = \mathbf{0}_{(r \times 1)}, \forall t \in \mathcal{T}$.

Moreover, following Theorem 2, a necessary and sufficient condition for spline co-breaking to occur is that the $(n \times T)$ matrix $(\varrho_1 : \dots : \varrho_T)$ has reduced row rank. The terminology in Definition 13 is intended to take account of the fact that it is not only the shifts ρ_t that are cancelled by Φ but the entire spline function ϱ_t . Chapman and Ogaki (1993) and Hatanaka and Yamada (2003), *inter alia*, consider similar settings, and call $\Phi' \mathbf{x}_t$ co-trending relationships. It seems more natural in empirical modelling, and especially in policy analysis and forecasting exercises, to examine location shifts, where there is no need for $\Phi' \mathbf{x}_t$ to have mean zero. Thus, we refer to the cancellation of location, or deterministic, shifts as co-breaking following Hendry (1996).

2.3 Cointegration co-breaking

This section considers two special instances of co-breaking arising naturally in a cointegrated vector equilibrium-correction model (VEqCM). To that end, recall the n -dimensional conditional process $\{\mathbf{x}_t | \mathbf{X}_{t-1}^1\}$ in equation (11) of the previous section, and define $\varpi_t = \pi_0 + \pi_t$ by analogy with ϱ_t in (14). The entire distribution may then be specified as:

$$\mathbf{x}_t | \mathbf{X}_{t-1}^1 \sim N_n(\varpi_t, \Sigma), \quad (15)$$

where $\varpi_t = E[\mathbf{x}_t | \mathbf{X}_{t-1}^1]$ is the conditional mean and $\Sigma = V[\mathbf{x}_t | \mathbf{X}_{t-1}^1]$ the conditional variance. More concretely, assume for simplicity that $\mathbf{X}_{t-1}^1 = \sigma(\mathbf{x}_{t-1}, \mathbf{x}_{t-2})$ and that $\varpi_t = \varpi$, such that the data generation process (DGP) over the period $\mathcal{T} = \{1, \dots, T\}$ is given by the second-order vector autoregressive process (VAR):

$$\mathbf{x}_t = \varpi + \Pi_1 \mathbf{x}_{t-1} + \Pi_2 \mathbf{x}_{t-2} + \epsilon_t \quad (16)$$

where $\epsilon_t | \mathbf{X}_{t-1}^1 \sim \text{IN}_n[\mathbf{0}, \Sigma]$ denotes an independent normal error. In (16), none of the roots of the autoregressive polynomial $\Pi(z)$ lies inside the unit circle. The DGP is assumed integrated of order unity, and satisfies $r < n$ cointegration relations such that $\Pi_1 + \Pi_2 - \mathbf{I}_n = \alpha\beta'$, where the following assumptions are made: first, α and β are $(n \times r)$ matrices of rank r such that α_\perp and β_\perp are $(n \times (n-r))$ matrices of rank $(n-r)$ with $\alpha'\alpha_\perp = \beta'\beta_\perp = \mathbf{0}_{(r \times (n-r))}$. Secondly, $\text{rk}(\alpha'_\perp \Psi \beta_\perp) = (n-r)$, where $\Psi = \mathbf{I}_n - \Psi_1$, with $\Psi_1 = -\Pi_2$, is the mean-lag matrix, to rule out the possibility that $\mathbf{x}_t \sim \text{I}(2)$. Then (16) can be reparametrized as the VEqCM:

$$\Delta \mathbf{x}_t = \varpi + \alpha\beta' \mathbf{x}_{t-1} + \Psi_1 \Delta \mathbf{x}_{t-1} + \epsilon_t \quad (17)$$

where $\Delta \mathbf{x}_t$ and $\beta' \mathbf{x}_t$ are $\text{I}(0)$. Pre-multiplying the intercept ϖ by the identity $\mathbf{I}_n = \beta_\perp (\alpha'_\perp \beta_\perp)^{-1} \alpha_\perp + \alpha (\beta' \alpha) \beta'$, where $\alpha_\perp \neq \beta$, yields $\varpi = \gamma - \alpha\mu$, where $\gamma = \beta_\perp (\alpha'_\perp \beta_\perp)^{-1} \alpha_\perp \varpi$ is $(n \times 1)$ and $\mu = -(\beta' \alpha) \beta' \varpi$ is $(r \times 1)$, and $\beta' \gamma = \mathbf{0}$ by construction. The VEqCM in deviations about means is therefore:

$$\Delta \mathbf{x}_t - \gamma = \alpha (\beta' \mathbf{x}_{t-1} - \mu) + \Psi_1 \Delta \mathbf{x}_{t-1} + \epsilon_t \quad (18)$$

where the system grows at the unconditional rate $\text{E}[\Delta \mathbf{x}_t] = \gamma$ with long-run solution $\text{E}[\beta' \mathbf{x}_t] = \mu$. Although the decomposition of ϖ is not orthogonal (since $\gamma' \alpha \mu \neq \mathbf{0}$), (18) is isomorphic to (17).

Any of the parameters α , β , γ , or μ could be subject to change, but we focus on the last two such that $\Pi_1 + \Pi_2 = \mathbf{I}_n + \alpha\beta'$ remains constant. Thus, suppose that up until time period $t = T$, the intercepts ϖ (and hence the equilibrium means and the unconditional growth rates) are determined by the relation $\pi_0 = \gamma_0 - \alpha\mu_0$ while, for $t > T$, they are subject to location shifts such that $\pi_t = \gamma_t - \alpha\mu_t$. Thus, the post-break intercept is given by $\varpi_t = \pi_0 + \pi_t = (\gamma_0 + \gamma_t) - \alpha(\mu_0 + \mu_t)$ such that, for $t > T$, the VEqCM in (18) is:

$$\Delta \mathbf{x}_t = (\gamma_0 + \gamma_t) + \alpha (\beta' \mathbf{x}_t - (\mu_0 + \mu_t)) + \Psi_1 \Delta \mathbf{x}_{t-1} + \epsilon_t = \widetilde{\Delta \mathbf{x}}_t + (\gamma_t - \alpha\mu_t), \quad (19)$$

where $\widetilde{\Delta \mathbf{x}}_t$ is the constant-parameter value of $\Delta \mathbf{x}_t$ in (18), and $\gamma_t - \alpha\mu_t$ is the composite intercept shift. In terms of the system in (19), consider the $(n \times r)$ matrix Φ and form the r linear combinations $\Phi' \Delta \mathbf{x}_t$:

$$\Phi' \Delta \mathbf{x}_t = \Phi' \widetilde{\Delta \mathbf{x}}_t + \Phi' (\gamma_t - \alpha\mu_t). \quad (20)$$

Then, r -dimensional equilibrium-mean co-breaking requires that $\Phi' \alpha\mu_t = \mathbf{0}_{(r \times 1)}$, $\forall t > T$, while r -dimensional drift co-breaking requires $\Phi' \gamma_t = \mathbf{0}_{(r \times 1)}$, $\forall t > T$. Two instances of Φ meet these conditions naturally.

First, the orthogonal complement of α , namely α_\perp , satisfies $\alpha'_\perp \alpha = \mathbf{0}_{((n-r) \times r)}$ by definition. Thus, in (20) choose $\Phi = \alpha_\perp$ to have

$$\alpha'_\perp \Delta \mathbf{x}_t = \alpha'_\perp \gamma_0 + \alpha'_\perp \Psi_1 \Delta \mathbf{x}_{t-1} + \alpha'_\perp \epsilon_t + \alpha'_\perp \gamma_t. \quad (21)$$

This result is stated in the following theorem.

Theorem 14 ‘Common trends’ are equilibrium-mean co-breaking.

Since α_\perp is the ‘selector’ of the equations which are not dependent on EqCMs, this result is close to explaining the effectiveness of differencing as a ‘solution’ to equilibrium-mean shifts as in Clements and Hendry (1998). Second differencing, or further co-breaking, however, would be required to remove the shift in the drift coefficient as well. To illustrate that this is feasible, suppose that $\alpha' = (\alpha'_{(r \times r)}; \mathbf{0}'_{((n-r) \times r)})$, such that an orthogonal complement α_\perp satisfying $\alpha' \alpha_\perp = \mathbf{0}$ is $\alpha'_\perp = (\mathbf{0}'_{(r \times (n-r))}; \mathbf{I}_{n-r})$. Then the co-breaking relationship in (21) would effectively be the lower block of $(n-p)$ equations, denoted by $\mathbf{x}_{b,t}$:

$$\Delta \mathbf{x}_{b,t} = \gamma_{0,b} + \Psi_{1,b} \Delta \mathbf{x}_{t-1} + \epsilon_{b,t} + \gamma_{t,b}. \quad (22)$$

It is now conceivable that there exists some linear combinations $\Phi' \Delta \mathbf{x}_{b,T+1}$ that are drift as well as equilibrium-mean break free. In particular, when $n - r > 1$ such a combination will always exist for a one-off change $\gamma_{t,b} \neq \mathbf{0}_{((n-r) \times 1)}$. Consequently, the drift and equilibrium-mean co-breaking vectors would be given by $\alpha_{\perp} \Phi$.

Secondly, recall that $\gamma \in c(\beta_{\perp})$ such that $\beta' \gamma = \mathbf{0}$, implying that the cointegrating vectors are trend free. As long as $\beta' \alpha \neq \mathbf{0}_{(r \times r)}$, therefore, choosing in (20) $\Phi = \beta$ yields:

$$\beta' \Delta \mathbf{x}_t = \beta' \alpha (\beta' \mathbf{x}_t - \mu_0) + \beta' \Psi_1 \Delta \mathbf{x}_{t-1} + \beta' \epsilon_t - \beta' \alpha \mu_t, \quad (23)$$

eliminating the shifts in the drift parameter. The following theorem summarizes.

Theorem 15 *The cointegration vector is drift co-breaking.*

3 Co-breaking in empirical research

The concept of co-breaking is relevant to many aspects of empirical research. Three such instances will be considered in this section. In section 3.1, the relationship between co-breaking and super exogeneity in policy analysis is examined, with the discussion drawing on Engle and Hendry (1993), Banerjee, Hendry and Mizon (1996) and Hendry and Mizon (1998). Secondly, building on [recent advances in](#) Hendry (2003), the rôle of co-breaking in impulse-response analysis is investigated in section 3.2. Finally, in section 3.3, we consider the [importance of co-breaking in forecasting macroeconomic time series](#).

3.1 Policy analysis and super exogeneity

Recall the unconditional mean of the n -dimensional independent Normal random variable \mathbf{x}_t as given in equation (14) of section 2.2.3, and assume its entire distribution is:

$$\mathbf{x}_t \sim \text{IN}_n [\boldsymbol{\varrho}_t, \boldsymbol{\Sigma}_t], \quad (24)$$

where the mean vector $\boldsymbol{\varrho}_t$ and the covariance matrix $\boldsymbol{\Sigma}_t$ are not constant over time (compare (24) to the conditional distribution in (15) of the previous section to clarify the notation). Suppose that an economic theory involves the two random variables \mathbf{y}_t and \mathbf{z}_t , of dimension $(n_1 \times 1)$ and $(n_2 \times 1)$, $n_1 + n_2 = n$, respectively, obtained by appropriate partitioning of \mathbf{x}_t :

$$\begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} \sim \text{IN}_n \left[\begin{pmatrix} \boldsymbol{\varrho}_{1,t} \\ \boldsymbol{\varrho}_{2,t} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11,t} & \boldsymbol{\Sigma}_{12,t} \\ \boldsymbol{\Sigma}'_{12,t} & \boldsymbol{\Sigma}_{22,t} \end{pmatrix} \right]. \quad (25)$$

In particular, the theory stipulates that the mean of \mathbf{y}_t is proportionally related to that of \mathbf{z}_t :

$$\boldsymbol{\varrho}_{1,t} = \Upsilon_t \boldsymbol{\varrho}_{2,t} \quad (26)$$

where Υ_t is the parameter of interest. Although (26) does not involve a constant term, the subsequent analysis is easily modified accordingly. An example of (26) would be the Permanent Income Hypothesis (PIH) as in [Friedman \(1957\)](#), according to which $\boldsymbol{\varrho}_{1,t}$ and $\boldsymbol{\varrho}_{2,t}$ denote permanent consumption and permanent income, respectively. The basic issue of the present analysis is to ask under what conditions the theory in (26) may be investigated by the empirical model:

$$\mathbf{y}_t = \Upsilon \mathbf{z}_t + \epsilon_t, \quad (27)$$

where $\epsilon_t \sim \text{IN}_n [\mathbf{0}, \boldsymbol{\Omega}]$, for $t = 1, \dots, T$.

To answer that question it is necessary to derive the expectation of \mathbf{y}_t conditional on \mathbf{z}_t , given their joint distribution in (25) and the relationship in (26). Following Engle and Hendry (1993), the conditional mean is:

$$E[\mathbf{y}_t|\mathbf{z}_t] = \mathbf{B}_t + \mathbf{\Gamma}_t\mathbf{z}_t \quad (28)$$

where $\mathbf{\Gamma}_t = \mathbf{\Sigma}_{12,t}\mathbf{\Sigma}_{22,t}^{-1}$ and $\mathbf{B}_t = \boldsymbol{\varrho}_{1,t} - \mathbf{\Gamma}_t\boldsymbol{\varrho}_{2,t}$, while the conditional variance is denoted by $\mathbf{\Omega}_t = \mathbf{\Sigma}_{11,t} - \mathbf{\Sigma}_{12,t}\mathbf{\Sigma}_{22,t}^{-1}\mathbf{\Sigma}_{21,t}$. Incorporating (26) in (28) yields $\mathbf{B}_t = (\mathbf{\Upsilon}_t - \mathbf{\Gamma}_t)\boldsymbol{\varrho}_{2,t}$ or rearranging:

$$E[\mathbf{y}_t|\mathbf{z}_t] = \mathbf{\Upsilon}_t\mathbf{z}_t + (\mathbf{\Gamma}_t - \mathbf{\Upsilon}_t)(\mathbf{z}_t - \boldsymbol{\varrho}_{2,t}). \quad (29)$$

Thus, the parameters of the conditional and marginal densities respectively are $\phi_{1,t} = (\mathbf{B}_t : \mathbf{\Gamma}_t : \mathbf{\Omega}_t^\epsilon)'$ and $\phi_{2,t} = (\boldsymbol{\varrho}_{2,t} : \mathbf{\Sigma}_{22,t}^\epsilon)'$, where $(\cdot)^\epsilon$ denotes column vectoring, retaining only non-redundant elements.

Three conditions must be satisfied for (27) to be a constant-parameter conditional empirical model:

- (a) $\mathbf{\Upsilon}_t = \mathbf{\Gamma}_t$,
- (b) $\phi_{1,t}$ is invariant to $\phi_{2,t}$,
- (c) $\mathbf{\Gamma}_t = \mathbf{\Gamma}$ and $\mathbf{\Omega}_t = \mathbf{\Omega}$, $\forall t$.

To see this, consider equation (29): Setting up an empirical model of \mathbf{y}_t given \mathbf{z}_t and $(\mathbf{z}_t - \boldsymbol{\varrho}_{2,t})$ would be fraught with perils since the two parameter sets $\phi_{1,t}$ and $\phi_{2,t}$ are not variation-free. This is essentially due to the cross-equation restriction given by the theory model in (26). Variation-freeness, however, may be ensured by imposing condition (a), implying $\mathbf{B}_t = \mathbf{0}_{(n_1 \times 1)}$ and thus $E[\mathbf{y}_t|\mathbf{z}_t] = \mathbf{\Upsilon}_t\mathbf{x}_t$. The marginal variable \mathbf{z}_t is thus weakly exogenous for $\phi_{1,t}$ and may safely be disregarded in the empirical analysis of \mathbf{y}_t given \mathbf{z}_t without loss of information. Condition (b) stipulates that $\phi_{1,t}$, i.e., $\mathbf{\Upsilon}_t$ and $\mathbf{\Omega}_t$, must not change in response to changes in $\phi_{2,t}$. This is crucial since, for instance, $\mathbf{\Gamma}_t$ is a function of $\mathbf{\Sigma}_{22,t}$ by definition, and if changes in the latter entailed changes in the former, analyzing the conditional model would be impossible despite weak exogeneity. In particular, if changes in $\phi_{2,t}$ are interpreted as policy interventions and the elements in $\phi_{1,t}$ as structural parameters it is condition (b) that makes policy analysis feasible; otherwise the conditional model may be prone to the Lucas critique, see Lucas (1976). The conjunction of (a) and (b) hence ensures that \mathbf{z}_t is super-exogenous for $\mathbf{\Upsilon}_t$. Finally, condition (c) demands that $\mathbf{\Sigma}_{12,t}\mathbf{\Sigma}_{22,t}^{-1}$ and $\mathbf{\Sigma}_{11,t} - \mathbf{\Sigma}_{12,t}\mathbf{\Sigma}_{22,t}^{-1}\mathbf{\Sigma}_{21,t}$ be constant over time. Note that this constancy is not synonymous with the invariance in (b), since non-constancy of $\mathbf{\Gamma}_t$ and $\mathbf{\Omega}_t$ could be inherited from, e.g., $\mathbf{\Sigma}_{12,t}$. Condition (c) is not strictly necessary for super exogeneity to hold, but it is not clear how an empirical analysis could proceed if the values of $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ changed over time in an unknown fashion. Conversely, $\mathbf{\Gamma}$ being constant may in fact be an indicator for it being invariant to changes in $\phi_{2,t}$. Note also that the existence of feedback from $\mathbf{Y}_{t-1}^1 = \sigma(\mathbf{y}_1, \dots, \mathbf{y}_{t-1})$ onto \mathbf{z}_t is irrelevant, so the issue of strong exogeneity of \mathbf{z}_t for $\mathbf{\Upsilon}$ does not arise. See Engle, Hendry and Richard (1983) for a thorough discussion of exogeneity.

Given, therefore, that the three conditions (a)–(c) are satisfied, the conditional expectation in (29) becomes:

$$E[\mathbf{y}_t|\mathbf{z}_t] = \mathbf{\Upsilon}\mathbf{z}_t, \quad (30)$$

the conditional variance is $\mathbf{\Omega}$, $\forall t$, and the joint density in (25) takes the form:

$$\begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} \sim \text{IN}_n \left[\begin{pmatrix} \mathbf{\Upsilon}\boldsymbol{\varrho}_{2,t} \\ \boldsymbol{\varrho}_{2,t} \end{pmatrix}, \begin{pmatrix} \mathbf{\Omega} + \mathbf{\Upsilon}\mathbf{\Sigma}_{22,t}\mathbf{\Upsilon}' & \mathbf{\Upsilon}\mathbf{\Sigma}_{22,t} \\ \mathbf{\Upsilon}\mathbf{\Sigma}_{22,t} & \mathbf{\Sigma}_{22,t} \end{pmatrix} \right]. \quad (31)$$

Under these conditions, the parameters $(\boldsymbol{\varrho}_{2,t}, \mathbf{\Sigma}_{22,t})$ can change in the marginal model $\mathbf{z}_t \sim \text{IN}_{n_2}[\boldsymbol{\varrho}_{2,t}, \mathbf{\Sigma}_{22,t}]$, with the agents who determine \mathbf{y}_t continuing to act as they planned, and as it is thought they would. Thus,

the empirical analogue of (30) is indeed given by the model in (27) which will not suffer from the Lucas critique and thereby deliver correct policy conclusion. Conversely, if \mathbf{z}_t is not super exogenous for Υ , then policy scenario analyses may yield misleading conclusions; Hendry (1995) elaborates on this issue.

To emphasize the link between super exogeneity and co-breaking recall that an equivalent requirement to condition (a) above was mentioned to be that $\mathbf{B}_t = \mathbf{0}_{(n_1 \times 1)}$, or $\boldsymbol{\varrho}_{1,t} - \Upsilon_t \boldsymbol{\varrho}_{2,t} = \mathbf{0}_{(n_1 \times 1)}$. In addition, conditions (b) and (c) establish that Υ_t is invariant to changes in $\boldsymbol{\phi}_{2,t} = (\boldsymbol{\varrho}_{2,t} : \Sigma_{22,t}^\epsilon)'$ and is also otherwise constant, so that an implication of conditions (a)–(c) is:

$$\boldsymbol{\varrho}_{1,t} - \Upsilon \boldsymbol{\varrho}_{2,t} = \mathbf{0}_{(n_1 \times 1)}. \quad (32)$$

This is precisely the condition for contemporaneous mean co-breaking established in section 2.2, with $\Phi' = (\mathbf{I}_{n_1} : -\Upsilon)$ being the co-breaking vectors, [see Definition 13 in particular](#). Put differently, although \mathbf{x}_t as a whole follows a multivariate Normal distribution with time-varying parameters $\boldsymbol{\varrho}_t$ and Σ_t , the linear combination $\Phi' \mathbf{x}_t$ has constant parameters Υ and Ω under the same conditions as are necessary to establish the super exogeneity of \mathbf{z}_t . Thus, although the marginal distributions of \mathbf{y}_t and \mathbf{z}_t are not constant, [see \(25\) and \(31\)](#), the conditional model of \mathbf{y}_t given \mathbf{z}_t is, as in (30). It is interesting that, in the present setting, conditions (a)–(c) also imply that, $\forall t$:

$$\Sigma_{12,t} - \Upsilon \Sigma_{22,t} = \mathbf{0}_{(n_1 \times n_2)}, \quad (33)$$

so that the means in (32) are interrelated by the same parameter Υ as the covariances $\Sigma_{12,t}$ are with the variances $\Sigma_{22,t}$. As a consequence, mean co-breaking and what may be called variance co-breaking occur simultaneously.

3.2 Impulse-response analyses

Impulse-response analysis is a widely-used method for evaluating the response of one set of variables to ‘shocks’ in another set of variables; see, e.g., Lütkepohl (1993), Runkle (1987), and Sims (1980). The finding that shifts in the parameters of dynamic reactions are not readily detectable is potentially disastrous for impulse-response analyses of economic policy based on closed systems, such as VARs. Since changes in VAR intercepts and dynamic coefficient matrices may not be detected – even when tested for – yet the full-sample estimates are a weighted average across different regimes, the resulting impulse responses need not represent the policy outcomes that will in fact occur. Indeed, this problem may be exacerbated by specifying VARs in first differences (as often occurs), since deterministic factors play a small role in such models: benefits in winning forecasting competitions are paid for by losses in coherent policy evaluation. Moreover, this is only one of a sequence of drawbacks to using impulse responses from models to evaluate policy, as emphasized over recent years: see, *inter alia*, Banerjee *et al.* (1996), Ericsson, Hendry and Mizon (1998a), and Hendry and Mizon (1998).

The results of impulse-response analyses are well known not to be invariant to the ordering of variables: yet ‘avoiding’ that problem by reporting orthogonalized impulse responses violates weak exogeneity for most orderings. Specifying a variable to be weakly or strongly exogenous alters the impulse responses, irrespective of whether or not that variable actually has the appropriate exogeneity status.

As an illustration, partition the n -dimensional \mathbf{x}_t as in section 3.1 above, assume that Σ is not time-variant, and suppose for simplicity that $n_1 = n_2 = 1$ such that y_t and z_t are scalar random variables; cf. equations (24) and (25). The objective of the analysis is to examine the effect of an ‘impulse’ to z_t on y_t , with $\Sigma_{12} \neq 0$. There are two ‘Choleski’ factorizations, and we take the first, based on the order in which the variables appear in \mathbf{x}_t . The conditional-marginal factorization is:

$$\begin{pmatrix} y_t | z_t \\ z_t \end{pmatrix} \sim \text{IN}_2 \left[\begin{pmatrix} \boldsymbol{\varrho}_{1,t} + \Gamma (z_t - \boldsymbol{\varrho}_{2,t}) \\ \boldsymbol{\varrho}_{2,t} \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} - \Gamma \Sigma_{yz} & 0 \\ 0 & \Sigma_{zz} \end{pmatrix} \right] \quad (34)$$

where $\Gamma = \Sigma_{yz}/\Sigma_{zz}$, compare the discussion surrounding equation (28). A perturbation to the $\{z_t\}$ process is not uniquely identifiable as deriving from a change to $\varrho_{2,t}$ or to the ‘error’ $z_t - \varrho_{2,t} = \epsilon_{2,t}$, although most impulse-response analyses ignore the former possibility. However, the effect on y_t need not be the same for these two possible sources.

To see the problem, let the underlying economics derive from agents’ optimization behaviour as the constant relationship $\varrho_{1,t} = \Upsilon \varrho_{2,t}$, an example of which is the PIH in (26) above. In other words, y_t and z_t co-break. Then from (34):

$$\begin{pmatrix} y_t | z_t \\ z_t \end{pmatrix} \sim \text{IN}_2 \left[\begin{pmatrix} \Upsilon \varrho_{2,t} + \Gamma \epsilon_{2,t} \\ \varrho_{2,t} \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} - \Gamma \Sigma_{yz} & 0 \\ 0 & \Sigma_{zz} \end{pmatrix} \right]. \quad (35)$$

Thus, unless $\Upsilon = \Gamma$, it matters which ‘part’ of the $\{z_t\}$ process is perturbed. But $\Upsilon = \Gamma$ is the condition for the weak exogeneity of z_t for Γ in the conditional model, see section 3.1, so unless weak exogeneity holds, impulse-response outcomes are not unique. Moreover, at most one direction is meaningful. The converse Choleski factorization is

$$\begin{pmatrix} y_t \\ z_t | y_t \end{pmatrix} \sim \text{IN}_2 \left[\begin{pmatrix} \varrho_{1,t} \\ \varrho_{2,t} + \Xi (y_t - \varrho_{1,t}) \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} & 0 \\ 0 & \Sigma_{zz} - \Upsilon \Sigma_{yz} \end{pmatrix} \right] \quad (36)$$

where $\Xi = \Sigma_{yz}/\Sigma_{yy}$. When, however, $\Upsilon = \Gamma$ then $\Xi \neq 1/\Gamma$ by the non-degeneracy of the joint distribution of $\mathbf{x}_t = (y_t : z_t)'$, so an incorrect reaction will be inferred.

The literature on ‘structural VARs’, see, e.g., Bernanke (1986), and Blanchard and Quah (1989), which also analyzes impulse responses for a transformed system, faces a similar difficulty. The lack of understanding of the crucial role of weak exogeneity in impulse-response analyses is puzzling in view of the obvious feature that any given ‘shock’ to the error and to the intercept are indistinguishable, yet the actual reaction in the economy will be the same only if the means and variances are linked in the same way – which is the weak exogeneity condition in the present setting.

3.3 Forecasting

Perhaps the role of greatest potential importance for co-breaking is in the arena of forecasting. Hendry and Massmann (1999) consider its application, and note the added value of having a block of constant relations even if other aspects of the system being modelled are non-constant, so all variables cannot be easily forecast. Factorize \mathbf{x}_t into $\Phi' \mathbf{x}_t$ and its orthogonal component $\Phi'_\perp \mathbf{x}_t$, where the former are the constant causal links and the latter isolate the non-constant relations. Then $\Phi' \mathbf{x}_{T+h}$ **should be forecastable based on its econometric model and data up until $t = T$** without any of the adaptive modifications, such as intercept corrections or additional differencing, needed following location shifts, as discussed in Hendry (2005). Even though these adaptations will be needed after breaks for $\Phi'_\perp \hat{\mathbf{x}}_{T+h|T}$, there will either be fewer intercept corrections, or less of a loss from unnecessary over-differencing.

Almost all econometric models fall in the class of equilibrium corrections (including regressions, VARs, VEqCMs, and simultaneous systems, as well as GARCH processes for variances), so the most pernicious break is a shift in the equilibrium mean: the data will move location, but the model will revert to the in-built equilibrium. Above, we considered the relation between co-breaking and cointegration, and established that ‘common trends’ based on α_\perp are equilibrium-mean co-breaking, whereas cointegration vectors β are drift co-breaking. Thus, following any shifts in equilibrium means, the unfortunate implication is either they have to be eliminated (which α_\perp induces), or the new equilibrium mean has to be determined rapidly (which intercept corrections can facilitate). Conversely, as noted in the previous section, other parameter shifts are less likely to induce forecast failure.

4 Co-breaking regressions

This section describes an intuitive way of operationalizing co-breaking. The basic idea is that regression models may be used, first, to ascertain whether a given vector of random variables is subject to location shifts and, secondly, to investigate whether the shifts vanish in a linear combination of the variables. These models are henceforth referred to as co-breaking regressions, in analogy to cointegrating regressions in Engle and Granger (1987) and co-feature regressions in Engle and Kozicki (1993). Importantly, co-breaking regressions allow one to investigate whether location shifts in individual variables remain present in a *given number* of linear combinations, be they predicted by economic theory or estimated. They are not concerned with the question of *how many* co-breaking relationships exist among the variables of interest. This issue will be dealt with in section 5.

To formalize the co-breaking regression approach, consider a variant of the distribution of $\mathbf{x}_t | \mathbf{X}_{t-1}^1$ given in (15) of section 2.3, namely $\mathbf{x}_t | \mathcal{F}_t \sim \text{ID}[\boldsymbol{\varpi}_t, \boldsymbol{\Sigma}]$, where \mathcal{F}_t is a general conditioning information set that may contain, for instance, the σ -field \mathbf{X}_{t-1}^1 or a set of exogenous variables \mathbf{w}_t . Note that $\mathbf{x}_t | \mathcal{F}_t$ is not necessarily Normally distributed. The conditional mean is again decomposed into the sum of an initial parameterization and deterministic shifts, i.e. $\boldsymbol{\varpi}_t = \boldsymbol{\pi}_0 + \boldsymbol{\pi}_t$. In line with the discussion of section 2.2.2, the shift vector $\boldsymbol{\pi}_t$ is assumed not to change in every time period but to comprise only $k > n$ distinct values such that $(\boldsymbol{\pi}_1 : \dots : \boldsymbol{\pi}_T) = \boldsymbol{\kappa} \mathbf{D}$, where $\boldsymbol{\kappa}$ is $(n \times k)$ and $\mathbf{D} = (\mathbf{d}_1 : \dots : \mathbf{d}_T)$ is $(k \times T)$; cf. equation (9). A linear regression model for \mathbf{x}_t may now be formulated as

$$\mathbf{x}_t = \boldsymbol{\pi}_0 + \boldsymbol{\kappa} \mathbf{d}_t + \boldsymbol{\delta} \mathbf{w}_t + \boldsymbol{\varepsilon}_t \quad (37)$$

where $\mathbf{w}_t \in \mathcal{F}_t$ and $\boldsymbol{\varepsilon}_t | \mathcal{F}_t \sim \text{IID}(\mathbf{0}_{(n \times 1)}, \boldsymbol{\Sigma})$. Assuming that $\text{rk}(\boldsymbol{\kappa}) = n - 1$, then $\boldsymbol{\kappa}$ may be decomposed into the product $\boldsymbol{\kappa} = \boldsymbol{\xi} \boldsymbol{\eta}'$, see (10), and there exists an $(n \times 1)$ vector $\boldsymbol{\xi}_\perp$ such that $\boldsymbol{\xi}'_\perp \boldsymbol{\xi} = \mathbf{0}_{(1 \times (n-1))}$. Thus, the shifts will no longer be present in the linear combination $\boldsymbol{\xi}'_\perp \mathbf{x}_t = \boldsymbol{\xi}'_\perp \boldsymbol{\pi}_0 + \boldsymbol{\xi}'_\perp \boldsymbol{\delta} \mathbf{w}_t + \boldsymbol{\xi}'_\perp \boldsymbol{\varepsilon}_t$. Put differently, the co-breaking regression:

$$y_t = \boldsymbol{\xi}'_{\perp,1} \mathbf{z}_t + \tilde{\boldsymbol{\pi}}_0 + \tilde{\boldsymbol{\delta}} \mathbf{w}_t + \tilde{\boldsymbol{\varepsilon}}_t \quad (38)$$

will be free of structural breaks. In (38), \mathbf{x}_t has been partitioned into a scalar component y_t and an $(n - 1)$ -dimensional vector \mathbf{z}_t , while the corresponding components of $\boldsymbol{\xi}_\perp$ are $(\xi_{\perp,0} : \boldsymbol{\xi}'_{\perp,1})'$ and $\xi_{\perp,0}$ has been normalized to unity; the remaining quantities are defined as, for instance, $\tilde{\boldsymbol{\pi}}_0 = \boldsymbol{\xi}'_\perp \boldsymbol{\pi}_0$.

In practice, the co-breaking regression procedure may be implemented in two steps: First, test whether the k shifts \mathbf{d}_t are in fact present in every one of the n components of \mathbf{x}_t , that is, estimate a regression model such as (37) and test for the significance of $\boldsymbol{\kappa}$. Secondly, augment the conditional model of y_t on \mathbf{z}_t in (38) by \mathbf{d}_t and test whether the shifts are now insignificant, with the co-breaking vector $\boldsymbol{\xi}_\perp$ either estimated or imposed.

A number of specific implementations of this general procedure have been suggested in the literature: see Chapman and Ogaki (1993), Hendry and Mizon (1998) and Morana (2002). These approaches differ in three main aspects. First, the conditional distribution $\mathbf{x}_t | \mathcal{F}_t$ is specified with respect to different information sets, resulting, correspondingly, in conditional or unconditional co-breaking relationships. Secondly, the definition of the shifts \mathbf{d}_t varies and, thirdly, the models place different restriction upon the dependence between y_t and \mathbf{z}_t , and thus different estimators and tests are employed to make inference in the conditional model in (38). Details of the three models will be provided in sections 4.1–4.3, emphasizing the generality of the regression approach to co-breaking, before they are assessed in section 4.4.

4.1 A spline function model

Setting $\mathcal{F}_t = \emptyset$ and thus effectively considering the unconditional distribution $\mathbf{x}_t \sim \text{ID}[\boldsymbol{\varrho}_t, \boldsymbol{\Sigma}]$, Chapman and Ogaki (1993) specify the regression model in (37) as the components model $\mathbf{x}_t = \boldsymbol{\varrho}_t + \boldsymbol{\varepsilon}_t$ or:

$$\mathbf{x}_t = \boldsymbol{\rho}_0 + \boldsymbol{\theta} \mathbf{d}_t + \boldsymbol{\varepsilon}_t, \quad (39)$$

where $\boldsymbol{\varrho}_t = \boldsymbol{\rho}_0 + \boldsymbol{\rho}_t = \boldsymbol{\rho}_0 + \boldsymbol{\theta} \mathbf{d}_t$ is a spline function. If, for instance, the spline is of order $s = 2$, then $\boldsymbol{\rho}_0$ is the sum of a baseline intercept $\boldsymbol{\rho}_{0,0}$, a baseline linear trend $\boldsymbol{\rho}_{1,0}$, and a baseline quadratic trend $\boldsymbol{\rho}_{2,0}$. Moreover, with q location shifts in each of the s components of $\boldsymbol{\rho}_t$ at times $T_1 < \dots < T_q$, the sample is effectively divided into $q + 1$ sub-samples, each potentially having a segmented trend of up to order s . As a consequence:

$$\boldsymbol{\theta} \mathbf{d}_t = \boldsymbol{\theta}_0 \mathbf{d}_{0,t} + \boldsymbol{\theta}_1 \mathbf{d}_{1,t} + \dots + \boldsymbol{\theta}_s \mathbf{d}_{s,t} \quad (40)$$

where $\mathbf{d}_{i,t} = (t_{T_1}^i : \dots : t_{T_q}^i)'$ is a $(q \times 1)$ vector with typical element $t_{T_j}^i = (t - T_j)^i 1_{\{t > T_j\}}$ while $\boldsymbol{\theta}_i = (\boldsymbol{\theta}_{i,1} : \dots : \boldsymbol{\theta}_{i,q})$ is $(n \times q)$, for $i \in \{0, 1, \dots, s\}$ and $j \in \{1, \dots, q\}$. It follows that $\boldsymbol{\theta}$ in (39) is of dimension $(n \times (s + 1)q)$ and \mathbf{d}_t is $((s + 1)q \times 1)$. The spline function in (40) is, without any zero restrictions on elements in $\boldsymbol{\theta}$ in (39), discontinuous at every single shift point; see the discussion in Massmann (2003a). Also, Chapman and Ogaki (1993) generalize this setup to spline functions that have different orders s_j in the $q + 1$ sub-samples.

To ascertain the order s of the polynomial, Chapman and Ogaki (1993) suggest fitting a spline of order $s + \varsigma$, $\varsigma > 0$, by means of least squares (LS) and using asymptotic F tests to decide on the significance of the corresponding coefficient vectors. The second step of the procedure consists in **concatenating the $\boldsymbol{\theta}_i$ into $\boldsymbol{\theta}_{\mathcal{I}}$ and $\boldsymbol{\theta}_{\mathcal{I}_c}$ where \mathcal{I} is a subset of $\{1, \dots, (s + 1)q\}$ and \mathcal{I}_c is its complement as in equation (13) of section 2.2.3, in assuming that $\text{rk}(\boldsymbol{\theta}_{\mathcal{I}}) = n - 1$ such that it may be decomposed into $\boldsymbol{\theta}_{\mathcal{I}} = \boldsymbol{\zeta} \boldsymbol{\nu}'$, and in estimating the following variant of the co-breaking regression in (38) by LS:**

$$y_t = \boldsymbol{\zeta}'_{\perp,1} \mathbf{z}_t + \boldsymbol{\zeta}'_{\perp} \boldsymbol{\rho}_0 + \tilde{\boldsymbol{\theta}} \mathbf{d}_t + \tilde{\boldsymbol{\varepsilon}}_t \quad (41)$$

where $\tilde{\boldsymbol{\theta}} = \boldsymbol{\zeta}'_{\perp} \boldsymbol{\theta}$ and $\tilde{\boldsymbol{\varepsilon}}_t = \boldsymbol{\zeta}'_{\perp} \boldsymbol{\varepsilon}_t$. If, using the asymptotic F test, $\boldsymbol{\zeta}_{\perp}$ is shown to be orthogonal to $\boldsymbol{\theta}_{\mathcal{I}}$ such that the corresponding coefficients of \mathbf{d}_t are insignificant, then co-breaking for the elements of \mathbf{d}_t in \mathcal{I} has occurred.

The trending behaviour of y_t , \mathbf{z}_t and \mathbf{d}_t will guarantee consistent estimation of the parameters in (41). Yet a problem arises because the LS approach suggested by Chapman and Ogaki (1993) neglects the endogeneity of \mathbf{z}_t . In the absence of a condition ensuring weak exogeneity of \mathbf{z}_t for $\boldsymbol{\zeta}_{\perp,1}$ and $\tilde{\boldsymbol{\theta}}$, the correlation between \mathbf{z}_t and $\tilde{\boldsymbol{\varepsilon}}_t$ will cause the asymptotic distribution of the least-squares estimator of $\boldsymbol{\zeta}_{\perp,1}$ to depend on nuisance parameters. This is analogous to the asymptotic median bias arising in cointegrating regressions; see Phillips and Loretan (1991) and Phillips (1989). As simulation studies by, for instance, Banerjee, Dolado, Hendry and Smith (1986) have shown, this bias may be severe in small samples and estimation or testing procedures corresponding to those suggested by Phillips and Hansen (1990) and Phillips and Park (1988), respectively, may be needed to ensure correct inference in (41).

Chapman and Ogaki (1993) use their algorithm to re-evaluate the stationarity of the real interest rate, as predicted by the Fisher relation. Using monthly US data for the nominal yield and the inflation rate between 1959-02 and 1990-12, the model in (39) was fitted to the data and the specification of \mathbf{d}_t was found to be a linear trend up until a location shift in $T_1 = 1979-10$ and a quadratic trend in the second sub-sample. In the second step, asymptotic F tests examined the significance of estimates of $\tilde{\boldsymbol{\theta}}$ with a view to reducing the order of the polynomials in t in the two sub-samples. Chapman and Ogaki (1993) circumvent the endogeneity problem by assuming the co-breaking vector to be known: $\boldsymbol{\zeta}_{\perp} = (1 : -1)'$. However, the null hypothesis that the linear combination $\boldsymbol{\zeta}'_{\perp} \mathbf{x}_t$ is stationary around (a) a breaking intercept, thus implying linear and quadratic trend co-breaking, and (b) a constant intercept, this being tantamount to intercept, linear trend and quadratic trend co-breaking, was rejected in both instances.

4.2 A multivariate Normal model

Hendry and Mizon (1998) suggest an alternative approach to the modelling of co-breaking regressions, that is motivated by the general-to-specific methodology as outlined in, for instance, Hendry (1995) and Mizon (1995). The basic idea of this methodology may be described by two modelling stages. At a first stage, a multivariate Normal model is estimated unrestrictedly and considered a valid baseline for subsequent analysis if it explains the salient data features sufficiently well. One such feature would be location shifts that are apparent in constituent elements of the vector of random variables. Once this data congruency is achieved, the second modelling stage proceeds by reducing the general unrestricted model by means of restrictions to yield a statistically acceptable, yet more parsimonious, model which is easier to interpret. An example restriction is that implied by the weak exogeneity of a subset of variables for the parameters of interest, implying that the equations describing these variables within that sub-system may be disregarded without the loss of information. The result is a model of the remaining variables conditional on the weakly exogenous ones.

In the present context, to implement the first stage of this strategy, consider the model in (37) and specify the information set as $\mathcal{F}_t = \mathbf{X}_{t-1}^{t-g}$ such that $\mathbf{w}_t = (\mathbf{x}'_{t-1} : \dots : \mathbf{x}'_{t-g})'$ and $\boldsymbol{\delta} = (\boldsymbol{\Pi}_1 : \dots : \boldsymbol{\Pi}_g)$. The complete system is then estimated by means of maximum likelihood and the shifts \mathbf{d}_t determined in the process. The second stage of the modelling procedure then consists in testing the weak exogeneity of \mathbf{z}_t for the parameters of interest and, if the reduction is valid, in transforming the system into:

$$y_t = \boldsymbol{\xi}_{\perp,1} \mathbf{z}_t + \tilde{\boldsymbol{\pi}}_0 + \tilde{\boldsymbol{\kappa}} \mathbf{d}_t + \sum_{i=1}^g \tilde{\boldsymbol{\Pi}}_i \mathbf{x}_{t-i} + \tilde{\boldsymbol{\varepsilon}}_t, \quad (42)$$

where the transformed parameters and variables are defined analogously to (38) above. If, in addition, \mathbf{z}_t is super exogenous in (42), then fewer dummy variables in \mathbf{d}_t are necessary to describe the data in the conditional model than in the original joint model and evidence for co-breaking has been found: section 3.1 provides a formal account of co-breaking in conditional models and exogeneity conditions.

The feasibility of economic policy analysis using co-breaking regressions is illustrated by Hendry and Mizon (1998), who estimate a bivariate model in money supply and interest rates. In particular, annual UK data ranging from 1871–1993 were available for the following variables: broad money stock M , real net national income Y , the price level P , and the short-term nominal interest rate Rna . With lower case letters denoting the logarithm of the data, the variables of interest were $mpy = m - p - y$ and Rna , where mpy may be interpreted either as the log of inverse velocity or as long-run equilibrium, money demand with unit income elasticity imposed; see Hendry and Ericsson (1991) and Ericsson, Hendry and Prestwich (1998b). For the period between 1871 and 1975, the data set was compiled by Friedman and Schwartz (1982), and for the remaining years by Attfield, Demery and Duck (1995): these data series are also used in the analysis discussed in section 5.3.

The modelling procedure followed the steps outlined at the beginning of this section. First, a bivariate VAR(5) in mpy and Rna , supplemented by dummy variables, was estimated and shown to be data-congruent. Specifically, impulse dummies to account for potential outliers as well as step dummies for ‘policy regimes’ were included in the initial specification; see Hendry and Mizon (1998, Section 9). The only step dummies remaining significant in the data-congruent specification were those for 1876–1913, 1934–1943, and 1984–1993. Next, following results obtained in Ericsson *et al.* (1998b), one cointegration relationship was imposed such that the VAR representation could be mapped into a cointegrated VEqCM form. Since weak exogeneity of ΔRna for the parameters in the Δmpy equation was found, the analysis continued on the basis of the conditional model of Δmpy given ΔRna . Importantly, only one of the three step dummies, viz. that for 1984–1993, remained significant in that model, so that the final parsimonious and data-congruent model is interpreted to be a co-breaking relationship for the other two regimes.

4.3 A Markov-switching model

A third implementation of the general procedure described above is given by Morana (2002) who specifies the general model in (37) as:

$$\mathbf{x}_t = \boldsymbol{\kappa} \mathbf{d}_t + \boldsymbol{\delta} \mathbf{w}_t + \boldsymbol{\varepsilon}_t \quad (43)$$

such that, effectively, $\boldsymbol{\pi}_0 = \mathbf{0}$. The variable \mathbf{d}_t is now a stochastic break process based on a Markov chain, while the variables \mathbf{w}_t are assumed to be weakly exogenous and to lie in the general information set \mathcal{F}_t . The error term $\boldsymbol{\varepsilon}_t$ are defined as before. In particular, every element d_{it} , $i \in \{1, \dots, n\}$, of \mathbf{d}_t is described by a univariate Markov-switching model with state-dependent intercept, i.e. $d_{i,t} = \mu_{i,s_t} + \varepsilon_{i,t}$, where $\varepsilon_{i,t} \sim N[0, \sigma_i^2]$ and s_t denotes the state of μ_i at time t . The number of states corresponds to the number of sub-periods delimited by the q locations shifts in \mathbf{x}_t . The advantage of this representation is that the timing of the breaks need not be assumed to be known with certainty. Instead, the break process $d_{i,t}$ may be approximated by $\widehat{d}_{i,t} = \sum_{j=1}^q \widehat{\rho}_{it,s} \widehat{\mu}_{i,s}$ where $\widehat{\rho}_{it,s}$ is the estimated smoothed probability that the intercept is in state s at time t , while $\widehat{\mu}_{i,s}$ is the estimated intercept for state s ; see Hamilton (1990) and Krolzig (1997) for estimation details.

Given the model in (43), Morana (2002) then uses the feature and common feature tests suggested by Engle and Kozicki (1993) to examine the existence of co-breaking. In particular, the feature test consists in estimating the univariate model $x_{i,t} = \kappa_i \widehat{d}_{i,t} + \delta_i \mathbf{w}_t + \varepsilon_{i,t}$ and in testing the null hypothesis of $H_0: \kappa_i = 0$, for every $i \in \{1, \dots, n\}$. If the estimated break process $\widehat{d}_{i,t}$ is statistically significant in all $x_{i,t}$ then the common feature test is employed to test the null hypothesis of $H_0: \text{rk}(\boldsymbol{\kappa}) = n - 1$ in (43). While Engle and Kozicki (1993) discuss various forms of that test, probably the most intuitive is based on estimating a variant of model (38):

$$y_t = \boldsymbol{\zeta}_{\perp,1} \mathbf{z}_t + \widetilde{\boldsymbol{\delta}} \mathbf{w}_t + \widetilde{\boldsymbol{\varepsilon}}_t \quad (44)$$

by means of 2SLS, with $\{\widehat{\mathbf{d}}_t, \mathbf{w}_t\}$ as the set of instruments, and testing the validity of the $\dim(\mathbf{z}_t) - \dim(\widehat{\mathbf{d}}_t) = 1$ over-identifying restriction in the system; see also Sargan (1958) and Hansen and Singleton (1982) for details of this test in different settings. The attraction in considering model (44) in conjunction with the common feature test is that the endogeneity of \mathbf{z}_t is fully taken into account. If the null hypothesis of a reduced-rank $\boldsymbol{\kappa}$ is not rejected, then Morana (2002) proposes to estimate the break process \mathbf{d}_t in a multivariate model $\mathbf{d}_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t$ in which the states are perfectly correlated across variables.

The objective of the empirical analysis in Morana (2002) is to estimate core inflation for the Eurozone. The data used for that purpose include the monthly harmonized index of consumer prices (HICP) and quarterly nominal M3. The former is unavailable for the Eurozone before 1995 and is extended backwards; the latter is interpolated to yield monthly figures. As a result, both data series range from 1980-01 to 2001-12 and are first-differenced to result in an inflation and a nominal money growth series. With the number of regimes fixed at 3, the estimated break processes for inflation and M3 growth show evidence of 3 sub-periods, that is, two location shifts, one of which takes places at around the end of 1983, the other roughly at the end of 1993. The feature test does not reject the significance of the break process in either series, and the common feature test suggests that real money growth is a co-breaking relationship, where the co-breaking vector is given by $(1 : -1)'$.

4.4 Discussion

This section described how co-breaking regressions may be used to estimate co-breaking relationships, taking their number as given. Co-breaking regressions are analogous to cointegrating regressions, and are effectively a special case of common feature regressions conceived by Engle and Kozicki (1993).

Three approaches to co-breaking regressions were discussed in this section, which differed in three aspects. They consider a stochastic process $\{\mathbf{x}_t\}$ with respect to different information sets, they make recourse to different specifications of the break function, and they handle the endogeneity problem in the regression model in alternative fashions. First, Chapman and Ogaki (1993) [model the unconditional process \$\{\mathbf{x}_t\}\$, assume the deterministic terms to be represented by a piecewise trend polynomial, and avoid the endogeneity issue by assuming the unconditional co-breaking vector to be known](#). Secondly, Hendry and Mizon (1998) [consider a VAR model of \$\mathbf{x}_t\$, use impulse and step dummies to model location shifts, and test the exogeneity of the marginal process before estimating the co-breaking relationship](#). Finally, Morana (2002) [examines a open system of \$\mathbf{x}_t\$ given \$\mathbf{w}_t \in \mathcal{F}_t\$, employs a Markov-switching model to estimate a stochastic break process, and uses instrumental variable estimation to allow for the endogeneity in the conditional co-breaking regression](#).

These three approaches reflect different modelling traditions in econometrics. At the one end of the spectrum, the analysis of Chapman and Ogaki (1993) in terms of polynomial trend functions and ARMA processes is mainly descriptive, and thus in the Box–Jenkins time-series tradition. At the other end, Hendry and Mizon (1998) and Morana (2002) attempt both descriptive and prescriptive data modelling, emphasizing data-driven inference that does not neglect the economic interpretation. The two approaches differ, however, in the degree to which they model the interdependence between the series of interest. Hendry and Mizon (1998) model the multivariate system as a whole while Morana (2002) neglects cross-equation correlation in the first stage and makes interdependence-robust inference at the second stage. Moreover, Hendry and Mizon (1998) establish approximate Normality of the residuals in the system, and thus engage in exact inference, while Morana (2002) relies on the asymptotic distribution of the test for over-identifying restrictions, although the small-sample behaviour of this test is unknown, especially in the presence of weak instruments.

5 Co-breaking rank analysis

In the present section, we relax the assumption that the number of co-breaking relationships is known to consider procedures that allow the estimation of the co-breaking rank. Once that has been inferred from the information contained in the data, a co-breaking regression analysis could in principle be used to estimate parameters of interest. As opposed to co-breaking regressions, however, co-breaking rank analysis is still in its infancy and, to the best of our knowledge, only two models that aim at estimating the order of co-breaking have so far been suggested in the literature. The one by Krolzig and Toro (2002) is based on a stationary vector autoregressive (VAR) model and uses canonical correlations to conduct a co-breaking rank test. This method will be reviewed in section 5.1. The second method, to be discussed in section 5.2, is proposed by Hatanaka and Yamada (2003) and is based on an unobserved components model with an $I(1)$ stochastic component and the co-breaking rank is tested by means of principal component analysis. To improve on these results, we suggest a new model for a co-breaking rank test in section 5.3 that combines aspects of these two methodologies. In particular, we also employ an unobserved components model, albeit with an $I(0)$ stochastic component, and use canonical correlations to perform our rank test. We provide theoretical details of the procedure, Monte Carlo simulation evidence on the rank test performance, and an empirical application that revisits the Stolper–Samuelson theorem.

5.1 A vector autoregressive approach

In their approach to co-breaking rank analysis, Krolzig and Toro (2002) analyze the n -dimensional conditional random variables $\mathbf{x}_t | \mathbf{X}_{t-1}^{t-g} \sim \text{NID}_n[\boldsymbol{\varpi}_t, \Sigma]$ with time varying conditional mean $\boldsymbol{\varpi}_t = \boldsymbol{\pi}_0 + \boldsymbol{\pi}_t =$

$\pi_0 + \kappa d_t$. The corresponding closed-form VAR(g) model in \mathbf{x}_t is:

$$\mathbf{x}_t = \pi_0 + \kappa d_t + \sum_{i=1}^g \Pi_i \mathbf{x}_{t-i} + \varepsilon_t, \quad (45)$$

as, for instance, in Hendry and Mizon (1998); see section 4.2. The term π_0 again comprises the deterministic components that have constant parameter representations while $\pi_t = \kappa d_t$ models q level shifts in ϖ_t , occurring in time periods $T_1 < \dots < T_q$. For convenience, these shifts are assumed to affect the intercept in (45) such that κ is of dimension $(n \times q)$ and d_t is $(q \times 1)$ with typical element $1_{\{t > T_1\}}$. Importantly, the $(n \times n)$ autoregressive coefficient matrices Π_i , $i = 1, \dots, q$, are such that the roots of the VAR polynomial are outside the unit circle so the vector process $\mathbf{x}_t \sim I(0)$.

As before, co-breaking in this setting is synonymous with κ being of reduced row rank $\text{rk}(\kappa) = n - r$, since this allows the decomposing of κ into $\kappa = \xi \eta'$ and the finding of an orthogonal complement ξ_\perp such that the r linear combinations $\xi_\perp' \mathbf{x}_t$, suitably normalized, no longer contain the deterministic shifts d_t . To estimate the rank of κ in (45), Krolzig and Toro (2002) suggest using canonical correlation analysis, similar to the strategy pursued by Johansen (1988) for estimating the cointegrating rank in a VEqCM. Rewrite (45) as the reduced-rank regression model:

$$\mathbf{x}_t = \pi_0 + \kappa d_t + \delta \mathbf{w}_t + \varepsilon_t,$$

where the autoregressive component is captured by $\delta \mathbf{w}_t$. Given the normality of ε_t , the null hypothesis of $H_0: \text{rk}(\kappa) \leq s$ may be tested against the unrestricted alternative of $H_1: \text{rk}(\kappa) = \min(n, q)$ by means of a likelihood ratio (LR) test, and Anderson (1951) shows that the test statistic reduces to the following expression:

$$\text{LR} = -T \sum_{i=s+1}^{\min(n,q)} \log(1 - \hat{\lambda}_i) \xrightarrow{d} \chi^2[(n-s)(q-s)] \quad (46)$$

where $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_{\min(n,q)}$ are the estimated squared partial canonical correlations between \mathbf{x}_t and d_t , corrected for the deterministic terms in π_0 and for \mathbf{w}_t : also see Hotelling (1936) and Bartlett (1938). The asymptotic distribution of LR is derived by Box (1949). Since the rank hypotheses are nested:

$$H_0(s=0) \subset H_0(s \leq 1) \subset \dots \subset H_0(s \leq \min(n, q) - 1) \quad (47)$$

the test procedure commences with the most restrictive hypothesis and continues until the first null cannot be rejected.

Krolzig and Toro (2002) examine the small-sample properties of the LR test by conducting a number of Monte Carlo experiments. Specifically, they consider two generating processes for their data, both being given by a bivariate version of (45) with $q = 2$ breaks in the intercept: DGP1 specifies $\text{rk}(\kappa) = 2$, DGP2 imposes the restriction $\text{rk}(\kappa) = 1$. For a given degree of autocorrelation and given break magnitudes, Krolzig and Toro (2002) test the null hypothesis of $H_0: \text{rk}(\kappa) \leq 1$ for both DGPs. The results may be summarized by saying that although the test is generally oversized and has power problems at $T = 50$, these deficiencies are much alleviated as the sample size increases to $T = 150$. Similar results are obtained by varying the degrees of autocorrelation and the break dates in the DGP.

Given the rank test in (46), Krolzig and Toro (2002) also devise two statistical tests for examining the hypothesis of whether the random variables conditioned upon in the co-breaking combinations $\xi_\perp' \mathbf{x}_t$ are super exogenous for the parameters of interest. Recall from section 3.1 that in addition to the reduced-rank condition on κ such that $\xi_\perp' \pi_t = 0, \forall t$, the crucial requirement for this to occur is that $\xi_\perp = (\mathbf{I}_r : -\Upsilon)$, where $\Upsilon = C(\mathbf{y}_t, \mathbf{z}_t) V(\mathbf{z}_t)^{-1}$. The tests by Krolzig and Toro (2002) have slightly better [small sample](#) properties than the original test for super exogeneity suggested by Engle and Hendry (1993).

5.2 An I(1) unobserved components model

Hatanaka and Yamada (2003) consider [what we call](#) an unobserved components (UC) representation of the n -dimensional random process $\{\mathbf{x}_t\}$:

$$\mathbf{x}_t = \boldsymbol{\varrho}_t + \mathbf{u}_t, \quad (48)$$

where $\boldsymbol{\varrho}_t = E[\mathbf{x}_t]$ is the time-varying unconditional mean of \mathbf{x}_t and \mathbf{u}_t is a mean-zero stochastic component. To refer to (48) as an UC model actually slightly abuses terminology, since $\boldsymbol{\varrho}_t$ is not unobserved. The model is, however, easily generalized to include random break processes $\boldsymbol{\varrho}_t$ as in Morana (2002), see section 4.3, thus warranting this descriptor. The stochastic component \mathbf{u}_t is assumed to follow a general ARIMA process with order of integration of unity such that there exist r^* cointegrating relationships $\boldsymbol{\beta}'\mathbf{u}_t \sim I(0)$, where $\boldsymbol{\beta}$ is $(n \times r^*)$. The deterministic component $\boldsymbol{\varrho}_t$ is generated by [the equilibrium-correction process](#):

$$\Delta \boldsymbol{\varrho}_t = \boldsymbol{\chi} \boldsymbol{\zeta}'_{\perp} \boldsymbol{\varrho}_{t-1} + \boldsymbol{\omega} \mathbf{d}_{0,t} \quad (49)$$

where the parameters $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}_{\perp}$ are $(n \times r)$ and of full rank $\text{rk}(\boldsymbol{\chi}) = \text{rk}(\boldsymbol{\zeta}_{\perp}) = r$ such that the eigenvalues of $(\mathbf{I}_r + \boldsymbol{\zeta}'_{\perp} \boldsymbol{\chi})$ are inside the unit circle. The coefficient matrix $\boldsymbol{\omega}$ is $(n \times q)$ such that $\text{rk}(\boldsymbol{\omega}' \boldsymbol{\chi}_{\perp}) = n - r$. The $(q \times 1)$ vector $\mathbf{d}_{0,t}$ is an ‘elementary trend function’ that takes the role of a disturbance term. Solving (49) for $\boldsymbol{\varrho}_t$ yields:

$$\boldsymbol{\varrho}_t = \boldsymbol{\zeta} \boldsymbol{\nu}' \mathbf{d}_{1,t} + \boldsymbol{\theta}_0 \mathbf{d}_{0,t} + \mathbf{e}_t \quad (50)$$

where the orthogonal complement of $\boldsymbol{\zeta}$, namely $\boldsymbol{\zeta}_{\perp}$, is of dimension $(n \times (n-r))$, $\boldsymbol{\nu}' = (\boldsymbol{\chi}'_{\perp} \boldsymbol{\zeta})^{-1} \boldsymbol{\chi}'_{\perp} \boldsymbol{\omega}$ is $((n-r) \times q)$ and $\boldsymbol{\theta}_0 = \boldsymbol{\chi} (\boldsymbol{\zeta}'_{\perp} \boldsymbol{\chi})^{-1} \sum_{j=0}^{\infty} (\mathbf{I}_r + \boldsymbol{\zeta}'_{\perp} \boldsymbol{\chi})^j \boldsymbol{\zeta}'_{\perp} \boldsymbol{\omega}$ is $(n \times q)$. Finally, $\mathbf{d}_{1,t} = \sum_{s=1}^t \mathbf{d}_{0,s}$, and \mathbf{e}_t is an asymptotically negligible remainder term. If now $\mathbf{d}_{0,t}$ comprises a constant term and indicator variables representing the timing of $(q-1)$ location shifts in time periods $T_1 < \dots < T_{q-1}$ such that $\mathbf{d}_{0,t} = (1 : 1_{\{t \geq T_1\}} : \dots : 1_{\{t \geq T_{q-1}\}})'$, then $\mathbf{d}_{1,t} = (t : (t - T_1) 1_{\{t \geq T_1\}} : \dots : (t - T_{q-1}) 1_{\{t \geq T_{q-1}\}})'$ constitutes a linear trend and trend shifts. From the expression in (50) it is clear that there are $(n-r)$ common deterministic trends $\boldsymbol{\nu}' \mathbf{d}_{1,t}$ in the system while (49) shows that there are r co-trending combinations $\boldsymbol{\zeta}'_{\perp} \boldsymbol{\varrho}_t$. In the ensuing analysis, the possibility that $r = n$ is disallowed since this would imply that there is no deterministic trend and no trend shifts in $\boldsymbol{\varrho}_t$, thereby defeating the purpose of the exercise.

The task set by Hatanaka and Yamada (2003) is to determine the co-breaking rank r as well as to estimate the number of co-trending vectors that are also cointegrating, say r_1 . To that end, they note that the principal components $\hat{c}_{i,t}$, $i = 1, \dots, n$, of \mathbf{x}_t , computed on the basis of the scaled data covariance matrix $\boldsymbol{\Omega} = T^{-3} \sum_{t=1}^T (\mathbf{x}_t - \bar{\mathbf{x}})(\mathbf{x}_t - \bar{\mathbf{x}})'$, where $\bar{\mathbf{x}} = T^{-1} \sum_{t=1}^T \mathbf{x}_t$, have different orders of magnitude, depending on whether their leading term contains a deterministic trend, a unit root or neither of the two. Specifically, defining $a_t \sim O_p(T^c)$ as meaning that $\sum_{t=1}^T a_t^2 \sim O_p(T^c)$ as $T \rightarrow \infty$, the authors show that

$$\hat{c}_{i,t} \sim \begin{cases} O_p(T^{3/2}), & \text{for } i = 1, \dots, n-r \\ O_p(T), & \text{for } i = n-r+1, \dots, n-r_2 \\ O_p(T^{1/2}), & \text{for } i = n-r_2, \dots, n. \end{cases} \quad (51)$$

The number of principal components that belong in each of these three groups is determined by the dimension of $c(\boldsymbol{\zeta})$, $c(\boldsymbol{\zeta}_{\perp,2}) = c(\boldsymbol{\zeta}_{\perp}) \cap c(\boldsymbol{\beta}_{\perp})$ and $c(\boldsymbol{\zeta}_{\perp,1}) = c(\boldsymbol{\zeta}_{\perp}) \cap c(\boldsymbol{\beta})$, respectively.

Given the difference in the principal components’ orders of magnitude, Hatanaka and Yamada (2003) suggest a two-stage procedure of tests for unit roots and deterministic trends to estimate $n-r$ and r_1 , and thereby r and r_2 . At the first stage, each principal component is examined for a unit root. It is shown that both univariate augmented Dickey–Fuller (ADF) tests, see Dickey and Fuller (1979), Said

and Dickey (1984) and Ng and Perron (1995), as well as system tests developed in Johansen (1988, 1991, 1994) remain asymptotically well-behaved despite the non-standard deterministic terms the principal components contain; see also Perron (1989), Inoue (1999) and Johansen *et al.* (2000). That is to say, the null hypothesis of a unit root is rejected with asymptotic probability one when the principal component lies either in $c(\zeta)$ and ζ are cointegrating vectors, or in $c(\zeta_{\perp,1})$. Similarly, the asymptotic acceptance probability of the null is equal to the significance level when the principal component lies either in $c(\zeta)$ and ζ are not cointegrating vectors, or in $c(\zeta_{\perp,2})$. However, the tests are unable to differentiate between either of the two sub cases, neither under the null nor under the alternative.

In order to sub-divide further each of these two categories Hatanaka and Yamada (2003) devise two tests for a deterministic trend, to be used at the second stage of the analysis. The first of these tests uses the fact that the $I(0)$ principal components that lie in $c(\zeta)$, when ζ are cointegrating vectors, and those that lie in $c(\zeta_{\perp,1})$ have different orders of magnitude, namely $O(T^{3/2})$ and $O(T^{1/2})$, respectively. Similarly, the second test relies on the $I(1)$ principal components in $c(\zeta)$, when the columns of ζ are not cointegrating vectors, to have a deterministic trend while those in $c(\zeta_{\perp,2})$ have not. A complication only arises because the asymptotic distribution of the second test statistic depends on $c(\nu')$ under the null. Estimating this quantity by minimizing the residual sum of squares of the reduced-rank regression in (50), with e_t assumed IID, the asymptotic distribution of the second test statistic may be tabulated although it is now data-dependent. Hatanaka and Yamada (2003) show, however, that both tests for a deterministic trend are consistent and have asymptotic power of unity.

As estimates of $n - r$ and r_1 , and thus r_2 , Hatanaka and Yamada (2003) suggest taking the values implied by the first hypothesis in the sequence $H_0(n - r = i \wedge r_1 = j)$, where j takes values $j = 0, \dots, n - i$, for every value $i = 1, \dots, q$. This sequence lists all theoretically consistent hypotheses about the three-way partitioning of x_t 's range space. In particular, the hypothesis $H_0(n - r = i \wedge r_1 = j)$, $i = 1, \dots, q$, and $j = 0, \dots, n - i$, means that $c(\zeta)$ is spanned by the principal components corresponding to the largest i eigenvalues, that $c(\zeta_{\perp,2})$ is spanned by the principal components corresponding to the $n - i - j$ next largest eigenvalues, and $c(\zeta_{\perp,1})$ is spanned by the remaining j principal components. Put differently, H_0 implies that each principal component in $c(\zeta)$ is $O(T^{3/2})$, each in $c(\zeta_{\perp,2})$ is $O_p(T)$ and each in $c(\zeta_{\perp,1})$ is $O_p(T^{1/2})$. Thus, to decide whether or not H_0 is true, each of the principal components needs to be subjected to the unit root and trend tests described above. Hatanaka and Yamada (2003) show that this decision rule rejects all false null hypotheses with probability one provided that T is sufficiently large.

In order to ascertain the merits of the above decision rule, Hatanaka and Yamada (2003) conduct some small-scale Monte Carlo simulations. The DGPs are of dimension $n = 2$, but differ in their values of r_1 and r^* . Since $n - r = 0$ is ruled out by assumption, the first principal component is a basis vector of $c(\zeta)$ by default, and only the second principal component needs classifying into one of $c(\zeta)$, $c(\zeta_{\perp,2})$ and $c(\zeta_{\perp,1})$. Thus, their sequence of hypotheses only consists in subjecting the second principal component to the unit-root null and the null of no deterministic trend. For a sample size of $T = 800$, the empirical rejection frequencies are close to their analytic asymptotic counterparts, under the null as well as under the alternative. For $T = 100$, however, serious small-sample problems become apparent. As an example, in the case of the DGP whose second principal component lies in $c(\zeta)$ and for which ζ is not cointegrating, the unit-root hypothesis is accepted far too often. It may be suspected that this is a reflection of the well-known problem of weak power of unit-root tests in small samples due to the presence of a deterministic trend. Moreover, in the case of the DGP whose second principal component contains a stochastic as well as a deterministic trend, the null hypothesis of no deterministic trend is rejected not often enough. The reason for this could be that the data-dependent critical values for the test are derived using the wrong rank of ν' in the estimation of the reduced-rank regression in (50).

In summary, Hatanaka and Yamada (2003) suggest a sophisticated and ambitious procedure for simultaneously determining the co-breaking rank and the cointegrating rank in the UC model in (48). A

particularly attractive feature is that the deterministic component $\boldsymbol{\varrho}_t$ follows the equilibrium-correction mechanism in (49), rendering unnecessary the assumption that more breaks are needed than the system has variables; see the discussion in section 2.2.2. On the downside, three aspects are unappealing. First, the setup of $\boldsymbol{\varrho}_t$ can only cope with location shifts in the trend component, and it is not clear how intercept breaks may be handled. Secondly, one of the tests for deterministic trend has a non-standard distribution whose critical values need to be tabulated for each given data set. Lastly, the testing procedures are all asymptotic in nature, and the Monte Carlo hints at some unsatisfactory small-sample properties.

5.3 An I(0) unobserved components model

5.3.1 Interpretation and estimation

The procedure we suggest in this section for estimating the co-breaking rank will draw on both Krolzig and Toro (2002) and Hatanaka and Yamada (2003), combining some of the appealing features in those analyses and [aiming to avoid](#) their drawbacks. In particular, we follow the latter authors in using an unobserved components (UC) model for the n -dimensional process \mathbf{x}_t , while we employ a likelihood ratio test based on canonical correlations to test for the co-breaking rank, as do the former authors.

Consider the UC model $\mathbf{x}_t = \boldsymbol{\varrho}_t + \mathbf{u}_t$, where $\boldsymbol{\varrho}_t$ is a general deterministic component and \mathbf{u}_t is a purely stochastic component. The attraction of this model formulation is that it is the process \mathbf{x}_t itself that is modelled, not a linear transformation of it, as in the case of a vector autoregressive representation of \mathbf{x}_t . Thus, $\boldsymbol{\Phi}$ would cancel out all or some constituent parts of the mean of \mathbf{x}_t and would constitute [unconditional co-breaking vectors](#).

The specification of $\boldsymbol{\varrho}_t$ could be as general as a spline function, as in the discussion of Chapman and Ogaki (1993), but, with a view to the empirical analysis below, we will restrict ourselves to considering q shifts in the linear trend of \mathbf{x}_t , occurring at times $T_1 < \dots < T_q$. Hence, we assume that $\boldsymbol{\varrho}_t = \boldsymbol{\rho}_0 + \boldsymbol{\theta}_1 \mathbf{d}_{1,t}$, where $\boldsymbol{\rho}_0$ consists of a constant intercept and a constant linear time trend, and $\boldsymbol{\theta}_1 \mathbf{d}_{1,t}$ models the trend breaks such that $\boldsymbol{\theta}_1$ is $(n \times q)$ and $\mathbf{d}_{1,t} = ((t - T_1)1_{\{T_1+1 \leq t \leq T_2\}} : \dots : (t - T_q)1_{\{T_q+1 \leq t\}})'$ is $(q \times 1)$. This definition of trend shifts is a linear transformation of that used in section 5.2, and allows the coefficient pertaining to the j th element of $\mathbf{d}_{1,t}$, $j \in \{1, \dots, q\}$, to be interpreted as the trend difference relative to the baseline. To avoid the difficulties in distinguishing between deterministic and stochastic trends we assume that the random component of \mathbf{x}_t follows a stationary AR(g) process: $\mathbf{u}_t = \sum_{i=1}^g \boldsymbol{\Pi}_i \mathbf{u}_{t-i} + \boldsymbol{\varepsilon}_t$ where $\boldsymbol{\varepsilon}_t \sim N[0, \boldsymbol{\Sigma}]$, and $\det(\boldsymbol{\Pi}(z)) = 0$ implies $|z| > 1$. The process $\{\mathbf{x}_t\}$ is thus modelled as stationary fluctuations around a segmented trend function, see also Rappoport and Reichlin (1989) and Johansen *et al.* (2000).

Following the discussion in previous sections, a $(n \times r)$ -dimensional matrix $\boldsymbol{\Phi}$ of trend co-breaking vectors such that $\boldsymbol{\Phi}'\boldsymbol{\theta}_1 = 0$ exists if $\text{rk}(\boldsymbol{\theta}_1) = r < \min(n, q)$. As a result, $\boldsymbol{\theta}_1$ may be decomposed into $\boldsymbol{\theta}_1 = \boldsymbol{\zeta}\boldsymbol{\nu}'$, and the trend co-breaking relationships are given by $\boldsymbol{\zeta}'_{\perp} \mathbf{x}_t$. Imposing the reduced-rank restriction on the deterministic component, and re-writing the dynamics of the UC model in terms of observables, yields the isomorphism:

$$\mathbf{x}_t = \sum_{i=1}^g \boldsymbol{\Pi}_i \mathbf{x}_{t-i} + \boldsymbol{\Pi}(1) \boldsymbol{\pi}_0 + \boldsymbol{\zeta}\boldsymbol{\nu}' \mathbf{d}_{1,t} + \sum_{i=1}^g \boldsymbol{\Pi}_i \boldsymbol{\zeta}\boldsymbol{\nu}' \mathbf{d}_{1,t-i} + \boldsymbol{\varepsilon}_t, \quad (52)$$

where all parameters are freely varying. This representation will be referred to as a restricted VAR (RVAR) model due to the restriction on the compound coefficients of the intercept and of $\mathbf{d}_{1,t-i}$, $i = 1, \dots, g$.

There are, however, two problems with estimating this UC model or, equivalently, the RVAR model in (52). First, given $\text{rk}(\boldsymbol{\theta}_1) = r$, the parameters are poorly identified because the regressors $\mathbf{d}_{1,t-i}$, $i = 0, 1, \dots, g$, are highly collinear. Secondly, r is unknown and it is not clear which analytical method

may be able to determine it. Numerical optimization procedures, in turn, are not appealing, given the multitude of reduced-rank conditions in (52).

To circumvent these problems, we suggest the following procedure: the RVAR in (52) is generalized to yield an unrestricted VAR (UVAR) in \mathbf{x}_t which is more easily estimated. Subsequently, by imposing restrictions on the UVAR parameters, the coefficients of interest may be retrieved. To be more specific, consider the UVAR model:

$$\mathbf{x}_t = \sum_{i=1}^g \mathbf{\Pi}_i \mathbf{x}_{t-i} + \psi_0 + \psi_{01} \mathbf{e}_{0,t} + \psi_1 \mathbf{e}_{1,t} + \psi_f \mathbf{f}_t + \varepsilon_t \quad (53)$$

where all parameters are unrestricted for the time being. The regressors $\mathbf{e}_{0,t}$ and $\mathbf{e}_{1,t}$ are of dimension $(q \times 1)$ while \mathbf{f}_t is $(qg \times 1)$, and their respective representative elements are $e_{0,i,t} = 1_{\{T_i+g+1 \leq t \leq T_{i+1}\}}$, $e_{1,i,t} = t 1_{\{T_i+g+1 \leq t \leq T_{i+1}\}}$ and $f_{i,j,t} = 1_{\{t=T_i+j\}}$, where $i = 1, \dots, q$ and $j = 1, \dots, g$. The idea of this representation is to divide the sample into $q + 1$ sub-samples, and model each as potentially having its own intercept and linear trend; see $\mathbf{e}_{0,t}$ and $\mathbf{e}_{1,t}$. Importantly, the first g observations of each sub-sample are regarded as transition periods following a location shift and are dummied out by \mathbf{f}_t . As a result, $\mathbf{e}_{1,t}$ effectively only models the slopes of the broken trend, while the intercept shifts associated with the trend shifts are captured by $\mathbf{e}_{0,t}$ and \mathbf{f}_t . The UVAR in (53) may thus be considered a generalized version of the segmented trend model discussed above.

The UVAR being more general than the RVAR, restrictions are imposed on the parameters of the former to obtain the latter. Using simple algebraic manipulations, the RVAR is obtained by setting $\psi_0 = \mathbf{\Pi}(1) \boldsymbol{\pi}_0$, $\psi_{01} = \psi_{01}(\boldsymbol{\pi}_0, \mathbf{\Pi}_1, \dots, \mathbf{\Pi}_g)$:

$$\psi_1 = \mathbf{\Pi}(1) \boldsymbol{\theta}_1, \quad (54)$$

and $\psi_f = \psi_f(\boldsymbol{\pi}_0, \mathbf{\Pi}_1, \dots, \mathbf{\Pi}_g)$: see Massmann (2003a) for details of the transformations. The coefficient matrices ψ_{01} and ψ_f are complicated functions of RVAR parameters. If, however, ψ_1 is of reduced rank r such that $\psi_1 = \boldsymbol{\zeta} \boldsymbol{\nu}'$, where $\boldsymbol{\zeta}$ and $\boldsymbol{\nu}$ are of full rank r and of dimension $(n \times r)$ and $(q \times r)$, respectively, then from (54), $\boldsymbol{\theta}_1 = \boldsymbol{\zeta} \boldsymbol{\nu}'$ is obtained by imposing $\boldsymbol{\zeta} = \mathbf{\Pi}(1) \boldsymbol{\zeta}$, provided $\mathbf{\Pi}(1)$ is non-singular.

The estimation of the UVAR model in (53) may be conducted along the lines of Krolzig and Toro (2002), see section 5.1. In particular, the sequence of rank hypotheses $H_0: \text{rk}(\psi_1) = r$, for $r = 0, \dots, \min(n, q) - 1$, may be implemented using the LR test based on the canonical correlations between \mathbf{x}_t and $\mathbf{e}_{1,t}$, corrected for the remaining explanatory variables in (53). The test statistic is analogous to LR given in equation (46). Once the co-breaking rank r has been determined, the parameters in the model may be estimated by maximum likelihood.

5.3.2 Monte Carlo experiments

To examine the small-sample distribution of the LR statistic, we carried out a set of Monte Carlo experiments. The artificial data were generated by the UC model in (53), with $n = 3$, $q = 4$, $g = 1$ and $r = 1$. Of particular interest is the impact on LR of varying degrees of autocorrelation in \mathbf{x}_t so that in addition to the baseline case of \mathbf{u}_t being a serially uncorrelated series with $\mathbf{\Pi}_1 = \mathbf{0}$, three magnitudes of positive autocorrelation were examined, namely $\mathbf{\Pi}_1 = \text{diag}(0.3 : 0.2 : 0.1)$, $\mathbf{\Pi}_1 = \text{diag}(0.6 : 0.5 : 0.4)$, and $\mathbf{\Pi}_1 = \text{diag}(0.9 : 0.8 : 0.7)$. The Monte Carlos were recursive with the following sample sizes: $T^* = \{25, 50, 75, 100, 125, 150, 175, 200, 300, 400, 500\}$. The magnitude of the trend breaks was chosen relative to noise in the system, see Hendry and Ericsson (1991) and Ericsson *et al.* (1998b), but not scaled by the sample size. Their timing was proportional to the sample size: $\{0.2T, 0.4T, 0.6T, 0.8T\}$.

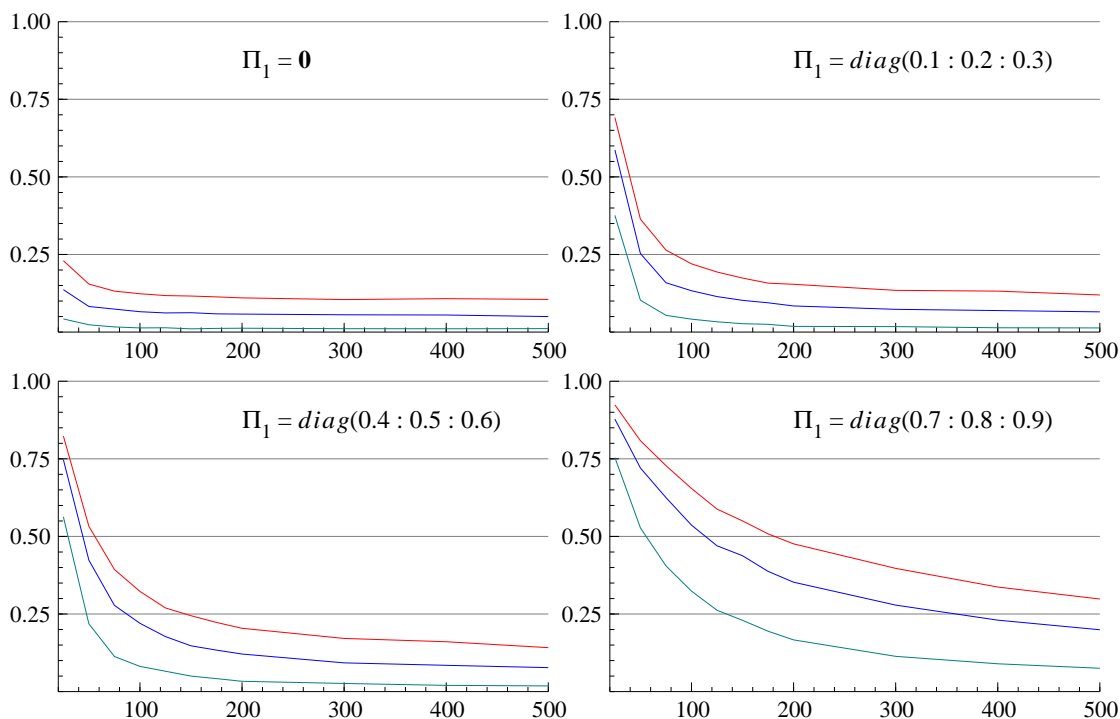


Figure 1: The four panels show the actual size of the LR test against the sample sizes in T^* for the specified autoregressive coefficient Π_1 . The three lines in each panel are for the 10%, 5% and 1% significance levels, respectively.

The number of replications was $M = 5,000$ and one stream of random numbers was used for all replications to facilitate inter-estimator comparability. The results were generated using Ox 3.30, see Doornik (2001).

The statistical models that were estimated were special cases of (53) above. They were correctly specified in that trend break dummies were included and the correct lag length was selected for the endogenous variable. The LR statistic was computed, evaluated at the 10%, 5% and 1% nominal critical values of its asymptotic distribution and the rejection frequencies recorded. The null $H_0: rk(\psi_1) = 0$ was rejected in all replications at all significance levels. This is reassuring as it means that the trend breaks dummies are detected as significant regressors in explaining the data. As regards the ability of the test to reject the correct null $H_0: rk(\psi_\ell) \leq 1$, however, the test is oversized, and increasingly so as the degree of autocorrelation in the data increases: see Figure 1. Thus, the greater the autocorrelation in the data, the greater is the probability of concluding that the rank of ψ_ℓ is not reduced, i.e. that no co-breaking is present.

5.3.3 Revisiting the Stolper–Samuelson Theorem

Our illustrative empirical application uses variables from the same UK data set as those used by Hendry and Mizon (1998) in section 4.2 above. The data series were also previously analyzed by Hendry (2000a, 2001). The variables of interest are the UK GDP deflator, an index of nominal UK wages, and world prices in sterling, such that the dimension of the system is $n = 3$. The three variables are measured in logarithms and denoted by pg , w , and pwe , respectively. Annual observations are available between

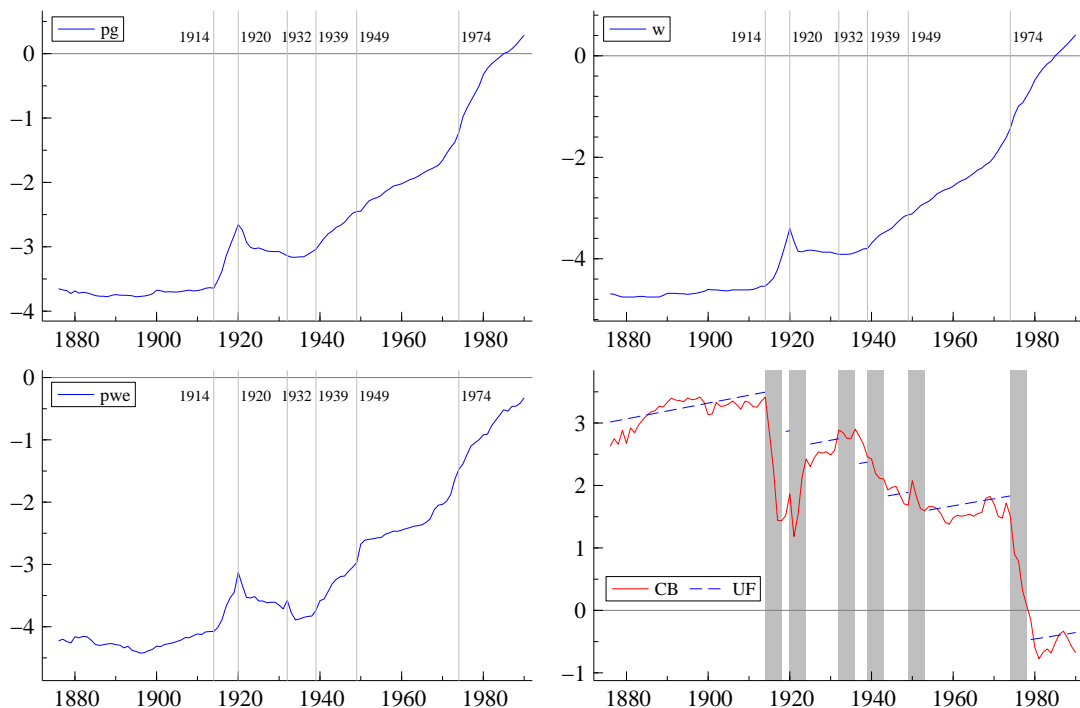


Figure 2: The top-left, top-right and bottom-left panels show the log of the UK GDP deflator (pg), nominal UK wages (w) and world prices (pwe), respectively. The six common break points in 1914, 1920, 1932, 1939, 1949 and 1974 are superimposed. In the bottom-right panel the solid line (CB) displays the estimated trend co-breaking relationship $\hat{\zeta}'_{\perp} \mathbf{x}_t$ while the dashed line (UF) shows its estimated mean unconditional fit $\hat{\zeta}'_{\perp} \hat{\psi}_0 + \hat{\zeta}'_{\perp} \hat{\psi}_{01} \mathbf{e}_{0,t}$ with the transition periods \mathbf{f}_t , shaded in grey, set to zero so as not to clutter the graph, see equation (53).

1872 and 1990 so that the sample size is $T = 119$. Modelling this data set has two attractions: first, it spans a time period that may arguably be described as ‘the long-term’, at least in an economic sense. It thus warrants the estimation of ‘long-run’ relations such as co-breaking combinations, and, at the same time, allows the identification of large ‘intermittent’ breaks. Secondly, the transformations $(w - pg)$ and $(pwe - pg)$ represent real UK wages and real world prices, respectively, two key ingredients in the Stolper–Samuelson Theorem. An interpretation of the results in this context, and a re-appraisal of the Theorem, may thus be attempted.

After graphing the data, $q = 6$ trend break points are identified in the following time periods, consistent with major ‘interventions’ in the past; see Box and Tiao (1975): 1914 and 1920 to denote World War I and its aftermath; 1932 to model the devaluation of the US dollar against gold; 1939 and 1949 for World War II; and, finally, 1974 to denote the oil crisis. In consequence, the break points are assumed known (deterministic), unlike in the analysis by, for instance, Hamilton (1989). For present purposes, testing for unknown break points as proposed in the literature by Bai and Perron (1998) is arguably not necessary since the model is meant to be illustrative: estimated break points with confidence intervals make the suggested procedure based on dummy variables difficult to adapt. The data are displayed in the first three panels of Figure 2, with the break points superimposed.

The empirical analysis proceeded in two stages. At the first stage, the UVAR model in (53) was fitted

	$\hat{\lambda}_i$	LR	df	p-val
$H_0: r_0 = 0$	0.2147	57.9775	18	0.0000
$H_0: r_0 \leq 1$	0.1775	30.1863	10	0.0008
$H_0: r_0 \leq 2$	0.0648	7.7135	5	0.1027

Table 1: *The table shows row-wise the results of the reduced rank tests on θ_1 . The second column presents the estimated squared partial canonical correlations, the third column shows the value of the associated LR test statistic, the fourth column displays the degrees of freedom of the corresponding asymptotic distribution, with the resulting p-value listed in column five.*

such that the residuals, individually and as vectors, were approximately Normally distributed and not autocorrelated; see Rao (1951) for an F-approximation to the asymptotic χ^2 distribution of the autocorrelation test, and Doornik and Hansen (1994) for the vector Normality test. This meant including four lags of the endogenous variable such that $g = 4$. At the second stage, the nested sequence of hypotheses $H_0: \text{rk}(\theta_1) \leq r$, for $r = 1, \dots, \min(n, q)$ was tested by applying partial canonical correlation analysis. The computations were carried out using COBRA, an OxPack module for GiveWin: see Massmann (2003b) and Doornik and Hendry (2001).

The estimated squared partial canonical correlations between \mathbf{x}_t and $\mathbf{d}_{1,t}$ as well as the resulting LR test statistics and associated p-values are shown in Table 1. Importantly, the residuals of the restricted model have remained approximately Normal and uncorrelated. The estimated rank of ψ_1 is $\hat{r} = 2$ such that the resulting dimensions of $\hat{\zeta}$ and $\hat{\nu}$ are (3×2) and (6×2) , respectively, so there are two common deterministic shifts $\nu' \mathbf{d}_{1,t}$ in the system. The eigenvalues of the estimated long-run matrix $\hat{\Pi}(1)$ are inside the unit circle, such that $\hat{\Pi}(1)$ is non-singular and the unconditional trend co-breaking vector $\hat{\zeta}_\perp$ may be derived as the orthogonal complement to $\hat{\zeta} = \hat{\Pi}(1)^{-1} \hat{\zeta}$, viz. $\hat{\zeta}_\perp = (-7.91 \quad 4.72 \quad 1.00)'$. Consequently, the trend co-breaking relationship is given by:

$$\hat{\zeta}'_\perp \mathbf{x}_t = -7.91pg_t + 4.72w_t + pwe_t \quad (55)$$

and is displayed in the [fourth panel](#) of Figure 2. The linear combination has a non-zero intercept and significant baseline trend. It is, however, no longer subject to slope breaks, but only affected by the intercept breaks implicit in the trend breaks. The hypothesis that the independent columns of ψ_{01} lie in the span of ζ was rejected.

Attempting an economic interpretation of the estimated co-breaking relationship in (55), suppose there existed a procedure which allowed us to test the hypothesis that the unconditional co-breaking space is spanned by the vector $\tilde{\zeta}'_\perp = (-6 : 5 : 1)$. If the hypothesis were not rejected, then the restricted co-breaking relationship would be given by:

$$\tilde{\zeta}'_\perp \mathbf{x}_t = 5 \cdot (w_t - pg_t) + 1 \cdot (pwe_t - pg_t). \quad (56)$$

Three conclusions could be drawn on the basis of (56). First and foremost, it is a long-run relationship between real wages in the UK and real world prices measured in sterling which has remained stable, i.e. unaffected by trend breaks, over the past 119 years. Secondly, the real wage and real world prices are inversely related, i.e. when there is a 1% increase in real world prices in sterling, then real wages decrease by 5% on average. Finally, if world prices are taken to be a proxy for import prices, then an interpretation of (56) in terms of the well-known theorem by Stolper and Samuelson (1941) may be attempted. This theorem says that an increase in import prices, perhaps due to the imposition of a tariff, leads to an increase in that factor's price which is used intensively in the import-competing industry. This is a statement about the long term, since the factors, usually taken to be labour and capital, are

assumed to be completely mobile; see Dixit and Norman (1980) for a detailed exposition. If the present data set is seen to support the hypothesis that an increase in the price of imports leads to a fall in wages, this supports the Stolper–Samuelson theorem provided that imports into the UK are considered labour-intensive: compare, for instance, Bentolila and Saint-Paul (1998, Figure 1) who estimate the UK labour share to fluctuate around 70% over the last 40 years.

6 Conclusion

We have overviewed the concept of co-breaking and its properties, as well as a number of approaches to determining and modelling its presence. None of these as yet is entirely satisfactory, but there remains much room for further analysis. Co-breaking relationships are linear combinations $\Phi'x_t$ that depend on fewer breaks in their deterministic components than x_t on its own. Although it is ambitious to attempt to account for both constant and breaking components of deterministic as well as stochastic trends, developing congruent models which are to remain constant over time necessitates doing so.

The unobserved components model provides a useful framework for modelling co-breaking, especially as the unrestricted VAR is convenient for estimation and testing purposes. The Monte Carlo results showed that the test for co-breaking rank was oversized in the presence of autocorrelation, but nevertheless, this approach to co-breaking was applied to modelling UK prices over the past 100 years, to re-visit the Stolper–Samuelson Theorem.

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