Cointegration Lecture III: Identifying the CVAR

Julia Giese
Nuffield College
julia.giese@economics.ox.ac.uk

Hilary Term 2008

Recap

Last week:

- Discussed *estimation* under the *reduced rank* hypothesis.
- Considered tests for *rank determination*.
- Briefly introduced tests to check for *parameter constancy*.

This week:

- Test (identifying) restrictions on *long-run structure* (β).
- Test restrictions on the *adjustment structure* (α).
- Discuss *short-run structure* (p equations in differences).

Outline

Testing restrictions

Consider simple model: $\underbrace{y_t}_{p\times 1} = \underbrace{\gamma'}_{p\times k}\underbrace{x_t}_{k\times 1} + \underbrace{\epsilon_t}_{p\times 1}$

To test whether elements in γ are equal to zero, we write $R'\gamma = 0$, e.g. for $k = 2$ and $\mathcal{H}_0 : \gamma_1 = 0$, we have $R' = [1, 0]$.

$$\underbrace{R'}_{m\times k}\gamma = 0 \quad \Leftrightarrow \quad \gamma = \underbrace{H}_{k\times(k-m)}\underbrace{\varphi}_{(k-m)\times p}, \qquad H = R_\perp$$

So we can write

$$y_t = \varphi' H' x_t + \epsilon_t,$$

and the maximum likelihood estimator given assumed H becomes

$$\begin{aligned} \hat{\varphi}_{ML} &= (H'\mathbf{S}_{xx}H)^{-1}H'\mathbf{S}_{xy} \\ \hat{\gamma}_{ML} &= H\hat{\varphi}_{ML} = H(H'\mathbf{S}_{xx}H)^{-1}H'\mathbf{S}_{xy} \end{aligned}$$

Now use LR test to test $\mathcal{H}_0 : \gamma = H\varphi$ against unrestricted model.

Hypotheses on β

$$\Delta \mathbf{x}_t = \alpha\beta'\mathbf{x}_{t-1} + \varepsilon_t$$

Linear restrictions on cointegration space

$$\beta = \underbrace{H}_{p\times s}\underbrace{\varphi}_{s\times r} \quad \Leftrightarrow \quad R'\beta = 0,$$

where $s = p - m$ and m is the number of restrictions imposed.

Some cointegrating vectors known

$$\beta = (\underbrace{b}_{p\times n}, \underbrace{H}_{p\times s}\underbrace{\varphi}_{s\times(r-n)})$$

For these restrictions, the likelihood can be maximised analytically.
Advantage: uniqueness of maximum can be proved.

Recall estimation of cointegrating vectors

$$\begin{aligned} L(\beta) &= \max_{\alpha,\Omega} L(\alpha,\beta,\Omega) = |\hat{\Omega}(\beta)|^{-T/2} \\ &= \left[|\mathbf{S}_{00}| \frac{|\beta'(\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01})\beta|}{|\beta'\mathbf{S}_{11}\beta|} \right]^{-T/2}. \end{aligned}$$

Solve $|\lambda \mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}| = 0$ for eigenvalues
$1 \geq \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p \geq 0$, and eigenvectors $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_p$ such that

$$\hat{\lambda}_i \mathbf{S}_{11}\hat{\mathbf{v}}_i = \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}\hat{\mathbf{v}}_i, \qquad \hat{\mathbf{v}}_i'\mathbf{S}_{11}\hat{\mathbf{v}}_i = 1, \qquad \hat{\mathbf{v}}_i'\mathbf{S}_{11}\hat{\mathbf{v}}_j = 0 \text{ for } i \neq j.$$

$$\hat{\beta} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_r), \qquad \max_{\alpha,\beta,\Omega} L(\alpha,\beta,\Omega) = \left[|\mathbf{S}_{00}| \prod_{i=1}^{r} \left(1 - \hat{\lambda}_i\right) \right]^{-T/2}.$$

Testing $\beta = H\varphi$

$$L(\varphi) = \left[|\mathbf{S}_{00}| \frac{|\varphi' H' (\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}) H\varphi|}{|\varphi' H \mathbf{S}_{11} H \varphi|} \right]^{-T/2}.$$

Solve $|\lambda H'\mathbf{S}_{11}H - H'\mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}H| = 0$ for eigenvalues $1 \geq \lambda_1^* \geq \cdots \geq \lambda_s^* \geq 0$, and eigenvectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_s^*$. Then:

$$\hat{\varphi} = (\mathbf{v}_1^*, \ldots, \mathbf{v}_r^*), \qquad \max_{\alpha,\varphi,\Omega} L(\alpha, \varphi, \Omega) = \left[|\mathbf{S}_{00}| \prod_{j=1}^{r} \left(1 - \lambda_j^*\right) \right]^{-T/2}.$$

Likelihood ratio test statistic:

$$LR(\beta = H\varphi | \mathcal{H}_r) = -T \sum_{j=1}^{r} \ln \left\{ \frac{(1 - \lambda_j^*)}{(1 - \hat{\lambda}_j)} \right\}.$$

Asymptotic theory

Theorem

Johansen, Theorems 7.2

Assumption 3.1 (A) , (B) , (C). *Then*

$$LR\left(\beta = H\varphi | \mathcal{H}_r\right) \xrightarrow{\mathsf{D}} \chi^2 \left\{r(p-s)\right\}$$

Nielsen (2000): In Johansen's Theorem 7.2

(i) Assumption 3.1 (B) , (C) *are not necessary*

(ii) However, distribution of LR $\left(\textit{restriction on } \beta\right)$ *does depend on* r.

Outline

Identification of β

Cointegrating *vectors* not unique but cointegrating *space* unique.
For any invertible $\xi \in \mathbf{R}^{r \times r}$ then

$$\alpha\beta' = \alpha\xi^{-1}\xi\beta'.$$

$\Rightarrow$ Unrestricted estimation under reduced rank gives span $(\beta) =$ span $(\beta\xi')$ but not a unique β.
Since unrestricted estimation only gives us cointegration space, we need to impose individual restrictions on each β_i in order to gain meaningful economic insights:

$$\beta = (\underbrace{H_1}_{p\times s_1}\underbrace{\varphi_1}_{s_1\times 1}, \ldots, \underbrace{H_r}_{p\times s_r}\underbrace{\varphi_r}_{s_r\times 1})$$

The likelihood needs to be maximised (locally?) by numerical procedures, e.g.

- “Switching algorithm”
- Newton-Raphson routines - as in PcGive.

A rank condition holds for an identified cointegrating relation:

$$\text{rank}(R_1'\beta_1, R_1'\beta_2, ..., R_1'\beta_r) = \text{rank}(R_1'H_1\varphi_1, R_1'H_2\varphi_2, ..., R_1'H_r\varphi_r) = r - 1$$

$\Rightarrow \quad \beta_2, \beta_3, ..., \beta_r$ can't be combined to form space spanned by β_1.
To check rank condition, need to know $\beta_i = H_i\varphi_i$, but to estimate coefficients, need to know whether restrictions are identifying... So check rank condition in terms of known H_i, e.g. for $r = 2$,

$$\text{rank}(R_i'H_j) \geq r - 1 = 1, \quad i \neq j.$$

The equality holds for *just-identifying* restrictions and the inequality for *over-identifying* restrictions.

- Just-identified system: achieved by linear combination of the long-run equations, and hence does not alter likelihood function.
- Over-identified system: constrains parameter space through additional restrictions, and hence changes likelihood function. Use LR test to test restrictions. Need at least r restrictions on each equation.

Outline

Tests for hypotheses on α [1]

Linear restrictions on cointegration space

$$\alpha = \underbrace{A}_{p\times s}\underbrace{\psi}_{s\times r} \quad \Leftrightarrow \quad R'\alpha = 0$$

Testing a known vector in α

$$\alpha = (\underbrace{a}_{p\times n}, \underbrace{A}_{p\times s}\underbrace{\psi}_{s\times(r-n)})$$

Estimation linked with estimation of β!

[1] see Doornik, Hendry and Nielsen (1998) and Johansen (1996), Ch. 8.

Asymptotic theory

Theorem

Johansen, Theorem 8.2, 8.4. *Assumptions A,B and C. Then*

$$LR\,(restriction\ on\ \alpha) \xrightarrow{D} \chi^2\,(n)$$

where

$$\begin{aligned} n &= (p-s)\,r && for\ \alpha = A\psi \\ n &= \#col(a) \times (p-r) && for\ \alpha = (a, A\psi) \end{aligned}$$

Degrees of freedom are computed numerically.

Hypotheses on α and common trends[2]

If last row of α is zero (and $p - r = 1$),

$$\begin{pmatrix} *_{(p-1\times r)} \\ 0_{(1\times r)} \end{pmatrix} \in \text{span}\,(\alpha) \Leftrightarrow \begin{pmatrix} 0_{(p-1\times 1)} \\ 1_{(1\times 1)} \end{pmatrix} \in \text{span}\,(\alpha_\perp)\,,$$

innovations of last equation in system cumulate to common stochastic trend in Granger-Johansen representation:

$$\mathbf{x}_t \approx \underbrace{\beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp}_{C}\sum_{s=1}^{t}\varepsilon_s + \text{stationary process} + A.$$

Also no adjustment of last equation to deviation from long-run:

$$\Delta\mathbf{x}_t = \alpha\beta'\mathbf{x}_{t-1} + \Gamma\Delta\mathbf{x}_{t-1} + \varepsilon_t.$$

Weak exogeneity: The last variable is then "weakly exogenous for β".

[2] See Juselius (2006), Ch. 14.3.

Weak exogeneity [3]

Suppose process X has two components Y and Z.
Q: Is it sufficient to look at conditional likelihood of Y given Z?
Joint density satisfy

$$f_{\lambda,\phi}(y,z) = f_{\phi}(y|z) f_{\lambda}(z).$$

Partition likelihood as

$$L_{Y,Z}(\lambda,\phi) = L_{Y|Z}(\phi) L_Z(\lambda)$$

Q: is

$$\max_{\lambda,\phi} L_{Y,Z}(\lambda,\phi) = \left\{ \max_{\phi} L_{Y|Z}(\phi) \right\} \left\{ \max_{\lambda} L_Z(\lambda) \right\}?$$

Sufficient condition: ϕ and λ vary freely.

[3] Hendry (1995), Ch. 5.

Exogeneity condition applies to one observation

$$\max_{\lambda,\phi} L_{Y,Z}(\lambda,\phi) = \left\{\max_{\phi} L_{Y|Z}(\phi)\right\}\left\{\max_{\lambda} L_{Z}(\lambda)\right\}$$

Suppose $X_1,\ldots,X_T$ is time series with two components Y_t and Z_t.

$$\begin{aligned} L_{X_1,\ldots,X_T}(\lambda,\phi) &= \prod_{t=1}^{T} L_{(X_t|past)}(\lambda,\phi) \\ &= \left\{\prod_{t=1}^{T} L_{(Y_t|Z_t,past)}(\phi)\right\}\left\{\prod_{t=1}^{T} L_{(Z_t|past)}(\lambda)\right\} \end{aligned}$$

Definition

According to Engle, Hendry and Richard (1983), Z_t is **weakly exogenous** for θ if

(i) λ,ϕ vary freely, and

(ii) θ is a function of ϕ only.

Weak exogeneity in cointegrated VAR[4]

Joint cointegrated VAR:

$$\begin{pmatrix} \Delta \mathbf{y}_t \\ \Delta \mathbf{z}_t \end{pmatrix} = \begin{pmatrix} \alpha_y \\ \alpha_z \end{pmatrix} \beta' \begin{pmatrix} \mathbf{y}_{t-1} \\ \mathbf{z}_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{y,t} \\ \varepsilon_{z,t} \end{pmatrix}$$

Conditional model for $\mathbf{y}_t$ given $\mathbf{z}_t$ is, with $\omega = \Omega_{yz}\Omega_{zz}^{-1}$,

$$\Delta \mathbf{y}_t = (\alpha_y - \omega\alpha_z)\,\beta'\mathbf{x}_{t-1} + \omega\Delta\mathbf{z}_t + (\varepsilon_{y,t} - \omega\varepsilon_{z,t})$$

Marginal model for $\mathbf{z}_t$ is

$$\Delta \mathbf{z}_t = \alpha_z\beta'\mathbf{x}_{t-1} + \varepsilon_{z,t}.$$

- $\mathbf{z}_t$ is weakly exogenous for β ('long-run weakly exogenous') if $\alpha_z = 0$.

[4] Juselius (2006), Ch. 11.

We then have parameter spaces

$$\begin{aligned} \lambda &= (\Omega_{zz}) \\ \phi &= (\alpha_y, \beta, \omega, \Omega_{yy} - \Omega_{yz}\Omega_{zz}^{-1}\Omega_{zy}) \end{aligned}$$

For long-run weak exogeneity of $\mathbf{z}_t$, our parameter of interest is β-matrix. Is $\alpha_z = 0$ sufficient for long-run weak exogeneity?

(i) It can be shown that parameters in λ and ϕ are variation free from each other, fulfilling condition (i).[5]

(ii) β only enters parameter space ϕ, so condition (ii) satisfied.

[5] See Johansen (1996), Ch. 8.

The partial model

Let $\mathbf{x}_t = (\mathbf{y}_t, \mathbf{z}_t)'$. If long-run weak exogeneity of $\mathbf{z}_t$ with respect to β holds, we can analyse the conditional model of $\mathbf{y}_t|\mathbf{z}_t$, also called the partial model, on its own, and obtain a fully efficient estimate of β:

$$\Delta\mathbf{y}_t = \omega\Delta\mathbf{z}_t + \Gamma_{1,y}\Delta\mathbf{x}_{t-1} + \ldots + \Gamma_{k-1,y}\Delta\mathbf{x}_{t-k+1} + \alpha_y\beta'\mathbf{x}_{t-1} + \mu_{0,y} + \varepsilon_{y.z,t}$$

Reasons to focus on the partial model:

- By conditioning on weakly exogenous variables, one can achieve a partial system with more stable parameters.
- Weak exogeneity may make sense *a priori*, e.g. it is unlikely that the Danish economy impacts on the US, but the US interest rate matters for the Danish economy.

Outline

Identification of the short-run structure[6]

Besides identifying the long-run structure, we can also restrict the short-run structure around lagged differences.

- We have p different equations in the system which determine the p current differences $\Delta \mathbf{x}_t$ through lagged differences, r long-run relations and deterministic terms.
- Identification of the simultaneous short-run structure requires $p - 1$ restrictions in each equation.
- If we have p_x weakly exogenous variables, we only need $p - p_x - 1$ restrictions in each equation.
- Estimating the reduced form model

$$\mathbf{A}_0 \Delta \mathbf{x}_t = \mathbf{A}_0 \alpha \beta' \mathbf{x}_{t-1} + \mathbf{A}_0 \Gamma_1 \Delta \mathbf{x}_{t-1} + \mathbf{A}_0 \mu_0 + \mathbf{A}_0 \varepsilon_t,$$

where $\mathbf{A}_0 = I$ imposes $p - 1$ restrictions on each equation by definition, i.e. we have a just-identified system.

[6] See Juselius (2006), Ch. 13.

Structural models

If we are interested in a structural model where shocks can be given a structural interpretation, we need to find restrictions such that the residuals are uncorrelated, for example:

- Establish a causal chain among variables, i.e. $\mathbf{A}_0$ becomes an upper triangular matrix, and restrict off-diagonal elements in the covariance matrix to zero.
- Orthogonalise the residuals directly, i.e. let $\mathbf{A}_0 = \hat{\Omega}^{-1/2}$.

Summer school: The cointegrated VAR model

If you are planning to do empirical time series modelling either in your DPhil or in your job, I would recommend a summer school at the University of Copenhagen, entitled:

"Econometric Methodology and Macroeconomics Applications - the Cointegrated VAR Model".

It is run by Søren Johansen, Katarina Juselius and Anders Rahbek and takes place from 4-24 August 2008. Besides technical lectures, you get the chance to work on your data set with the experts for three weeks – and have a paper written by the end of it!

For further information talk to me or visit `www.econ.ku.dk/Summerschool`.

Outline

The bivariate normal distribution

Suppose $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ independent

$$\mathsf{N}_2\left\{\begin{pmatrix}\mu_y\\ \mu_z\end{pmatrix}, \begin{pmatrix}\Omega_{yy} & \Omega_{yz}\\ \Omega_{yz} & \Omega_{zz}\end{pmatrix}\right\}.$$

Let $\omega = \Omega_{yx}\Omega_{xx}^{-1}$ be population regression coefficient. Then

$$\begin{pmatrix}1 & -\omega\\ 0 & 1\end{pmatrix}\begin{pmatrix}Y_i\\ Z_i\end{pmatrix} \overset{\mathsf{D}}{=} \mathsf{N}_2\left\{\begin{pmatrix}\mu_y - \omega\mu_z\\ \mu_z\end{pmatrix}, \begin{pmatrix}\Omega_{yy} - \Omega_{yz}\Omega_{zz}^{-1}\Omega_{zy} & 0\\ 0 & \Omega_{zz}\end{pmatrix}\right\}$$

so

$$(Y_i|Z_i) \overset{\mathsf{D}}{=} \mathsf{N}(\mu_y + \omega(Z_i - \mu_z), \Omega_{yy} - \Omega_{yz}\Omega_{zz}^{-1}\Omega_{zy}), \qquad Z_i \overset{\mathsf{D}}{=} \mathsf{N}(\mu_z, \Omega_{zz})$$

Exercises

1. Exam 2007, Question 6 (iii)-(vi).
2. Exam 2006, Question 5 (v), (vi).
3. Exam 2004, Question 8.
4. Exercise 5.3, Johansen (1996).
5. Exercise 5.6, Johansen (1996).

References

- Doornik, J.A., Nielsen, B. and Hendry, D.F. (1998) Inference in cointegrating models: UK M1 revisited, *Journal of Economic Surveys*, 12, 533–572.
- Engle, F., Hendry, D.F., Richard, J-F. (1983) Exogeneity, *Econometrica*, 51, 277–304.
- Hendry, D.F. (1995) *Dynamic Econometrics*, Oxford: Oxford University Press.
- Nielsen, B. (2000). The asymptotic distribution of likelihood ratio test statistics for cointegration in unstable vector autoregressive processes. Nuffield College working paper no. 24.