Cointegration Lecture IV: Recap and extensions

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Recap

Last week:

- Tested (identifying) restrictions on long-run structure (β).
- Tested restrictions on the adjustment structure (α).
- Looked at short-run structure (p equations in differences).

This week:

- Review of CVAR.
- Discuss extensions to CVAR, like:
  1. I(2) models
  2. Specific to general regarding cointegrating relations
  3. Global VAR

Summer school: The cointegrated VAR model

If you are planning to do empirical time series modelling either in your DPhil or in your job, I would recommend a summer school at the University of Copenhagen:

"*Econometric Methodology and Macroeconomics Applications - the Cointegrated VAR Model*".

It is run by Sœren Johansen, Katarina Juselius and Anders Rahbek and takes place from 4-24 August 2008. Besides technical lectures, you get the chance to work on your data set with the experts for three weeks – and have a paper written by the end of it!

For further information talk to me or visit `www.econ.ku.dk/Summerschool`.

Outline

Review of cointegrated VAR

I(2) data

Specific-to-general vs general-to-specific

Global VAR

Review of cointegrated VAR so far

We want to study relationships between a limited number of macroeconomic variables, e.g. variables influencing the money market, determining inflation, or relating to exchange rate developments.[1] First, formulate unrestricted VAR model,

$$\mathbf{x}_t = \Pi_1 \mathbf{x}_{t-1} + \ldots + \Pi_k \mathbf{x}_{t-k} + \phi \mathbf{D}_t + \varepsilon_t, \qquad t = 1, 2, \ldots, T$$

where $\mathbf{x}_t$ is a vector of our variables of interest. We then determine the lag length, consider what deterministic terms we need (e.g. constant, trend, dummies) and check that the model is well specified, i.e. that residuals satisfy assumptions $\varepsilon_t \sim$ iid $\mathrm{N}_p(\mathbf{0},\Omega)$ and parameters are constant (recursive tests). Our preferred model may then be:

$$\mathbf{x}_t = \Pi_1 \mathbf{x}_{t-1} + \Pi_2 \mathbf{x}_{t-2} + \mu_0 + \mu_1 t + \varepsilon_t,$$

which may be re-written as (with $\Pi = \Pi_1 + \Pi_2 - I$ and $\Gamma_1 = -\Pi_2$)

$$\Delta \mathbf{x}_t = \Pi \mathbf{x}_{t-1} + \Gamma_1 \Delta \mathbf{x}_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t.$$

[1] See References for applied papers and Juselius (2006), Ch. 20, 21.

Macroeconomic data tends to be non-stationary, i.e. the system contains unit roots. In this case, Π is of reduced rank r, and we can write $\Pi = \alpha\beta'$. Using beautiful identity, we decompose $\mu_1 \equiv \alpha\beta_1 + \gamma_1$, where we typically set $\gamma_1 = 0$ as it would cumulate to quadratic trend at odds with data. Our model is now:

$$\Delta \mathbf{x}_t = \alpha(\beta', \beta_1) \begin{pmatrix} \mathbf{x}_{t-1} \\ t \end{pmatrix} + \Gamma_1 \Delta \mathbf{x}_{t-1} + \mu_0 + \varepsilon_t.$$

We interpret $\beta' \mathbf{x}_{t-1}$ as deviations from stationary long-run equilibrium relations between the variables (e.g. money demand equation, Taylor rule, Phillips curve, PPP), α as speed of adjustment to the equilibrium and $\Gamma_1 \Delta \mathbf{x}_{t-1}$ as short term effects. Note that if we knew (β', β_1), we could estimate α, Γ_1, μ_0 and Ω by maximum likelihood.

Assumptions

(A) $\text{rank}(\Pi) = r \leq p$,
(B) number of unit roots is $p - r$,
(C) remaining r roots are stationary.

The corresponding Granger-Johansen representation is given by:

$$\mathbf{x}_t = \mathbf{C}\sum_{i=1}^{t}\varepsilon_i + \mathbf{C}\mu_0 t + \mathbf{C}^*(L)\varepsilon_t + \mathbf{C}^*(L)\mu_0 + \mathbf{C}^*(L)\alpha\beta_1 t + \tilde{\mathbf{X}}_0$$

where $\mathbf{C} = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp$ with $\Gamma = I - \Gamma_1$, $\beta'\tilde{\mathbf{X}}_0 = 0$ and $\mathbf{C}^*(L)\varepsilon_t$ is a stationary process.

To find estimates for β we solve an eigenvalue problem of the concentrated likelihood function (where short-run effects and unrestricted deterministic components are concentrated out using Frisch-Waugh theorem).

$$\begin{aligned} L(\beta) &= \max_{\alpha,\Omega} L(\alpha,\beta,\Omega) = |\hat{\Omega}(\beta)|^{-T/2} \\ &= \left[|\mathbf{S}_{00}| \frac{|\beta'(\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01})\beta|}{|\beta'\mathbf{S}_{11}\beta|} \right]^{-T/2}. \end{aligned}$$

Solve $|\lambda \mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}| = 0$ for eigenvalues $1 \geq \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p \geq 0$, and eigenvectors $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_p$ such that

$$\hat{\lambda}_i \mathbf{S}_{11}\hat{\mathbf{v}}_i = \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}\hat{\mathbf{v}}_i, \qquad \hat{\mathbf{v}}_i'\mathbf{S}_{11}\hat{\mathbf{v}}_i = 1, \qquad \hat{\mathbf{v}}_i'\mathbf{S}_{11}\hat{\mathbf{v}}_j = 0 \text{ for } i \neq j.$$

$$\hat{\beta} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_p), \qquad \max_{\alpha,\beta,\Omega} L(\alpha,\beta,\Omega) = \left[|\mathbf{S}_{00}| \prod_{i=1}^{p} \left(1 - \hat{\lambda}_i\right) \right]^{-T/2}.$$

Now substitute $\hat{\beta}$ into the maximum likelihood estimators of α, Γ_1, μ_0 and Ω as found before in terms of β.

Having maximised the concentrated likelihood, we want to find rank r of $\Pi = \alpha\beta'$, and we test a model with rank $r \leq p$ against alternative of full rank, starting with

$$\mathcal{H}_r : \text{rank}\Pi \leq 0 \text{ against } \mathcal{H}_p : \text{rank}\Pi \leq p$$

and setting rank r at first $\mathcal{H}_r$ not rejected. Note that under $\mathcal{H}_r$ we ignore the $p - r$ non-stationary eigenvalues, but estimation of the models is identical. The LR test statistic for the rank (also called *trace* test because of its asymptotic distribution) is then found as:

$$\begin{aligned} LR(\mathcal{H}_r|\mathcal{H}_p) &= -2\ln Q(\mathcal{H}_r/\mathcal{H}_p) \\ &= T\ln\left\{\frac{|\mathbf{S}_{00}|(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)\ldots(1-\hat{\lambda}_r)}{|\mathbf{S}_{00}|(1-\hat{\lambda}_1)(1-\hat{\lambda}_2)\ldots(1-\hat{\lambda}_r)\ldots(1-\hat{\lambda}_p)}\right\} \\ &= -T\sum_{j=r+1}^{p}\ln(1-\hat{\lambda}_j). \end{aligned}$$

Its asymptotic distribution is $\mathsf{DF}_l\,(p-r)$, but note that it changes depending on what deterministic terms are included in the model.

Since $\beta' \mathbf{x}_{t-1}$ may be interpreted as deviations from long-run equilibria between variables, we would like to test hypotheses on β by imposing restrictions.

We can impose linear restrictions on the whole cointegration space, for example to test whether a variable is excludable from the long run:

$$\beta = \underbrace{H}_{p \times s} \underbrace{\varphi}_{s \times r} \quad \Leftrightarrow \quad R'\beta = 0,$$

where $s = p - m$ and m is the number of restrictions imposed, and $R'H = 0$, i.e. $H = R_{\perp}$.

For these restrictions, the likelihood can be maximised analytically. *Advantage*: uniqueness of maximum can be proved.

To test these restrictions, we again maximise the concentrated likelihood, but substituting $\beta = H\varphi$ (note that H is known under the null hypothesis).

$$L(\varphi) = \left[|\mathbf{S}_{00}| \frac{|\varphi' H' (\mathbf{S}_{11} - \mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}) H\varphi|}{|\varphi' H \mathbf{S}_{11} H\varphi|} \right]^{-T/2}.$$

Solve $|\lambda H'\mathbf{S}_{11}H - H'\mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}H| = 0$ for eigenvalues $1 \geq \lambda_1^* \geq \cdots \geq \lambda_s^* \geq 0$, and eigenvectors $\mathbf{v}_1^*, \ldots, \mathbf{v}_s^*$. Note that $s \geq r$, and that we discard the non-stationary eigenvalues $\lambda_s^*, \lambda_{s-1}^*, ..., \lambda_{r+1}^*$:

$$\hat{\varphi} = (\mathbf{v}_1^*, \ldots, \mathbf{v}_r^*), \qquad \max_{\alpha,\varphi,\Omega} L(\alpha, \varphi, \Omega) = \left[|\mathbf{S}_{00}| \prod_{j=1}^{r} \left(1 - \lambda_j^*\right) \right]^{-T/2}.$$

$$LR(\beta = H\varphi | \mathcal{H}_r) = -T \sum_{j=1}^{r} \ln \left\{ \frac{(1 - \lambda_j^*)}{(1 - \hat{\lambda}_j)} \right\}.$$

The asymptotic distribution of this test statistic is $\chi^2 \{ r(p-s) \}$.

If we assume that we know some cointegrating vectors, b, we test:

$$\beta = (\underbrace{b}_{p \times n}, \underbrace{\varphi}_{p \times (r-n)})$$

Imposing certain vectors b leaving φ unrestricted, we can maximise the likelihood analytically by concentrating out $\alpha_1 b'$ from the model as it is known under the null hypothesis.

If we want to test restrictions only on certain parameters in a vector of β, leaving other parameters in the same vector unrestricted, we write:

$$\beta = (\underbrace{H}_{p \times s}\underbrace{\varphi}_{s \times n}, \underbrace{\psi}_{p \times (r-n)})$$

In this case, the likelihood function can only be maximised numerically.

Remember that α and β are only jointly, but not individually identified. In order to test economically meaningful relationships between variables, we need to impose *r over-identifying* restrictions on vectors in β:

$$\beta = (\underbrace{H_1}_{p\times s_1}\underbrace{\varphi_1}_{s_1\times 1}, \ldots, \underbrace{H_r}_{p\times s_r}\underbrace{\varphi_r}_{s_r\times 1})$$

Furthermore, we can test restrictions on α, e.g. whether one row is equal to zero:

$$\alpha = \underbrace{A}_{p\times s}\underbrace{\psi}_{s\times r} \quad \Leftrightarrow \quad R'\alpha = 0$$

In this case, the variable corresponding to the zero row does not react to deviations from long-run equilibrium, and is said to be weakly exogenous with respect to β. A zero row in α also implies a unit vector in $\alpha_\perp$, and hence a common trend $\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i$ made up only of the cumulated residuals of the variable in question.

Outline

Signs of I(2) in the I(1) model[2]

Consider the model:

$$\Delta \mathbf{x}_t = \alpha\beta' \mathbf{x}_{t-1} + \Gamma_1 \Delta \mathbf{x}_{t-1} + \phi \mathbf{D}_t + \varepsilon_t$$

- Clearly, if $\Delta \mathbf{x}_t$ displays non-stationary behaviour this is evidence for I(2).
- When for a given r near unit roots remain in the data, and these don't disappear when decreasing r, this is a sign that there may be additional unit root(s) in $\Gamma = I - \Gamma_1$.
- Also, if $\beta_i' \mathbf{x}_t$ appears non-stationary in a graphical inspection, but $\beta_i' \mathbf{R}_{it}$ stationary, this is a sign of an I(2) trend as β_i only takes $\mathbf{x}_t$ from I(2) to I(1).

[2] See Juselius (2006), Ch. 16.

Testing the nominal-to-real transformation

We typically transform nominal I(2) data to real data, assuming that for example nominal money supply and prices share the same I(2) trend, resulting in I(1) variables.

$$\begin{pmatrix} m_t \\ p_t \\ y_t \\ R_t \end{pmatrix} \sim I(2) \to \begin{pmatrix} (m-p)_t \\ \Delta p_t \\ y_t \\ R_t \end{pmatrix} \sim I(1).$$

If $(m_t - ap_t) \sim I(1)$ and $a \neq 1$, but $a = 1$ is imposed in the transformation, the real variable will contain a small I(2) component that may lead to difficulties in the estimation.

We can test the *long-run price homogeneity* assumption between nominal money supply and prices by estimating the model in *nominal variables*, and impose the same restrictions on all cointegrating relations, such that for $p = 4$, $r = 2$, $m = 1$, $s = 3$:

$$\beta = H\varphi = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ -\varphi_{11} & -\varphi_{12} \\ \varphi_{21} & \varphi_{22} \\ \varphi_{31} & \varphi_{32} \end{pmatrix}$$

So this can just be tested by an LR test for same restrictions on all β_i as seen in Lecture 3.

The I(2) model[3]

If the nominal-to-real transformation is rejected, we need to work with the I(2) model. Re-writing a VAR(2) in second differences, we find cointegrating relations of different orders:

$$\begin{aligned}\underbrace{\Delta^2 \mathbf{x}_t}_{I(0)} &= \Pi \mathbf{x}_{t-1} + \Gamma \Delta \mathbf{x}_{t-1} + \phi \mathbf{D}_t + \varepsilon_t \\ &= \alpha \underbrace{[\underbrace{\beta' \mathbf{x}_{t-1}}_{I(1)} + \underbrace{\delta' \Delta \mathbf{x}_{t-1}}_{I(1)}]}_{I(0)} + \zeta \underbrace{\tau' \Delta \mathbf{x}_{t-1}}_{I(0)} + \phi \mathbf{D}_t + \varepsilon_t\end{aligned}$$

This model will have s_1 I(1) trends (equivalent to reduced rank of $\alpha'_{\perp} \Gamma \beta_{\perp}$), s_2 I(2) trends and in total $r + s_1$ cointegrating relations. Hypotheses may be formulated through restrictions on parameters similarly to I(1) case (Juselius (2006), Ch. 18; Kongsted (2003)).

[3] See Juselius (2006), Ch. 17.

Outline

Specific-to-general vs general-to-specific[4]

Typically, we would like to follow a general-to-specific modelling approach to ensure that we have not omitted any crucial variables.

$\Rightarrow$ The selection of variables is motivated by economic theory, and we then test which variables actually matter.

However, in VAR modelling we face a problem of dimensionality:

- Adding one variable introduces $(2p + 1)k$ parameters...
- While it is possible to estimate models with more parameters than observations, interpretability is lost.
- Hence, two approaches may be followed:
  1. Gradually increasing the information set.
  2. Combining partial systems.

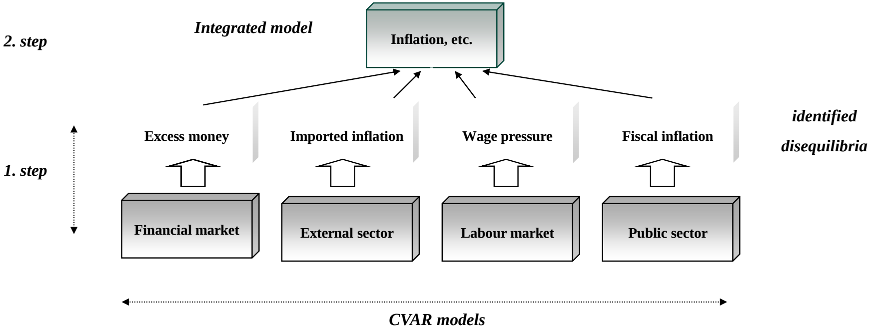
[4] See Juselius (2006), Ch. 19.

Gradually extending the information set

- Since cointegrating relations are stationary linear combinations between variables, they are invariant to an extension of the information set.
- If a variable is added to the system, the rank may increase. The previously found cointegrating relations remain valid, but an additional one may be needed.
- This gradual approach greatly facilitates the identification of cointegrating relations.

Combining partial systems

Model inflation as interplay of disequilibria in different markets:



- Deviations from cointegrating relations represent disequilibria.
- Identify partial equilibrium effects by estimating separate CVARs for different markets.
- Include deviations from all cointegrating relations in new model, say of (differenced) inflation (but could include other variables) and estimate the short run structure. See Juselius (2006), Ch. 22.

Outline

Review of cointegrated VAR

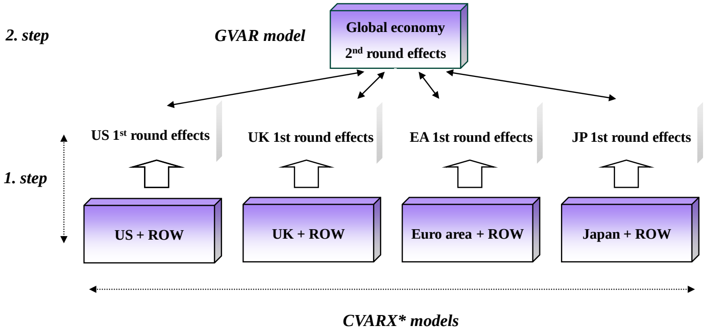
I(2) data

Specific-to-general vs general-to-specific

Global VAR

The GVAR framework

The GVAR, developed by Pesaran *et al.*, is built in two stages:



1. Estimate CVAR models for **individual countries** including country-specific ROW variables (CVARX*).
2. **Combine** country models in accordance with weighting scheme for purposes of impulse responses and forecasting (GVAR).

Country CVARX* models

For each country i, estimate CVARX*:

$$\Delta \mathbf{x}_{it} = \Pi_i(\mathbf{x}_i, \mathbf{x}_i^*)'_{t-1} + \Gamma_i(\Delta \mathbf{x}_i, \Delta \mathbf{x}_i^*)'_{t-1} + \Upsilon_i \Delta \mathbf{x}_{it}^* + \Theta_i \mathbf{D}_{it} + u_{it},$$

Endogenous **domestic** variables, $\mathbf{x}_i$, included in all i country models:

$$\mathbf{x}_{it} = (m_{ri}, y_{ri}, \Delta p_i, I_{si}, I_{li}, s_{ri}, h_{ri}, ppp_i)'_t$$

Ideally, use following country-specific **ROW** variables, $\mathbf{x}_i^*$:

$$\mathbf{x}_{it}^* = (m_{ri}^*, y_{ri}^*, \Delta p_i^*, I_{si}^*, I_{li}^*, s_{ri}^*, h_{ri}^*)'_t$$

But degrees of freedom problem ⇒ test ROW variables down by general-to-specific (GETS) modelling strategy to obtain appropriate model (using Autometrics).

Combining country models

Re-writing country CVARs in levels, taking weakly exogenous variables to left-hand side and stacking all countries gives:

$$\begin{aligned}\mathbf{A}_0\mathbf{z}_t &= \mathbf{h}_0+\mathbf{h}_1\mathbf{t} + \mathbf{A}_1\mathbf{z}_{t-1}+\mathbf{A}_2\mathbf{z}_{t-2}+\mathbf{u}_t \\ &\Updownarrow \\ \mathbf{A}_0\mathbf{W}\mathbf{x}_t &= \mathbf{h}_0+\mathbf{h}_1\mathbf{t} + \mathbf{A}_1\mathbf{W}\mathbf{x}_{t-1}+\mathbf{A}_2\mathbf{W}\mathbf{x}_{t-2}+\mathbf{u}_t\end{aligned}$$

where $\mathbf{z}_t = (\mathbf{x}_1, \mathbf{x}_1^*, \mathbf{x}_2, \mathbf{x}_2^*, ..., \mathbf{x}_N, \mathbf{x}_N^*)'_t = \mathbf{W}\mathbf{x}_t$. Dividing by $\mathbf{A}_0\mathbf{W}$, we get:

$$\mathbf{x}_t = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{F}_1\mathbf{x}_{t-1} + \mathbf{F}_2\mathbf{x}_{t-2} + \varepsilon_t$$

MA representation and impulse responses

Re-write equation in MA form: $$\mathbf{x}_t = \mathbf{c}_t + \sum_{j=0}^{\infty} \mathbf{C}_j \epsilon_{t-j},$$

$$\begin{aligned} \text{where } \mathbf{C}_j &= \mathbf{F}_1 \mathbf{C}_{j-1} + \mathbf{F}_2 \mathbf{C}_{j-2}, \quad j = 1, 2, ..., \\ \text{with } \mathbf{C}_0 &= I, \mathbf{C}_j = \mathbf{0} \text{ for } j < 0. \end{aligned}$$

The corresponding GIRFs of a one-standard-deviation shock to the ℓ^{th} element of $\mathbf{x}_t$ on its $\jmath^{\text{th}}$ element is given by:

$$\begin{aligned} GIRF(\mathbf{x}_t; u_{\ell t}, h) &= \frac{e'_\jmath \mathbf{C}_h (\mathbf{A}_0 \mathbf{W}_{t-1})^{-1} \Sigma_u e_\ell}{\sqrt{e'_\ell \Sigma_u e_\ell}} \\ h &= 0, 1, 2, ..., H; \ell, \quad \jmath = 1, 2, ..., k, \end{aligned}$$

where e_j and e_l are unit vectors and Σ_u is a covariance matrix.

Exercises

1. Exercise 3.2, Johansen (1996).
2. Exam 2001, Question 5.

References to applied papers

Check `www.econ.ku.dk/okokj/` for papers by Katarina Juselius:

- Juselius, K. and MacDonald, R. (2003), "International Parity Relationships between Germany and the United States: a Joint Modelling Approach".
- Johansen, S. and Juselius, K., "Controlling inflation in a cointegrated vector autoregressive model with an application to US data".

`www.econ.ku.dk/massimo/Conference/programX.htm`:

- Giese , J.V., "Level, slope, and curvature: The yield curve's derivatives and their relations to macro variables".
- Juselius, M., "Testing the new Keynesian model on US and European data".
- Mœller, N.F., "Linking simple economic theory models and the cointegrated VAR model".
- Reade, J. and Stehn, J., "Estimating the interactions between monetary and fiscal policy using the cointegrated VAR methodology".
- Tuxen, C.K., "An empirically relevant model of inflation in the Euro area".

References

- Des, S., Holly, S., Pesaran, M.H., Smith, L.V. (2007) Long Run Macroeconomic Relations in the Global Economy. Economics: The Open-Access, Open-Assessment E-Journal, Vol. 1, 2007-3.
- Des, S., di Mauro, F., Pesaran, M.H., Smith, L.V. (2007) Exploring the International Linkages of the Euro area: A Global VAR Analysis. Journal of Applied Econometrics, 22, 138.
- Kongsted, H.C. (2003) An I(2) cointegration analysis of small-country import prices determination. Econometrics Journal, 6, 53-71.