Multidimensional Inequality Measurement: A Proposal[1]

Christian List
Nuffield College
Oxford OX1 1NF, U.K.

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Abstract. Two essential intuitions about the concept of multidimensional inequality have been highlighted in the emerging body of literature on this subject: first, multidimensional inequality should be a function of the *uniform* inequality of a multivariate distribution of goods or attributes *across people* (Kolm, 1977); and, second, it should also be a function of the *cross-correlation* between distributions of goods or attributes *in different dimensions* (Atkinson and Bourguignon, 1982; Walzer, 1983). While the first intuition has played a major role in the design of fully-fledged multidimensional inequality indices, the second one has only recently received attention (Tsui, 1999); and, so far, multidimensional generalized entropy measures are the only inequality measures known to respect *both* intuitions. The present paper proposes a general method of designing a wider range of multidimensional inequality indices that also respect *both* intuitions, and illustrates this method by defining two classes of such indices: a generalization of the Gini coefficient, and a generalization of Atkinson's one-dimensional measure of inequality.

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1. Introduction

The concern of the present paper is the problem of multidimensional inequality measurement. Suppose we are asked to evaluate the overall level of inequality in society not just on the basis of one good/attribute -- or a one-dimensional item of information -- for each person or household (e.g. each person's or household's income), but on the basis of several goods/attributes -- or a multidimensional vector of information -- for each person or household (e.g. a vector whose different components represent a person's or households's income, level of education, level of access to health care etc.). Given different multidimensional distributions (each of which assigns to each person or household in society a corresponding vector of goods/attributes), the problem of multidimensional inequality measurement is, in essence, to specify what it means to say that one such distribution is more unequal than another and, as far as possible, to rank different distributions in an order of inequality.

To illustrate, consider the following four distributions:

$$X_1 = \begin{array}{c} \\ \text{person 1} \\ \text{person 2} \\ \text{person 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{education} & \text{health} & \text{income} \end{array} \\ \begin{pmatrix} 9 & 10 & 11 \\ 5 & 7 & 2 \\ 3 & 5 & 1 \end{pmatrix} \end{array}, \qquad X_2 = \begin{array}{c} \\ \text{person 1} \\ \text{person 2} \\ \text{person 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{education} & \text{health} & \text{income} \end{array} \\ \begin{pmatrix} 5 & 10 & 1 \\ 3 & 5 & 11 \\ 9 & 7 & 2 \end{pmatrix} \end{array},$$

$$Y_1 = \begin{array}{c} \\ \text{person 1} \\ \text{person 2} \\ \text{person 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{education} & \text{health} & \text{income} \end{array} \\ \begin{pmatrix} 7.0 & 8.4 & 7.2 \\ 5.4 & 7.2 & 3.6 \\ 4.6 & 6.4 & 3.2 \end{pmatrix} \end{array}, \qquad Y_2 = \begin{array}{c} \\ \text{person 1} \\ \text{person 2} \\ \text{person 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{education} & \text{health} & \text{income} \end{array} \\ \begin{pmatrix} 5.4 & 8.4 & 3.2 \\ 4.6 & 6.4 & 7.2 \\ 7.0 & 7.2 & 3.6 \end{pmatrix} \end{array}.$$

What can we say about the relative levels of inequality in these distributions? Using traditional one-dimensional methods of inequality measurement (for a comprehensive survey, see Sen, 1997), we can probably say that, if we consider *each* of the three dimensions of education, health and income *separately*, the Y-distributions are less unequal than the X-distributions. But can we say more?

The problem of multidimensional inequality measurement can be -- and, historically, has been -- approached in two stages. Stage (i) is to state certain *dominance criteria* that specify the conditions under which one multidimensional distribution should be taken to be "clearly" at least as equal as another. In many cases, however, these dominance criteria may induce only *partial* orderings on the set of alternative distributions. So, *if* one requires rankings even in cases which are left undecided by the specified dominance criteria, stage (ii) is to define an *inequality index*, consistent with these dominance criteria, that maps each multidimensional distribution to a real number and thereby induces a *complete* ordering on the set of alternative distributions.

The study of multidimensional inequality was pioneered by Fisher (1956), who developed the idea of a multidimensional distribution matrix, and, more recently, by the seminal contributions of Kolm (1977), Atkinson and Bourguignon (1982) and Walzer (1983). Amongst Kolm's proposals with regard to stage (i) are the criteria that have become known as *uniform majorization* (in essence, a multidimensional generalization of the well-known Pigou-Dalton criterion) and *directional/price majorization* (a criterion that involves multiplying multidimensional distributions by price vectors and comparing the resulting one-dimensional distributions). These criteria are primarily sensitive to the *uniform* inequality of a multidimensional distribution *across people*. Kolm's criteria capture the idea that distribution Y_1 is less unequal than distribution X_1, and that distribution Y_2 is less unequal than distribution X_2. Atkinson and Bourguignon have drawn our attention to the intuition that multidimensional inequality also depends on how systematic the correlation between distributions *of different goods/attributes* (and especially between inequalities *in different dimensions*) is and have developed appropriate dominance criteria. Since inequalities in different dimensions are more systematically cross-correlated in distributions X_1 and Y_1 than in distributions X_2 and Y_2, respectively, distribution X_2 should thus be considered less unequal than distribution X_1, and distribution Y_2 should be considered less unequal than distribution Y_1. In a similar spirit, the political theorist Walzer developed the conception of *complex equality*, according to which overall equality consists not so much in *local equality within each dimension (distributive sphere* in Walzer's terminology*)*, but in the extent to which *local inequalities in different dimensions* cancel each other out, by advantaging and disadvantaging *different* people in *different* dimensions. So complex equality would be better realized in distributions X_2 and Y_2 than in distributions X_1 and Y_1, respectively. Note that separate one-dimensional inequality evaluation in each dimension is insufficient to take account of problems of cross-correlation: distributions X_1 and X_2 have identical *local levels of inequality* in each of the three separate dimensions, and so do distributions Y_1 and Y_2.

Although these pioneers primarily addressed stage (i), their work inspired subsequent proposals as to how one could approach stage (ii) and construct fully-fledged multidimensional inequality indices (e.g. Maasoumi, 1986; Tsui, 1995, 1999; Koshevoy and Mosler, 1997). While all these proposed inequality measures make use of the dominance criteria proposed by Kolm, the Atkinson-Bourguignon-Walzer intuition that a multidimensional inequality measure should also be sensitive to the cross-correlation between inequalities in different dimensions has only recently received explicit attention in the design of such measures[2]. Tsui (1999) formally introduced a correlation-sensitive majorization criterion into the debate and showed that this new criterion, together with Kolm's uniform majorization criterion and a standard set of axioms, leads to the class of multidimensional generalized entropy measures. However, Tsui's result uses a version of the somewhat controversial axiom of (additive) decomposability, which, by requiring us to ignore some -- arguably useful -- information in a distribution, is known to rule out all but entropy-based measures in various economic and information-theoretic contexts (see Sen, 1997, chapter A.5, for a discussion of this axiom).

It is therefore worth asking whether it is possible to design *other* multidimensional inequality measures that satisfy *both* Kolm's criteria *and* the correlation-sensitive majorization criterion introduced by Tsui, thereby respecting the intuition that (a) uniform inequalities across people and (b) cross-correlations between inequalities in different dimensions matter (i.e. the intuition that (a) Y_1 is more equal than X_1, and Y_2 is more equal than X_2, and (b) X_2 is more equal than X_1, and Y_2 is more equal than Y_1).

The present paper seeks to answer this question. Its purpose is methodological and substantive. On the methodological side, the paper presents a rather general way of defining multidimensional inequality indices by first transforming multidimensional distributions into suitable 'welfare-concentration curves' (a term from Kolm, 1977) and then constructing a multidimensional inequality index on the basis of a suitable one-dimensional aggregation function that takes these 'welfare-concentration curves' as its input and that respects the generalized Lorenz-ordering of these curves. On the substantive side, this method is then used to construct two examples of multidimensional inequality indices, and it is shown that these indices satisfy all of the above mentioned desiderata. One example is a generalization of the Gini coefficient, the other is a generalization of Atkinson's one-dimensional measure of inequality (Atkinson, 1970). It is also shown that an extension of Kolm's less frequently invoked

[2] In terms of the conditions stated below, it can easily be verified that condition (CIM) is violated by the inequality indices proposed in Tsui's 1995 paper subtitled, somewhat misleadingly in view of Atkinson & Bourguignon (1982), "The Atkinson-Kolm-Sen Approach": Tsui's relative inequality index (1995, theorem 1.), for instance, can assign a *lower* value to X_1 than to X_2 and a *lower* value to Y_1 than to Y_2, contrary to the Atkinson-Bourguignon-Walzer intuition that systematic cross-correlations between inequalities in different dimensions *increase* overall inequality.

criterion, *directional/price majorization*, namely *non-negative directional/price majorization*, already captures the Atkinson-Bourguignon-Walzer intuition about cross-correlation: we shall prove that the correlation-sensitive majorization criterion introduced by Tsui (1999) is in fact a (proper) sub-criterion of *non-negative directional/price majorization*.

After some basic definitions (section ***2.***), it will be requisite to survey various dominance criteria and explore their logical interrelations (section ***3.***); we shall then explain the present use of 'welfare-concentration curves' (section ***4.***), and we shall finally turn to the construction of fully-fledged inequality indices (section ***5.***).

2. Definitions and Basic Axioms

Let $N = \{1, 2, ..., n\}$ be a set of persons or households (for simplicity, hereafter 'persons'), and $K = \{1, 2, ..., k\}$ a set of goods/attributes, dimensions or distributive spheres.

A *multidimensional distribution* is an $n \times k$ matrix $X = (x_{ij})$ over the non-negative real numbers such that the sum of each column is non-zero, where x_{ij} represents person i's share of good/attribute j. Let $M(n,k)$ be the set of all such matrices. The row vectors $x_1, x_2, ..., x_n$ represent different persons' vectors of goods/attributes. The distributions X_1, X_2, Y_1 and Y_2 above are examples of multidimensional distributions for n=3 and k=3.

A *multidimensional inequality index* is a function $I^n : M(n,k) \to \mathrm{R}$, where $I^n(X) \geq I^n(Y)$ is interpreted to mean "the overall level of multidimensional inequality in distribution X is at least as great as that in distribution Y".

The following basic axioms are straightforward generalizations of their familiar one-dimensional counterparts (see Tsui, 1999):

CONTINUITY (C). I^n is a continuous function.

ANONYMITY (A). For any $n \times n$ permutation matrix P permuting the rows of X, $I^n(X) = I^n(\Pi X)$.

NORMALIZATION (N). If all rows of a distribution X are identical (i.e. the distribution in each dimension is perfectly equal), $I^n(X) = 0$.

REPLICATION INVARIANCE (RI). Given a $n\times k$ distribution matrix X, let Y be the $n*r\times k$ distribution matrix defined by

$$Y = \begin{pmatrix} X \\ X \\ \cdots \\ X \end{pmatrix} \text{ (with r 'replications' of } X\text{).}$$

Then $I^{n*r}(Y) = I^n(X)$.

RATIO-SCALE INVARIANCE (RS). For any $n\times n$ diagonal matrix $L=diag(l_1, l_2, ..., l_n)$ (with each l_i>0), $I^n(LX) = I^n(X)$.

These axioms by themselves, however, are insufficient to guarantee that a multidimensional inequality index is in any substantive sense 'egalitarian', i.e. that it respects the various intuitions about multidimensional inequality briefly introduced in the introduction. For this reason, our present list of axioms needs to be supplemented with the dominance criteria capturing these intuitions.

3. Dominance Criteria

In the present section, we will survey some of the most important dominance criteria proposed in the literature and explain how they are logically interrelated. In this context, we will prove a new result showing that the correlation-sensitive criterion introduced by Tsui (1999) is a subcriterion of non-negative directional majorization.

Each of the dominance criteria to be stated represents a proposed answer to the question of when a distribution X is *"clearly" at least as equal as*, and can therefore be said to *(at least weakly) dominate*, a distribution Y.

The first two dominance criteria to be stated have been suggested by Kolm (1977) and are essentially generalizations of the one-dimensional Pigou-Dalton criterion, according to which any transfer from a poorer person to a richer person increases inequality, other things remaining equal. Accordingly, if a distribution X can be obtained from a distribution Y by uniformly redistributing attributes so as to reduce the 'inequality-gap' between two or more persons, then X dominates Y. Define a *Pigou-Dalton matrix* to be an $n\times n$ matrix $P = l*E + (1-l)*Q$, where E is the $n\times n$

identity matrix and Q is a permutation matrix which transforms other matrices by interchanging two rows.

UNIFORM PIGOU-DALTON MAJORIZATION (UPD). $(X,Y)\in$ UPD and X º$_{\mathrm{UPD}}Y$ ("distribution X dominates distribution Y according to (UPD)") if and only if $X = TY$ where T is a finite product of Pigou-Dalton matrices. X Â$_{\mathrm{UPD}}Y$ ("distribution X *strictly* dominates distribution Y according to (UPD)") if, in addition, X cannot be derived from Y by permuting the rows of Y.

Define a *bistochastic matrix* to be an $n\times n$ matrix $B = (b_{ij})$ such that, for all j, $\sum_i b_{ij}=1$, and for all i, $\sum_j b_{ij}=1$.

UNIFORM MAJORIZATION (UM). $(X,Y)\in$ UM and X º$_{\mathrm{UM}}Y$ if and only if $X = BY$, where B is a bistochastic matrix. X Â$_{\mathrm{UM}}Y$ if, in addition, X cannot be derived from Y by permuting the rows of Y.

It is easily verified that, for the examples of multidimensional distributions given in the introduction, Y_1 Â$_{\mathrm{UM}}X_1$ and Y_2 Â$_{\mathrm{UM}}X_2$.

Moreover, (UPD) is a subrelation of (UM):

Proposition *3.1*. (Kolm, 1977; Tsui, 1999) UPD$\subseteq$UM (whenever $k\leq 2$, UPD=UM).

To introduce Kolm's criterion of *directional/price majorization* (1977), we first need to introduce the one-dimensional concept of *generalized Lorenz-dominance*, in short *GL-dominance*.

We shall say that the vector $(s_1, s_2, ..., s_n) \in \mathrm{R}^n$ *GL-dominates*[3] *the vector* $(t_1, t_2, ..., t_n) \in \mathrm{R}^n$ if, for all j,

$$\sum_{i\in\{1, 2, ..., j\}} s'_i \geq \sum_{i\in\{1, 2, ..., j\}} t'_i ,$$

[3] This is the concept of *generalized* Lorenz-dominance because, when we compare the vectors $(s_1, s_2, ..., s_n)$ and $(t_1, t_2, ..., t_n)$ here, we do not consider their *normalized* Lorenz curves as in the standard definition of Lorenz-dominance, i.e. we do not normalize $(s_1, s_2, ..., s_n)$ and $(t_1, t_2, ..., t_n)$ by multiplying them by the inverses of the means of the s_i and of the t_i, respectively. For a discussion of generalized Lorenz dominance, see Shorrocks (1983) and Sen (1997, appendix A.3).

where $(s'_1, s'_2, ..., s'_n)$ and $(t'_1, t'_2, ..., t'_n)$ are permutations of $(s_1, s_2, ..., s_n)$ and $(t_1, t_2, ..., t_n)$, respectively, such that $s'_1 \leq s'_2 \leq ... \leq s'_n$ and $t'_1 \leq t'_2 \leq ... \leq t'_n$. The relation of GL-dominance is said to be *strict* if at least one of the above inequalities is strict.

Intuitively, the *generalized Lorenz curve* of an (income) vector $(s_1, s_2, ..., s_n)$ can be obtained by, firstly, rewriting the vector $(s_1, s_2, ..., s_n)$ as $(s'_1, s'_2, ..., s'_n)$ such that the incomes of the n persons are arranged in an increasing order $s'_1 \leq s'_2 \leq ... \leq s'_n$; secondly, by plotting the proportion of persons $^j/_n$, ranging from $0 = {}^0/_n$ to $1 = {}^n/_n$, on the x-axis against the total income $\sum_{i \in \{1, 2, ..., j\}} s'_i$ controlled by the poorest $^j/_n$ of society (the poorest j persons) on the y-axis and connecting these points with line-segments. Then $(s_1, s_2, ..., s_n)$ GL-dominates $(t_1, t_2, ..., t_n)$ if the generalized Lorenz curve of $(s_1, s_2, ..., s_n)$ lies nowhere below that of $(t_1, t_2, ..., t_n)$, and the dominance is strict if the two curves do not coincide.

DIRECTIONAL/PRICE MAJORIZATION (DM). $(X,Y) \in$ DM and X º$_{\mathrm{DM}}Y$ if and only if, for all price vectors $a \in \mathrm{R}^k$, the vector Xa GL-dominates the vector Ya. X Â$_{\mathrm{DM}}Y$ if, in addition, X cannot be derived from Y by permuting the rows of Y.

The logical connection between the previous dominance criteria and directional/price majorization is characterized by the following proposition:

Proposition *3.2*. (Kolm, 1977; Bhandari, 1995) UM⊂DM.

We can extend the dominance relation of (DM) by restricting the set of relevant price vectors to all non-negative ones.

NON-NEGATIVE DIRECTIONAL/PRICE MAJORIZATION (DM+). $(X,Y) \in$ DM+ and X º$_{\mathrm{DM+}}Y$ if and only if, for all price vectors $a \in \mathrm{R}_+^k$, the vector Xa GL-dominates the vector Ya. X Â$_{\mathrm{DM+}}Y$ if, in addition, X cannot be derived from Y by permuting the rows of Y.

Obviously, DM⊆DM+. Below we shall in fact see that DM⊂DM+.

Tsui (1999) introduced a dominance criterion that explicitly captures the Atkinson-Bourguignon-Walzer intuition about cross-correlation. If the only difference between two multidimensional distributions X and Y is that there is a stronger positive correlation between advantaged positions within different dimensions and also between disadvantaged positions within different dimensions under Y than under X

(i.e. under Y, someone who is well-off in one dimension is more likely to be well-off across the board than under X; and, under Y, someone who is badly off in one dimension is more likely to be badly off across the board than under X), then X dominates Y:

Define a *correlation increasing transfer* as follows (Boland & Proschan, 1988). Given two row vectors $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$, let $x \wedge y = (min(x_1, y_1), min(x_2, y_2), ..., (min(x_k, y_k))$, and let $x \vee y = (max(x_1, y_1), max(x_2, y_2), ..., max(x_k, y_k))$. A distribution Y can be derived from a distribution X by a *correlation increasing transfer* if, for some row indices i and j ($i \neq j$), $y_i = x_i \wedge x_j$ and $y_j = x_i \vee x_j$, and, for all $m \notin \{i, j\}$, $x_m = y_m$. Such a transfer is *strict* if $Y \neq X$ and Y is not just the result of swapping the rows i and j in X.

CORRELATION INCREASING MAJORIZATION (CIM). $(X,Y) \in$ CIM and $X \succsim_{\text{CIM}} Y$ if and only if Y can be derived from X by a permutation of rows and a finite sequence of correlation increasing transfers. $X \succ_{\text{CIM}} Y$ if, in addition, at least one of these correlation increasing transfers is strict.

For the examples of multidimensional distributions given in the introduction, X_1 and Y_1 can be derived, respectively, from X_2 and Y_2 by a sequence of strict correlation increasing transfers, whence $X_2 \succ_{\text{CIM}} X_1$ and $Y_2 \succ_{\text{CIM}} Y_1$.

The following proposition shows that (UM) (including its subrelation (UPD)) and (CIM) are logically independent.

Proposition *3.3.* (Tsui, 1999) UM$\cap$CIM=$\{(X,Y)$: X can be derived from Y by a permutation of rows$\}$, i.e. there exists no pair of distributions X and Y such that $X \succ_{\text{UM}} Y$ *and* $X \succ_{\text{CIM}} Y$.

We will now prove that (CIM) defines a subrelation of (DM+), but not of (DM), a logical connection that may not be at first sight obvious (whence DM$\neq$DM+ and, since DM$\subseteq$DM+, DM$\subset$DM+).

Proposition *3.4.* CIM$\subset$DM+.

Proof. Since UM$\not\subset$CIM, but UM$\subset$DM+, CIM$\neq$DM+. It is thus sufficient to prove that CIM$\subseteq$DM+. Suppose $X \succsim_{\text{CIM}} Y$, i.e. there exists a finite sequence $X = QX_0, X_1, ..., X_m = Y$ such that, for each i, X_{i+1} can be derived from X_i by a correlation increasing transfer and Q is a row-permutation matrix. We need to show that, for all price vectors $a \in \mathbb{R}_+^k$,

Xa GL-dominates Ya. Let $a \in R_+^k$. Since X can be obtained from X_0 by a permutation of rows, the generalized Lorenz curves of Xa and X_0a are identical, and, trivially, Xa (weakly) GL-dominates X_0a. We will now show that, for each i, X_ia GL-dominates $X_{i+1}a$. Write $X_i = (b_{ij})$ and $X_{i+1} = (c_{ij})$. Now there exist row-indices p and q ($p \neq q$) such that, for each j, $c_{pj} = min(b_{pj}, b_{qj})$ and $c_{qj} = max(b_{pj}, b_{qj})$ and, for all $r \notin \{p, q\}$ and all j, $c_{rj} = b_{rj}$. Then, for all $r \notin \{p, q\}$, the r^{th} components of X_ia and $X_{i+1}a$ conincide and equal $b_{r1}a_1+b_{r2}a_2+...+b_{rk}a_k$. However,

$$p^{\text{th}} \text{ component of } X_{i+1}a = min(b_{p1},b_{q1})a_1+min(b_{p2},b_{q2})a_2+...+min(b_{pk},b_{qk})a_k$$
$$\leq p^{\text{th}} \text{ component of } X_ia = b_{p1}a_1+b_{p2}a_2+...+b_{pk}a_k\,,$$
$$q^{\text{th}} \text{ component of } X_ia = b_{q1}a_1+b_{q2}a_2+...+b_{qk}a_k$$
$$\leq q^{\text{th}} \text{ component of } X_{i+1}a = max(b_{p1},b_{q1})a_1+max(b_{p2},b_{q2})a_2+...+max(b_{pk},b_{qk})a_k.$$

Hence the generalized Lorenz curve of $X_{i+1}a$ lies nowhere above that of X_ia, and X_ia GL-dominates $X_{i+1}a$. But GL-dominance is transitive, and so X GL-dominates Y. Since this holds for any $a \in R_+^k$, X º$_{\text{DM+}}Y$. If, in addition, X Â$_{\text{CIM}}Y$, then X and Y cannot be permutations of each other, and thus X Â$_{\text{DM+}}Y$, too. **Q.E.D.**

Proposition ***3.5.*** CIM$\not\subset$DM. In fact, whenever Y can be obtained from X by a strict correlation-increasing transfer, $(X,Y) \notin$ DM.

Proof. Suppose Y can be obtained from X by a strict correlation-increasing transfer. Then there exist row indices i and j ($i \neq j$) such that $y_i = x_i \wedge x_j$ and $y_j = x_i \vee x_j$, and, for all $m \notin \{i, j\}$, $x_m = y_m$. Moreover, $Y \neq X$ and Y is not just the result of swapping the rows i and j in X. Then, for at least two column indices, p and q, it must be the case that $x_{ip} > x_{jp}$ and $x_{iq} < x_{jq}$ (if necessary swap the labels i and j). Assume, for a contradiction, $(X,Y) \in$ DM. Then, for all $a \in R^k$, Xa GL-dominates Ya. Consider the price vector a whose p^{th} and q^{th} components equal $(-a_p)$ and a_q, respectively, where a_p, $a_q > 0$ (e. g. $a_p = a_q = 1$) and whose other entries are all 0. By assumption, Xa GL-dominates Ya (note that Xa and Ya differ from each other only in rows i and j). This implies that
either

$$\text{row } i \text{ of } Ya = y_{ip}(-a_p) + y_{iq}a_q = x_{jp}(-a_p) + x_{iq}a_q$$
$$\leq \text{row } i \text{ of } Xa = x_{ip}(-a_p) + x_{iq}a_q\,,$$
$$\text{row } j \text{ of } Xa = x_{jp}(-a_p) + x_{jq}a_q$$
$$\leq \text{row } j \text{ of } Ya = y_{jp}(-a_p) + y_{jq}a_q = x_{ip}(-a_p) + x_{jq}a_q$$

or

$$\text{row } j \text{ of } Ya = y_{jp}(-a_p) + y_{jq}a_q = x_{ip}(-a_p) + x_{jq}a_q$$
$$\leq \text{row } i \text{ of } Xa = x_{ip}(-a_p) + x_{iq}a_q\,,$$
$$\text{row } j \text{ of } Xa = x_{jp}(-a_p) + x_{jq}a_q$$
$$\leq \text{row } i \text{ of } Ya = y_{ip}(-a_p) + y_{iq}a_q = x_{jp}(-a_p) + x_{iq}a_q$$

From the first set of inequalities, we get

(i) $x_{jp} \geq x_{ip}$, $x_{iq} \leq x_{jq}$,

and, from the second set of inequalities, we get

(ii) $x_{jq} \leq x_{iq}$, $x_{ip} \geq x_{jp}$.

Now (i) contradicts $x_{ip} > x_{jp}$, and (ii) contradicts $x_{iq} < x_{jq}$. Consequently, $(X,Y) \notin$ DM. **Q.E.D.**

To summarize, the logical connections between the stated dominance criteria are as follows:

$$\begin{matrix} \text{UPD} \subseteq \text{UM} \subset \text{DM} \\ \text{CIM} \end{matrix} \subset \text{DM+} \qquad \text{(with UPD=UM whenever } k \leq 2\text{).}$$

In particular, (DM+) is the only one of the stated dominance criteria that is sensitive *both* to the uniform inequality of a multidimensional distribution across people *and* to the cross-correlation between inequalities in different dimensions of goods/attributes.

4. Welfare Concentration Curves

We have already defined what it means to say that one vector in R^n GL-dominates another. Given a distribution matrix X, the basic idea of the present section is, first, to use a suitable function $u : R_+^k \rightarrow R_+$ to aggregate each person's row-vector of goods/attributes into an overall evaluation figure for this person (representing how well-off this person is in terms of his or her share of goods *across the different dimensions*) and, second, to assess the resulting vector of evaluation figures by considering its generalized Lorenz curve, to be called the *welfare concentration curve* of the distribution X for the aggregation function u.

For any two distribution matrices X and Y and an aggregation function u, we can then ask whether the corresponding welfare concentration curve of X lies nowhere below that of Y, i.e. whether $(u(x_1), u(x_2), ..., u(x_n))$ GL-dominates $(u(y_1), u(y_2), ..., u(y_n))$.

The following three propositions give us some important information about what properties the aggregation function $u : R_+^k \rightarrow R_+$ must satisfy in order for the relation of GL-dominance between corresponding welfare concentration curves to include UM (including UPD), CIM and DM+ (including DM).

Let X and Y be two multidimensional distribution matrices with row vectors x_1, x_2, ..., x_n and y_1, y_2, ..., y_n, respectively. Given a function $u : R_+^k \to R_+$, we shall say that X *(strictly) u-dominates* Y if ($u(x_1)$, $u(x_2)$, ..., $u(x_n)$) (strictly) GL-dominates ($u(y_1)$, $u(y_2)$, ..., $u(y_n)$).

Proposition *4.1.* (Kolm, 1977) Let $u : R_+^k \to R_+$ be continuous, increasing and strictly concave. If X º$_{UM}Y$, then X u-dominates Y; and if X Â$_{UM}Y$, then X strictly u-dominates Y.

A function $u : R_+^k \to R_+$ is said to be *L-superadditive* if, for any two vectors x and y in R_+^k, $u(x \wedge y) + (x \vee y) \geq u(x) + u(y)$. It can be shown that, if the second partial derivatives of u exist, u is L-superadditive if and only if, for all i, j ($i \neq j$),

$$\frac{\partial^2 u(t_1, t_2, ..., t_k)}{\partial t_i \partial t_j} \geq 0$$

(Marshall & Olkin, 1979; Tsui, 1999).

Proposition *4.2.* (Tsui, 1999) Let $u : R_+^k \to R_+$ be increasing, L-superadditive and of the form $u(t) = f(t_1) + f(t_2) + ... + f(t_k)$, for all $t \in R_+^k$ (with $f : R_+ \to R_+$). If X º$_{CIM}Y$, then X u-dominates Y; and if X Â$_{CIM}Y$, then X strictly u-dominates Y.

Proposition *4.3.* Let $u : R_+^k \to R_+$ be continuous, increasing, strictly concave and of the form $u(t) = f_1(t_1) + f_2(t_2) + ... + f_k(t_k)$, for all $t \in R_+^k$ (with $f_j : R_+ \to R_+$, for each j). If X º$_{DM+}Y$, then X u-dominates Y; and if X Â$_{DM+}Y$, then X strictly u-dominates Y.

Proof. Let $u : R_+^k \to R_+$ be any increasing concave function of the form $u(t) = f_1(t_1) + f_2(t_2) + ... + f_k(t_k)$, for all $t \in R_+^k$. Suppose that X dominates Y according to (DM+). Then, for all price vectors $a \in R_+^k$, the vector Xa GL-dominates the vector Ya. In particular, for each j in {1, 2, ..., k}, putting $a_j = (a_1^j, a_2^j, ..., a_k^j)$ with $a_i^j = 1$ for $i=j$ and $a_i^j = 0$ for all $i \neq j$, $Xa_j = (x_{1j}, x_{2j}, ..., x_{nj})$ GL-dominates $Ya_j = (y_{1j}, y_{2j}, ..., y_{nj})$, and, for any increasing and concave function f_j, $\sum_i f_j(x_{ij}) \geq \sum_i f_j(y_{ij})$. Then $\sum_i f_1(x_{i1}) + \sum_i f_2(x_{i2}) + ... + \sum_i f_k(x_{ik}) \geq \sum_i f_1(y_{i1}) + \sum_i f_2(y_{i2}) + ... + \sum_i f_k(y_{ik})$, and thus $\sum_i u(x_i) \geq \sum_i u(y_i)$. But since this holds for any increasing concave u of the form $u(t) = f_1(t_1) + f_2(t_2) + ... + f_k(t_k)$, X is "weakly more equal" than Y according to Kolm's definition (1977), and by Kolm's theorem 7., for the relation "weakly more equal" (see Kolm's remark on p. 8), ($u(x_1)$, $u(x_2)$, ..., $u(x_n)$) GL-dominates ($u(y_1)$, $u(y_2)$, ..., $u(y_n)$) for any such u, including any u satisfying the conditions of proposition ***4.3.***. If, in addition, u is *strictly* concave, as assumed in proposition ***4.3.***, the generalized Lorenz curves of ($u(x_1)$, $u(x_2)$, ..., $u(x_n)$)

and $(u(y_1), u(y_2), ..., u(y_n))$ coincide only if X and Y are permutations of each other. **Q.E.D.**

Is it possible to find a function u such that the corresponding relation of u-dominance includes *all of* UM (including UPD), DM+ (including DM) and CIM? By propositions ***4.1.***, ***4.2.*** and ***4.3.***, a function u has the required properties if it is continuous, increasing, strictly concave, L-superadditive and of the form $u(t) = f(t_1) + f(t_2) + ... + f(t_k)$, for all $t \in R_+^k$. More generally, since UM, CIM $\subset$ DM+ (see section ***3.***), whenever u satisfies the conditions of proposition ***4.3.***, the relation of u-dominance already includes all of DM+, CIM, DM, UM, UPD.

Are these conditions satisfiable? The answer to this question is positive: the function $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^r$ (with $0 < r < 1$) satisfies the conditions of propositions ***4.1.***, ***4.2.*** and ***4.3.***[4], and $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^{r_j}$ (with $0 < r_j < 1$, for each j), satisfies the conditions of proposition ***4.3.***.

We are now in a position to define a partial ordering on the set of all multidimensional distributions which includes all of the majorization criteria discussed in section ***3.***: let $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^{r_j}$ (with $0 < r_j < 1$, for each j) and define X to be "at least as equal as" ("more equal than") Y if X (strictly) u-dominates Y, i.e. if $(u(x_1), u(x_2), ..., u(x_n))$ (strictly) GL-dominates $(u(y_1), u(y_2), ..., u(y_n))$.

5. Defining Multidimensional Inequality Indices

For each of the dominance criteria (UPD), (UM), (DM), (CIM) and (DM+), we shall say that a multidimensional inequality index I^n satisfies the given criterion if, for any two distributions X and Y, $I^n(Y) \geq I^n(X)$ whenever X dominates Y according to the given criterion, and $I^n(Y) > I^n(X)$ whenever X strictly dominates Y according to this criterion.

The main question of this paper can now be formulated more precisely: How, if at all, can we define a multidimensional inequality index satisfying all of (C), (A), (N), (RI), (RS), (UPD), (UM), (CIM), (DM) and (DM+)? From Tsui (1999), we know that this set of axioms -- excluding (DM) and (DM+), which Tsui did not consider -- is

[4] The function u is clearly increasing, continous and of the required additive form. Its strict concavity can be shown by observing that its Hessian matrix $D^2u(t)$ is a diagonal matrix which has strictly negative eigenvalues and is thus negative definite for every $t \in R_+^k$. Its L-superadditivity can be shown by observing that, for all $i \neq j$, $\partial^2 u(t_1, t_2, ..., t_k)/\partial t_i \partial t_j = 0$.

consistent, for it is possible to define a suitable class of multidimensional inequality measures satisfying all of them, Tsui's example being a class of multidimensional generalized entropy measures. But, as briefly mentioned above, Tsui also invokes an axiom of *decomposability* which requires that, for any partition of the set of persons N into two subgroups N_1 and N_2, overall inequality be a function of (weighted) *within-group* inequality for each of N_1 and N_2 and *between-group* inequality determined on the basis of the mean distributions (vectors of column means) for each of N_1 and N_2. While useful for many purposes, inequality indices satisfying decomposability must ignore certain types of information about a distribution (see Sen, 1997, chapter A.5, for a discussion). In particular, in the case of multidimensional inequality measurement (and especially on a Walzerian conception of (in)equality), the question of how well-off each person in each dimension of goods/attributes is in comparison with *every* other person may be as important as the question of how well-off a person is in relation to a subgroup of society and how well-off this group, in aggregate, is in relation to other groups. A decomposable inequality index, however, cannot use the former type of information. For this reason, the present section seeks to explain how to construct multidimensional inequality indices other than those derived by Tsui using decomposability, yet respecting all of the above stated desiderata.

Essentially, the idea is to use the above defined function u to convert each multidimensional distribution X into a one-dimensional distribution $(u(x_1), u(x_2), ..., u(x_n))$ and then to apply a suitable generalized-Lorenz-consistent aggregation function[5] to map $(u(x_1), u(x_2), ..., u(x_n))$ to a real number, to be interpreted as the overall level of inequality $I^n(X)$ under X. By the generalized-Lorenz-consistency of the aggregation function, we would have $I^n(Y) \geq I^n(X)$ whenever $(u(x_1), u(x_2), ..., u(x_n))$ GL-dominates $(u(y_1), u(y_2), ..., u(y_n))$, i.e. whenever X is considered to be at least as equal as Y by the partial ordering defined at the end of the previous section.

However, an inequality index constructed like this would violate (RS): it would not be invariant under positive linear transformations of the column vectors of a distribution. In order to capture the idea of *relative* inequality measurement represented by (RS), an inequality index must be sensitive *only* to the *relative* distribution of goods within each dimension and *not* to the *total* size of the 'cake' in each dimension. Thus, when we evaluate the overall level of inequality in a distribution X, what we are really

[5]An aggregation function $f : \mathrm{R}^n \to \mathrm{R}$ is *generalized-Lorenz-consistent* if, for all vectors $(s_1, s_2, ..., s_n)$ and $(t_1, t_2, ..., t_n)$, whenever $(s_1, s_2, ..., s_n)$ GL-dominates $(t_1, t_2, ..., t_n)$, $f(s_1, s_2, ..., s_n) \leq f(t_1, t_2, ..., t_n)$ (in fact, this is the definition for an 'inequality'-context; in a 'welfare'-context, '≤' would be replaced with '≥').

looking at is the level of inequality in an adjusted matrix X^c, where X^c is the result of scaling the column vectors in X in such a way that the mean of each column equals 1:

Given a distribution $X = (x_{ij})$, let $m_1, m_2, ..., m_k$ be the means of the k columns of X, i.e. for each j, $m_j = {}^1/_n * \sum_{i \in \{1, 2, ..., n\}} x_{ij}$. Then X induces a *compensation matrix* X^c defined by $X^c = (x_{ij}/m_j)$. The i,j-th entry in X^c represents the proportion of good/attribute j held by person i.

We shall say that an inequality index $I^n : M(n,k) \rightarrow$ R is *u-dominance-consistent* with respect to a given function $u : \mathrm{R}_+^k \rightarrow \mathrm{R}_+$ if, for all distributions X and Y, $I^n(X) \geq I^n(Y)$ whenever Y^c u-dominates X^c and $I^n(X) > I^n(Y)$ whenever Y^c strictly u-dominates X^c.

It is important to note that, for each of the dominance criteria (UPD), (UM) and (CIM), the dominance of a matrix X over a matrix Y is logically sufficient for the dominance of the adjusted matrix X^c over the adjusted matrix Y^c.

Proposition *5.1.* For each of the dominance criteria (UPD), (UM) and (CIM) and any two distributions X and Y, if X (strictly) dominates Y according to the chosen criterion, then X^c (strictly) dominates Y^c according to the same criterion.

Proof. First of all, note that, under each of (UPD), (UM) and (CIM), a *necessary* (but clearly not sufficient) condition for a distribution X to dominate a distribution Y is that the means of the k columns of X, $m_1, m_2, ..., m_k$, are identical to those of Y (the transformations induced by Pigou-Dalton matrices, bistochastic matrices and correlation increasing transfers preserve the sums of column vectors).
If X can be obtained by permuting the rows of Y, the same row-permutation(s) can be used for transforming Y^c into X^c. We may therefore turn directly to strict dominance.
Suppose X Â$_{\mathrm{UPD}}Y$ (or X Â$_{\mathrm{UM}}Y$). Then $X = TY$, where T is a finite product of Pigou-Dalton matrices (or T is a bistochastic matrix) and X cannot be obtained by permuting the rows of Y. Let $x_{\bullet 1}, x_{\bullet 2}, ..., x_{\bullet k}$ and $y_{\bullet 1}, y_{\bullet 2}, ..., y_{\bullet k}$ be the column vectors of X and Y, respectively. For each j, $x_{\bullet j} = Ty_{\bullet j}$, and hence $(1/m_j)x_{\bullet j} = (1/m_j)Ty_{\bullet j} = T((1/m_j)y_{\bullet j})$, i.e. $X^c = TY^c$, hence X^c ° $_{\mathrm{UPD}}Y^c$ (or X^c ° $_{\mathrm{UM}}Y^c$) Moreover, this dominance is strict: $X^c = (x^c_{ij})$ cannot be obtained by permuting the rows of $Y^c = (y^c_{ij})$, since the same row-permutation(s) would then be suitable for transforming $Y = (y^c_{ij}m_j)$ into $X = (x^c_{ij}m_j)$, contradicting the assumption that X Â$_{\mathrm{UPD}}Y$ (or X Â$_{\mathrm{UM}}Y$).
Suppose X Â$_{\mathrm{CIM}}Y$. Then Y can be derived from X by a permutation of rows and a finite sequence of correlation increasing transfers at least one of which is strict. Now, for each column vector $x_{\bullet j}$ of X, the ordering of this vector's components is invariant

under multiplication by $(1/m_j)$, and hence $X^c = (x_{ij}/m_j)$ can be transformed into $Y^c = (y_{ij}/m_j)$ by the same correlation increasing transfers (one of which is strict) and row-permutations by which X can be transformed into Y. Hence X^c Â$_{CIM}$ Y^c. **Q.E.D.**

If X dominates Y according to (DM) (or (DM+)), on the other hand, this does not in general imply that X^c also dominates Y^c according to (DM) (or (DM+)). For instance, if Y=2X, then Y Â$_{DM+}$$X$, but X^c=Y^c, whence it is not the case that Y^c Â$_{DM+}$$X^c$. However, if our main focus is on the *relative* distribution of goods within each dimension rather than the *total* amount of goods in each dimension, we can use the following criteria instead of (DM) and (DM+):

DIRECTIONAL/PRICE MAJORIZAION OF COMPENSATION MATRICES (DM^C). $(X,Y)\in$ DM^C if and only if $(X^c, Y^c)\in$ DM.

DIRECTIONAL/PRICE MAJORIZAION OF COMPENSATION MATRICES (DM+^{C}). $(X,Y)\in$ DM+^{C} if and only if $(X^c, Y^c)\in$ DM+.

Proposition ***5.1.*** and the results of section ***3.*** are easily seen to imply that

$$\begin{matrix} \text{UPD} \subseteq \text{UM} \subset \text{DM}^C \\ \text{CIM} \end{matrix} \subset \text{DM+}^C \qquad \text{(with UPD=UM whenever } k\leq 2\text{)}.$$

But then proposition ***5.1.*** and the results of section ***4.*** imply that, whenever $u : R_+^k \rightarrow R_+$ is continuous, increasing, strictly concave and of the form $u(t) = f_1(t_1) + f_2(t_2) + ... + f_k(t_k)$, for all $t\in R_+^k$, a u-dominance-consistent inequality index satisfies (UM) (including (UPD)), (CIM) and (DM+^{C}) (including (DM^C))[6].

We will now define two u-dominance-consistent inequality indices satisfying (C), (A), (N), (RI), (RS), (UPD), (UM), (DM^C), (DM+^{C}) and (CIM).

The first one is a generalization of the well-known one-dimensional Gini coefficient. As before, let $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^{r_j}$ (with $0 < r_j < 1$, for each j).

First note the following lemma:

[6]Such an inequality index will not, in general, satisfy (DM) and (DM+). But if -- as emphasized -- our focus is on *relative* distributions, rather than the *total* amounts, of goods, and if compensation matrices rather than unadjusted distribution matrices should therefore form the basis for inequality comparisons, it is plausible (indeed requisite) to replace (DM) and (DM+) with (DM^C) and (DM+^{C}), respectively.

Lemma *5.2.* If $u : R_+^k \to R_+$ is continuous, increasing and strictly concave,

$$max\{{}^1/_n * \sum_i u(x^c_i) : x^c_1, x^c_2, ..., x^c_n \text{ are the row-vectors of some } X^c\}$$

exists and equals $m_{u\text{-}max} = u(t)$, where $t = (1, 1, ..., 1)$.

Proof. Let B be the bistochastic $n \times n$ matrix all of whose entries equal $1/n$. Given any $Y=(y_{ij}) \in M(n,k)$, note that $BY^c = X = (x_{ij})$, where X ($=X^c$) is the $n \times k$ matrix all of whose entries equal 1. But, since u is continuous, increasing and strictly concave, theorem 3 in Kolm (1977) implies that $\sum_i u(x^c_i) \geq \sum_i u(y^c_i)$. Thus $n*u(1, 1, ..., 1) \geq \sum_i u(y^c_i)$, and $u(1, 1, ..., 1) \geq {}^1/_n * \sum_i u(y^c_i)$. **Q.E.D.**

Given a distribution X, with row-vectors $x_1, x_2, ..., x_n$, let $x^c_1, x^c_2, ..., x^c_n$ be the row-vectors of the compensation matrix X^c induced by X. Consider the area between the generalized Lorenz curve of $(u(x^c_1), u(x^c_2), ..., u(x^c_n))$ and the line of perfect equality, also definable as the generalized Lorenz curve of $(u(1, 1, ..., 1), u(1, 1, ..., 1), ..., u(1, 1, ..., 1))$, representing a perfectly equal distribution (from lemma ***5.2.***, we can infer that the former generalized Lorenz curve lies below that line); then define the generalized Gini-coefficient to be the ratio between this area and the entire triangular area underneath the line of perfect equality.

Some algebraic manipulation yields the following definition:

Definition *5.3.* The *multidimensional generalized Gini coefficient* is the function $I^n : M(n,k) \to R$, defined by

$$I^n(X) = 1 - \frac{m_u}{m_{u\text{-}max}} * (1\text{-}G(u(x^c_1), u(x^c_2), ..., u(x^c_n)))$$

(where $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^r$, $m_u = {}^1/_n * \sum_i u(x^c_i)$, $m_{u\text{-}max} = u(1, 1, ..., 1)$, and $G : R_+^n \to R_+$ is the one-dimensional Gini-coefficient, defined by

$$G(t_1, t_2, ..., t_n) = 1 - \frac{1}{n^2 * {}^1/_n * \sum_i t_i} * \sum_{i\in\{1, 2, ..., n\}} \sum_{m\in\{1, 2, ..., n\}} min(t_i, t_m),$$

see Sen, 1997, chapter 2)

$$= 1 - \frac{{}^1/_n * \sum_i (\sum_{j\in\{1, 2, ..., k\}} f_j(x_{ij}/m_j))}{\sum_{j\in\{1, 2, ..., k\}} f_j(1)} * (1\text{-}G(u(x^c_1), u(x^c_2), ..., u(x^c_n))$$

(expanding the Gini-coefficient)

$$= 1 - \frac{\sum_{i\in\{1, 2, ..., n\}}\sum_{m\in\{1, 2, ..., n\}} min(\sum_{j\in\{1, 2, ..., k\}} f_j(x_{ij}/m_j), \sum_{j\in\{1, 2, ..., k\}} f_j(x_{mj}/m_j))}{\sum_{j\in\{1, 2, ..., k\}} f_j(1)*n^2}$$

(putting $u(t) = \sum_{j\in\{1, 2, ..., k\}} t_j^{r_j}$, with $0 < r_j < 1$, for each j)

$$= 1 - \frac{\sum_{i\in\{1, 2, ..., n\}}\sum_{m\in\{1, 2, ..., n\}} min(\sum_{j\in\{1, 2, ..., k\}} (x_{ij}/m_j)^{r_j}, \sum_{j\in\{1, 2, ..., k\}} (x_{mj}/m_j)^{r_j})}{k*n^2} .$$

Theorem *5.4.* The multidimensional generalized Gini coefficient satisfies (C), (A), (N), (RI), (RS), (UPD), (UM), (DM^C), ($DM+^C$) and (CIM).

Proof. (C): Consider the formulation

$$I^n(X) = 1 - \frac{m_u}{m_{u\text{-}max}} * (1 - G(u(x^c_1), u(x^c_2), ..., u(x^c_n))),$$

and note that the function which maps each X to X^c and the functions which map each X^c to the vector $(u(x^c_1), u(x^c_2), ..., u(x^c_n))$ and to m_u, and G are all continuous (and $m_{u\text{-}max}$ is constant), and hence, by the chain rule for continuity, so is I^n.
(A): Given an $n\times n$ permutation matrix P permuting the rows of X, first note that $(PX)^c = PX^c$. But now it is sufficient to observe that both $m_u = {}^1/_n*\sum_i u(x^c_i)$ and $G(u(x^c_1), u(x^c_2), ..., u(x^c_n))$ are invariant under permutations of $x^c_1, x^c_2, ..., x^c_n$.
(N): If all rows of a distribution X are identical, X^c is the matrix all of whose entries equal 1. Then $m_u = m_{u\text{-}max}$ and $G(t, t, ..., t) = 0$ with $t = u(1, 1, ..., 1)$, whence $I^n(X) = 0$.
(RI): Given a $n\times k$ matrix X, let Y be the $n*r\times k$ matrix defined by

$$Y = \begin{pmatrix} X \\ X \\ \cdots \\ X \end{pmatrix} \text{ (with } r \text{ 'replications' of } X),$$

and first note that

$$X^c = \begin{pmatrix} X^c \\ X^c \\ \cdots \\ X^c \end{pmatrix} \text{ (with } r \text{ 'replications' of } X^c).$$

Now let $y^c_1, y^c_2, ..., y^c_{r*n}$ be the row-vectors of Y^c; then, for each $j\in\{0, 1, ..., r-1\}$ and each $i\in\{1, 2, ..., n\}$, $y^c_{j*n+i} = x^c_i$, and

$$I^{r*n}(Y) = 1 - \frac{{}^1/_{r*n}*\sum_{i\in\{1, ..., r*n\}} u(y^c_i)}{\sum_{j\in\{1, 2, ..., k\}} f_j(1)} * (1 - G(u(y^c_1), u(y^c_2), ..., u(y^c_{r*n}))),$$

$$= 1 - \frac{{}^{1}/_{n*r}*(r*\sum_{i\in\{1,\dots,n\}}u(x^c_i))}{\sum_{j\in\{1,2,\dots,k\}}f_j(1)} * (1-G(u(x^c_1), u(x^c_2), \dots, u(x^c_n))),$$

(since G is replication invariant -- see Sen (1997), pp. 139 / 140)

$$= I^n(X).$$

(RS): It is sufficient to observe that, for any $n\times n$ diagonal matrix $L=diag(l_1, l_2, \dots, l_n)$ (with each $l_i>0$), $(LX)^c = X^c$.

To prove that I^n satisfies (UPD), (UM), (DM^C), (DM+^{C}) and (CIM), it is sufficient to prove that I^n is u-dominance-consistent. Given our definition of u, I^n will then satisfy (DM+^{C}), including (DM^C), (UM), (UPD) and (CIM), as established above. But it is known that, for any $s = (s_1, s_2, \dots, s_n)$, $t = (t_1, t_2, \dots, t_n) \in R^n$, whenever s GL-dominates t, $({}^1/_n*\sum_i s_i)*(1-G(s_1, s_2, \dots, s_n)) \geq ({}^1/_n*\sum_i t_i)*(1-G(t_1, t_2, \dots, t_n))$ ('>' if the dominance is strict) (see Sen (1997), pp. 136 / 137). This implies that ${}^1/_n*\sum_i u(x^c_i)*(1-G(u(x^c_1),u(x^c_2),\dots,u(x^c_n))) \geq {}^1/_n*\sum_i u(y^c_i)*(1-G(u(y^c_1),u(y^c_2),\dots,u(y^c_n)))$ whenever X^c u-dominates Y^c ('>' if the u-dominance is strict); and hence $I^n(X)\leq I^n(Y)$ whenever X^c u-dominates Y^c ('>' if the u-dominance is strict). Thus I^n is u-dominance-consistent as required. **Q.E.D.**

The second u-dominance-consistent inequality index to be defined can be interpreted as a generalization of Atkinson's one-dimensional measure of inequality. In the present case, the idea is to define a social evaluation function W which maps each compensation matrix X^c to an 'equally distributed equivalent compensation figure', i.e. a strictly positive real number m_e such that $W(X^c) = m_e = W(Y)$, where Y is the $n\times k$ matrix all of whose entries equal m_e. The overall level of inequality under a distribution X is then identified with the normalized difference between the 'equally distributed equivalent compensation figure' of a perfectly equal distribution and the 'equally distributed equivalent compensation figure' of X^c.

The function W will be required to be a suitable 'social extension' of the (personal) aggregation function $u : R_+^k \rightarrow R_+$. Define $W : M(n,k) \rightarrow R_+$ as follows. This time, let $u(t) = \sum_{j\in\{1,2,\dots,k\}} t_j^r$ (with $0 < r < 1$). Given a distribution matrix X with row-vectors x_1, x_2, ..., x_n, let

$$W(X) = ({}^1/_n*\sum_i({}^1/_k*u(x_{ij}))^s)^{1/(r*s)} = ({}^1/_n*\sum_{i\in\{1,2,\dots,n\}}({}^1/_k*\sum_{j\in\{1,2,\dots,k\}}x_{ij}^{\ r})^s)^{1/(r*s)}$$

where $0 < r, s < 1$. Then W satisfies the demanded properties: in particular, for any matrix X, $W(X) = m_e = W(Y)$, where Y is the $n\times k$ matrix all of whose entries equal m_e.

Before we can define the inequality index, we need to state two lemmas:

Lemma *5.5.* If $w : R_+ \rightarrow R_+$ is continous, increasing and strictly concave and C is a fixed strictly positive constant,

$$max\{{}^1/_n * \sum_i w(t_i) : t = (t_1, t_2, ..., t_n) \in R_+{}^n, \text{ where } \sum_i t_i \leq C\}$$

exists and equals $w({}^C/_n)$.

Proof. Let B be the bistochastic $n \times n$ matrix all of whose entries equal $1/n$. Given any $t = (t_1, t_2, ..., t_n)$ such that $\sum_i t_i \leq C$, let $e = (C-\sum_i t_i)/n$, and let $t' = (t'_1, t'_2, ..., t'_n)$ with $t'_i = t_i + e$. Then $\sum_i t'_i = C$, and since w is increasing, ${}^1/_n * \sum_i w(t'_i) \geq {}^1/_n * \sum_i w(t_i)$. Note that $B(t'_1, t'_2, ..., t'_n) = ({}^C/_n, {}^C/_n, ..., {}^C/_n)$, where the vectors are interpreted as column vectors. But since w is continuous, increasing and strictly concave, standard results (e.g. Sen, 1997, theorem 3.1) imply that $\sum_i w({}^C/_n) \geq \sum_i w(t'_i)$, and therefore $w({}^C/_n) = {}^1/_n * \sum_i w({}^C/_n) \geq {}^1/_n * \sum_i w(t'_i) \geq {}^1/_n * \sum_i w(t_i)$. **Q.E.D.**

Lemma *5.6.* $W_{max} := max\{W(X^c) : X^c$ is a compensation matrix$\} = 1$.

Proof. By lemma ***5.2.***,

$$max\{\sum_i u(x^c{}_i) : x^c{}_1, x^c{}_2, ..., x^c{}_n \text{ are the row-vectors of some } X^c\}$$
$$= u(1, 1, ..., 1) = n*k,$$

whence, by lemma ***5.5.***,

$$max\{{}^1/_n * \sum_i u(x^c{}_i)^s : x^c{}_1, x^c{}_2, ..., x^c{}_n \text{ are the row-vectors of some } X^c\}$$
$$= max\{{}^1/_n * \sum_i w(t_i) : t = (t_1, t_2, ..., t_n) \in R_+{}^n, \text{ where } \sum_i t_i \leq n*k\}$$
$$= ({}^{n*k}/_n)^s = k^s, \text{ (with } w(t) = t^s)$$

and some easy manipulation yields the desired result. **Q.E.D.**

Lemma ***5.6.*** confirms our intuition that the maximal value attained by the function W for some compensation matrix X^c equals 1, which is the 'equally distributed equivalent compensation figure' of the matrix all of whose entries equal $m_1 = m_2 = ... = m_k = 1$.

Definition *5.7.* A multidimensional generalization of Atkinson's one-dimensional inequality index is given by the function $I^n : M(n,k) \rightarrow R$, where

$$I^n(X) = 1 - \frac{W(X^c)}{W_{max}} = 1 - ({}^1/_n * \sum_{i \in \{1, 2, ..., n\}} ({}^1/_k * \sum_{j \in \{1, 2, ..., k\}} (x_{ij}/m_j)^r)^s)^{1/(r*s)},$$

where $0 < r, s < 1$.

Theorem *5.8.* The above defined multidimensional generalization of Atkinson's one-dimensional inequality index satisfies (C), (A), (N), (RI), (RS), (UPD), (UM), (DM^C), ($DM+^C$) and (CIM).

Proof. (C): Consider the formulation

$$I^n(X) = 1 - ({}^1/_n * \sum_{i\in\{1, 2, ..., n\}}({}^1/_k * \sum_{j\in\{1, 2, ..., k\}}(x_{ij}/m_j)^r)^s)^{1/(r*s)}$$

and note that the function which maps each X to X^c, as well as all other 'components' of this function are themselves continuous functions; by the chain rule for continuity, I^n is continuous.

(A): The invariance of I^n under permutations of the row-vectors x_1, x_2, ..., x_n, equivalent to permutations of the n terms $({}^1/_k * \sum_{j\in\{1, 2, ..., k\}}(x_{ij}/m_j)^r)$, is obvious.

(N): If all rows of a distribution X are identical, again note that X^c is the matrix all of whose entries equal 1. But we have seen above that, in this case, $W(X^c) = 1$, and hence $I^n(X) = 0$.

(RI): Given a $n\times k$ matrix X, define Y and Y^c as in the proof of theorem ***5.4.*** (just use p instead of r to denote the number of replications of X). Then

$$\mathrm{I}^{r*n}(Y) = 1 - ({}^1/_{p*n} * \sum_{i\in\{1, ..., p*n\}}({}^1/_k * \sum_{j\in\{1, 2, ..., k\}}(y^c_{ij})^r)^s)^{1/(r*s)}$$

$$= 1 - ({}^1/_{p*n} * p * \sum_{i\in\{1, ..., n\}}({}^1/_k * \sum_{j\in\{1, 2, ..., k\}}(x^c_{ij})^r)^s)^{1/(r*s)}$$

$$= 1 - ({}^1/_n * \sum_{i\in\{1, ..., n\}}({}^1/_k * \sum_{j\in\{1, 2, ..., k\}}(x^c_{ij})^r)^s)^{1/(r*s)} = I^n(X).$$

(RS): As before, it is sufficient to observe that, for any $n\times n$ diagonal matrix $L=diag(l_1, l_2, ..., l_n)$ (with each $l_i>0$), $(LX)^c = X^c$.

To prove that I^n satisfies (UM), (UPD), ($DM+^C$), (DM^C) and (CIM), it is again sufficient to prove that I^n is u-dominance-consistent. First define $E : \mathrm{R}_+^n \to \mathrm{R}_+$ be the function

$$E(t) = {}^1/_n * \sum_i t_i^s, \text{ with } 0 < s < 1.$$

Now a result by Shorrocks (1983) implies that, since $E(t)$ is symmetric, replication invariant, increasing, strictly concave, and additive, the following holds: for any $s, t \in \mathrm{R}_+^n$, $E(s) \geq E(t)$ whenever s GL-dominates t ('>' if the dominance is strict). But this means that ${}^1/_n * \sum_i u(x^c_i)^s \geq {}^1/_n * \sum_i u(y^c_i)^s$ (and hence $I^n(Y) \geq I^n(X)$) whenever X^c u-dominates Y^c ('>' if the dominance is strict), and thus I^n is u-dominance consistent. **Q.E.D.**

6. Conclusion

In the present paper, I have first surveyed a number of dominance criteria representing different answers to the question of when one multidimensional distribution is more unequal than another: uniform Pigou-Dalton majorization (UPD), uniform majorization (UM), directional/price majorization ($DM^{(c)}$), non-negative directional/price majorization ($DM+^{(c)}$) and correlation increasing majorization (CIM), and I have shown that they are logically interrelated in the following way ("$\subseteq$" ("$\subset$") means "is a (proper) subrelation of"):

$$\begin{array}{l} \text{UPD} \subseteq \text{UM} \subset \text{DM}^{(c)} \\ \qquad\qquad\qquad\qquad \subset \text{DM+}^{(c)} \\ \qquad\qquad\quad \text{CIM} \end{array} \qquad \text{(with UPD=UM whenever } k \leq 2\text{)}.$$

It is important to note that, whilst (UPD), (UM) and ($DM^{(c)}$) are sensitive to the *uniform* inequality of a multidimensional distribution *across people*, only (CIM) and, as I have shown, ($DM+^{(c)}$) are sensitive to the *cross-correlation* between inequalities in *different dimensions*.

I have secondly proposed a new method of constructing multidimensional inequality indices. Given a multidimensional distribution matrix (subsequently normalized such that the mean of each column, i.e. dimension of goods/attributes, equals 1), the first step is to use a suitable function $u : R_+^k \rightarrow R_+$ to aggregate each person's row-vector of goods/attributes into an overall evaluation figure for this person (representing how well-off this person is in terms of his or her share of goods *across the different dimensions*) and thus to transform a multidimensional distribution into a one-dimensional distribution of evaluation figures. The second step is to note that, for our definition of *u*, the generalized Lorenz ordering of these one-dimensional distributions of evaluation figures respects *all* of the dominance criteria (for the original multidimensional distributions) surveyed above. The third step is to define a multidimensional inequality index (and thereby to extend the dominance-induced partial orderings on the set of all multidimensional distributions to a complete ordering) by using a suitable generalized-Lorenz-consistent aggregation function to map each one-dimensional distribution of evaluation figures to a single real number, representing the overall level of inequality in the given multidimensional distribution.

To illustrate the proposed method, I have defined two new classes of multidimensional inequality indices:

(a) a multidimensional generalization of the Gini-coefficient:
for each distribution $X = (x_{ij}) \in M(n,k)$ with column means $m_1, m_2, ..., m_k$,

$$I^n(X) = 1 - \frac{\sum_{i\in\{1, 2, ..., n\}}\sum_{m\in\{1, 2, ..., n\}} min(\sum_{j\in\{1, 2, ..., k\}}(x_{ij}/m_j)^{r_j}, \sum_{j\in\{1, 2, ..., k\}}(x_{mj}/m_j)^{r_j})}{k*n^2},$$

where $0 < r_j < 1$, for each j; and

(b) a multidimensional generalization of Atkinson's one-dimensional inequality index:
for each distribution $X = (x_{ij}) \in M(n,k)$ with column means $m_1, m_2, ..., m_k$,

$$I^n(X) = 1 - ({}^1/_n*\sum_{i\in\{1, 2, ..., n\}}({}^1/_k*\sum_{j\in\{1, 2, ..., k\}}(x_{ij}/m_j)^r)^s)^{1/(r*s)},$$

where $0 < r, s < 1$.

Both (a) and (b) satisfy continuity, anonymity, normalization (in fact, they always take values in the interval [0, 1]), replication invariance and ratio-scale invariance. Moreover, they respect all of (UM), (UPD), (DM^C), ($DM+^C$) and (CIM) and thereby capture both Kolm's and Atkinson, Bourguignon and Walzer's intuitions about multidimensional inequality: firstly, they are sensitive to how uniformly unequal the distribution of goods/attributes across people is (by virtue of satisfying (UPD), (UE) and (DM^C)), and, secondly, they are sensitive to how systematically inequalities in different dimensions are cross-correlated (by virtue of satisfying (CIM) and ($DM+^C$)).

The results of this paper may thus point towards new ways of operationalizing the ideas pioneered by Kolm (1977), Atkinson and Bourguignon (1982) and Walzer (1983).

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