Multiproduct Haggling

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1 Introduction

Firms show a great deal of ingenuity in targeting certain products at certain consumers. A multiproduct monopolist can screen consumers over quality (see Rochet and Choné (1998)). Alternatively many firms screen consumers by bundling their products - a sophisticated form of quantity discounts (see McAfee et al. (1989), the work of Thanassoulis (2001) Chapter 2 and Bakos and Brynjolfsson (1999)). On top of these tools firms could use the probability of delivery as another screening instrument. Many firms commit to fixed take it or leave it prices, however, examples of firms offering consumers a lottery or a 'prize draw' also exist.[1] Still more common are firms haggling or bargaining with potential consumers. In terms of calculating the monopolist's expected profit, Riley and Zeckhauser (1983) showed that both haggling and prize draws are equivalent to lotteries in which either the goods the consumer receives or the price she pays for them is random. These selling strategies are screening consumers using the probability of delivery. The question is, if a multiproduct monopolist can commit to any selling strategy, when is the probability of delivery a profitable screening instrument? Can lotteries enhance monopoly profits or are fixed prices best?

In making her choice between fixed prices and lotteries a seller is attempting to balance two basic considerations. On the one hand consumers who do not value a product highly might not be prepared to pay for it outright, but are prepared to pay something less for the *chance* of receiving it. Similarly consumers who do value the products will be less keen to lose them through a lottery or by haggling. The lotteries therefore act to separate the consumers and so would allow the seller to increase her profits. On the other hand, the availability of the lottery might encourage some consumers who were going to purchase a product to try and save money by taking a gamble on receiving it through the lottery.

[1] For example, package tour operators will offer consumers a holiday at a reduced rate in a chosen country but the exact resort is ex ante random, depending on which has most space left.

This effect would act against the seller and tend to depress profits. Which of these effects dominates has thus far remained an open question. Does the cost to the seller of using the screening instrument at all outweigh any benefits she can then receive through enhanced price discrimination? Can we characterise when this is or isn't the case?

1.1 A Brief Survey of the Literature

The first major inroad into this problem was made by Riley and Zeckhauser (1983). They considered a one good monopolist selling to a population of consumers with unit demands. This restriction to one good had the effect of causing all consumers to differ in only one variable. In this context Riley and Zeckhauser (1983) were able to tackle the problem directly using an integration by parts technique coined in Mirrlees (1971) which is common to the screening literature in one dimension. With this tool Riley and Zeckhauser (1983) construct an elegant geometrical proof that the optimal probability of sale function takes only the values 0 or 1 and so must correspond to a take-it-or-leave-it offer. In other words, lotteries do not help a single good monopolist. However, the integration by parts technique famously does not hold in multiple dimensions.[2] This led to speculation that the Riley and Zeckhauser (1983) result might be specific to the case in which all consumers can be ordered on a line.

The work of Rochet and Choné (1998) seemed to support this view. They consider the general multiple dimensional screening problem in which a multiproduct monopolist sells her goods with a continuous range of product qualities to consumers who are characterised by a multiple dimensional type vector. Rochet and Choné (1998) reduce the monopolist's profit maximisation problem to a set of partial differential equations. These partial differential equations often can only be solved numerically. Rochet and Choné (1998) were able to show that in most such problems the monopolist will offer large sections of the population a personalised quality schedule. This made it seem very unlikely indeed that in the complicated multiproduct case lotteries could be ruled out with a Riley and Zeckhauser (1983) type result.

Attempts were however made to extend the 'no lottery' result of Riley and Zeckhauser (1983) to the multiproduct monopoly case. In particular McAfee and McMillan (1988) attempted to extend the Riley and Zeckhauser (1983) proof to multiple consumer characteristics directly. As mentioned above, the integration by parts method of Mirrlees (1971) does not extend to multiple dimensions. Instead, McAfee and McMillan (1988) considered the problem as the sum of one-dimensional elements. To allow the problem to be broken up in this way McAfee and McMillan (1988) restrict the distribution of consumers' valuations to satisfy a specific hazard rate condition. Having done this McAfee

[2] See Rochet and Choné (1998).

and McMillan (1988) reduce their restricted problem to a series of mathematical conditions. However, they acknowledge that these conditions are difficult to interpret and so restrict the problem further to the case of a two good monopolist. Here, they are able to claim that lotteries are not beneficial to the monopolist and so the Riley and Zeckhauser (1983) result does extend, albeit in their special case. This chapter will throw some doubt on the scope of this result.

Work in a different area by Rasul and Sonderegger (2000) tilted the question of lotteries in a different direction. They looked at the context of consumers as agents contracting with the monopolist or, in their work, the principal. What is crucial now is that the consumers (agents) have outside options which depend on their types. This is because Rasul and Sonderegger (2000) model the situation in which the agents must make relationship specific investments before they can contract with the principal.[3] Specifically, consumers with high valuations (agents of high type) who do not trade with the principal lose a great deal more than consumers with low valuations. In the Rasul and Sonderegger (2000) model this is because of the differing opportunity costs of no trade across the agents. This assumption differs with the standard approach to monopoly situations in which if a consumer receives nothing their utility is zero. Rasul and Sonderegger (2000) now find that using lotteries is profit maximising for the principal, even in the one dimensional case. So what has changed? Previously consumers with high valuations might have been tempted to pay less for a monopolist's products and take a gamble on receiving them at all. In the Rasul and Sonderegger (2000) context when these high valuation consumers lose the lottery they don't just receive nothing, they are actively hurt. This of course deters high valuation consumers from using the lottery option and so prevents profit loss along this avenue. On the other side of the equation low valuation consumers who wouldn't have participated if no lottery is offered are still tempted to try it. If they lose they are hardly hurt at all as Rasul and Sonderegger (2000) have a type dependent outside option. The balance of the two forces we discussed above has therefore clearly been pushed in favour of lotteries and consequently lotteries are found to be beneficial.

1.2 Outline of the Chapter

This chapter will show that even without tipping the scales in the sense of Rasul and Sonderegger (2000) the Riley and Zeckhauser (1983) no lottery result does not extend to the multiproduct case. In addition this chapter will begin to characterise the benefits of lotteries to the multiproduct monopolist. Previous multiproduct work on lotteries (see

[3] The Rasul and Sonderegger (2000) work is motivated by the automobile industry in which the component manufacturing 'agent' firms make relationship specific investments before they can trade with the principal who actually puts the cars together.

McAfee and McMillan (1988)) has been impeded by the fact that determining even the optimal non-lottery (or bundling) tariff is incredibly difficult.[4] Clearly therefore answering the question of whether there is any improvement in profitability with lotteries over and above the best bundling tariff is more complicated still. We are able to begin our analysis sidestepping these computational issues by introducing a model of substitutable goods in which bounds can be found for the optimal prices. We suppose that the monopolist is selling two substitutable goods and consumers demand at most one unit of either good. However, no consumer will choose to buy both goods. This model applies to a large class of markets, the market for a TV for example. In addition the results of this model will have direct implications for the more general multiproduct monopolist case analysed by McAfee and McMillan (1988).

The chapter begins with a simple example. We show that a monopolist selling substitutes to uniformly distributed consumers can benefit from using lotteries. In other words, the optimal selling strategy is more complicated than take it or leave it prices. Section 3 introduces the multiproduct monopoly model of substitutable goods which allows us to address when a general lottery will be more profitable than the best fixed prices. By looking in turn at the case of symmetry and asymmetry in Section 4 we find key applicability conditions on consumer taste which are sufficient to guarantee that lotteries are part of the most profitable selling strategy. The conditions are captured formally in Propositions 4 and 6 and discussed intuitively in Section 5.1. We see that one vehicle through which lotteries become profitable is if there are few consumers in the population who value *both* component goods relatively highly: it is these consumers who already purchase who would be tempted to make use of a lottery option and so would create a profit loss for the firm. Secondly we see that lotteries become profitable if margins are high: in this case profit increases from a small gain in the proportion of the market served outweigh the profit loss from other consumers swapping to the new lottery. More subtly we see that for lotteries to be profitable along this avenue we require there to be some consumers who would be tempted by the lottery. We will see that this might not be the case if the two substitutable goods differ very greatly in their quality. A further applicability condition derives from the fact that the lotteries can have asymmetric effects across a boundary. These effects must be considered individually to ensure a lottery is profitable.

In Section 5.2 we discuss the implication for selling strategies. It may, on the surface, seem that lotteries are not practical sales tools. However, we will see that profitable lotteries can be introduced by (a) claiming capacity constraints or (b) ensuring that a seller remains haggling over more than one single good, keeping negotiations open on several fronts. These insights have applications in industries as diverse as car sales, tourism and

[4] The work of McAfee et al. (1989) and Thanassoulis (2001) Chapter 2 on bundling considers deviations from optimal pure component pricing rather than the fully optimal tariff as a result.

telecoms amongst many others.

Having established the relevance of lotteries to the two substitutable goods seller, Section 6 considers the general two good monopolist serving consumers with no complementarities in demand. Once again lotteries are found to be generally useful as exhibited through a class of cases. This family of cases forms the core of a counter-example to the result of McAfee and McMillan (1988) documented above. We exhibit a case, satisfying the conditions of their no lottery result, in which lotteries are profitable. This example shows that the McAfee and McMillan (1988) result is not fully stated and at least needs some extra restrictions on consumer types to be valid.

Section 7 returns to the two substitutable goods case and considers how to determine the fully optimal pricing strategy. Sadly, as discussed in Rochet and Choné (1998), analytical solutions are hard to come by. The section therefore describes how the problem can be solved using linear programming techniques. The fully specified problem contains non-linear constraints which are hard to solve. Section 7 documents how the non-linear constraints can be relaxed to linear ones using implications of the convexity constraints satisfied by the solution. This technique is used to determine the fully optimal solution of a class of examples with uniformly distributed consumers. Proposition 12 shows us that in this case the optimal selling strategy is very simple and consists of take it or leave it prices in combination with only one lottery. This particular class of cases only leads to very modest profit gains. However, to illustrate that this is not always the case, Section 7.3.2 includes an example in which the profit gains of lotteries over the best fixed prices are over 8%.

Having established a model in which the profitability of lotteries can be studied, Section 8 considers the welfare implications of the profitable use of lotteries. As the substitutes model is new in the literature as a forum for the discussion of the profitability of lotteries so it is also new as a forum to assess the welfare implications. We show that if the introduction of a lottery which only alters consumer behaviour slightly is profit enhancing then it is also welfare enhancing. We show that this result not only holds in the substitutes model of Section 3 but also in the no complementarities model of McAfee and McMillan (1988). However, we show that this welfare result does not hold once the monopolist moves to the fully optimal lottery pricing strategy. In this case the welfare implications of allowing lotteries/haggling are ambiguous. We depict this result through a numerical example using the substitutes paradigm.

Thus far we have considered the case for lotteries in a monopoly context. Section 9 considers the role lotteries have to play amongst firms in strong competition. Using a result of Armstrong and Vickers (2001) we show that lotteries do not have a role to play in a strongly competitive market. Here prices have been driven too close to cost to make the lottery option appealing to consumers. Section 10 concludes with Appendix A giving the proofs of technical results discussed in the chapter.

2 Motivating Example

Riley and Zeckhauser (1983) showed that lotteries do not form part of the most profitable selling strategy for a single product monopolist. This example shows that this result does not extend to sellers of more than one good.

Suppose a monopolist is selling two substitutable products such as two types of TV set. We assume that no consumer will choose to purchase both goods together. For simplicity we assume that the monopolist has a zero unit cost of production for each good. Suppose that consumers' valuations for the two goods, (x, y) are uniformly distributed on the square $[5,6]^2$. The size of the population is normalised to 1.

Offering only take it or leave it prices the firm would offer price p for either good and receive a profit of

$$\pi(p) = 2 \cdot \underbrace{p}_{\text{revenue}} \cdot \underbrace{(6-p)\frac{1}{2}(1+p-5)}_{\substack{\text{proportion of consumers}\\ \text{buying one particular good}}}$$

The first order condition for the price p can then be derived and so the optimal fixed price found to be:

$$p^{opt} = \frac{10+\sqrt{28}}{3} = 5.097 \text{ (to 3 s.f.)}$$

Producing a profit with no lottery of

$$\pi\left(p^{opt}\right) = 5.049 \text{ (to 3 s.f.)} \tag{1}$$

Now suppose that the monopolist introduces a lottery offer of $\left(\frac{1}{2}, \frac{1}{2}\right)$ for a price of $p^{opt} - 0.04 = 5.057$. In words the new pricing strategy could be expressed as:

> "You can purchase good 1 at a price of p^{opt}. Alternatively you can purchase good 2 at the same price, p^{opt}. Finally you might instead decide to purchase the lottery which will deliver to you a good with certainty, with probability $\frac{1}{2}$ it will be good 1, alternatively it will be good 2. This lottery can be bought for a price of $p^{opt} - 0.04$."

The offering of this lottery will cause a set of people with valuations close to equal for the two goods to swap to the lottery option. Those consumers who only value one of the goods highly will not want to take a gamble on receiving the other good. Finally the number of consumers served will be increased as there will be some who will now decide

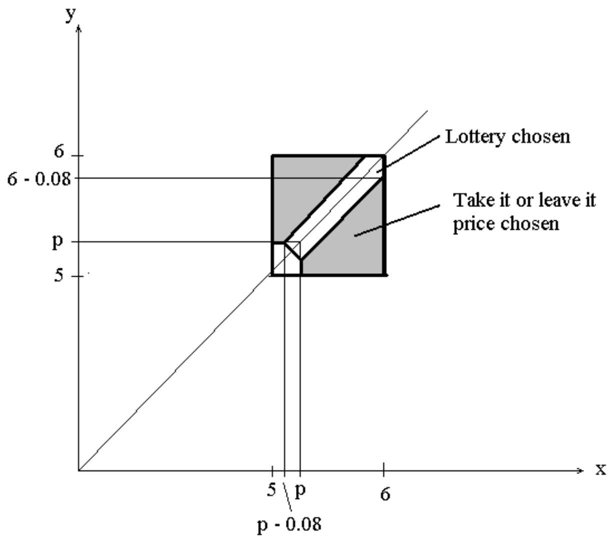


Figure 1: Figure depicting the behaviour of consumers in response to the new lottery offer

to buy given the slightly cheaper price. The consumers can therefore be broken down into three regions, separated by the lines

Indifferent between the lottery and a good with certainty	$x - p = \frac{x}{2} + \frac{y}{2} - p + 0.04 \Rightarrow y = x - 0.08$
Indifferent between the lottery and the outside option	$\frac{x}{2} + \frac{y}{2} - p + 0.04 = 0 \Rightarrow x + y = 2p - 0.08$

These regions are shown in Figure 1.

We can therefore calculate the new profit from this new tariff. The proportion of consumers buying one good with certainty is given by

$$\left[2 \cdot \frac{1}{2}(6-p)(1-0.08+p-0.08-5)\right]_{p=5.097} = 0.846$$

The proportion of consumers purchasing the lottery is given by

$$\left[2 \cdot (6-p) \cdot 0.08 + \frac{1}{2} \cdot 0.08^2\right]_{p=5.097} = 0.148$$

The new profit is therefore given by

$$\begin{aligned} \pi(p, p-0.04) &= [p \cdot 0.846 + (p-0.04) \cdot 0.148]_{p=5.097} \\ &= 5.060 \text{ (to 3 s.f.)} \end{aligned} \tag{2}$$

Therefore comparing the profit without a lottery (1) to that with a lottery (2) we have achieved a profit gain. That is, using a lottery has made the monopolist more profitable.

In this case by the very modest margin of 0.2%. This is small in this case but significant as we now see that lotteries can be profitable. This chapter will address when this is the case, how profitable lotteries might be and the welfare implications of using lotteries at all.

3 The Substitutes Model

To address the wider applicability of lotteries we consider a formal model of substitutable goods. Each individual consumer is characterised by two variables, x and y. These variables capture the consumer's willingness to pay for good 1 and good 2 respectively. The goods are substitutable and so we assume that the consumer will not choose to purchase both. This crucial simplification of the general multiproduct monopolist model allows us to determine many insights governing the profitability of lotteries over and above the best fixed prices can do in the multiproduct context. We suppose that consumers are distributed in the population according to an exogenous density function $f(x, y)$ which is known to the seller. For technical reasons we assume that the support of f is convex and that the density is bounded with bounded derivative. We normalise the size of the population to 1. The consumers are all risk neutral.

Each consumer will make the purchase decision which maximises her utility which is her willingness to pay less the amount paid. Each consumer also has the option of not making a purchase at all. This outside option is normalised to provide a utility of 0. The seller cannot differentiate between consumers.

We model the seller as a monopolist who has unit costs of production c_1 and c_2 for goods 1 and 2 respectively. The seller has no economies of scale or scope in serving this market. The seller has no commitment problem and so can offer take it or leave it prices or prices for a lottery of the sort described in the example of Section 2 above absent credibility issues. We suppose that initially the seller is offering her goods for sale at the optimal take it or leave it (tioli) prices, $\underline{p} = (p_1, p_2)$. The seller is considering introducing the lottery (q_1, q_2) such that $q_1 + q_2 = 1$ priced at $\underline{q} \cdot \underline{p} - \eta$ where $\eta > 0$ is to be determined. This is the lottery which awards the consumer good 1 with probability q_1 and failing that good 2 with probability q_2.[5] We will establish sufficient conditions for this to be a profitable strategy. I have decided to focus on lotteries which provide the consumer with a good with probability 1, $(q_1 + q_2 = 1)$, as these seem to have greater practical relevance.[6]

[5] The lottery is therefore like a financial asset: with probability q_1 the state of nature will be such that good 1 is delivered and with probability q_2 the state of nature will be such that good 2 is delivered.

[6] For a seller to be able to claim to take consumers money and then commit to randomise honestly and provide them with no good in return only some of the time is placing a great reliance on the sellers assumed full credibility.

We will see that in a large class of cases such lotteries will be the only ones involved in the fully optimal selling strategy.

The key to this chapter is that we can apply a derivatives approach to the firm's profit as a function of η above. The reason for this is that the choke price of a lottery $\underline{q}$ is at most $\underline{q} \cdot \underline{p}$. That is, no consumer will ever be willing to purchase the lottery $\underline{q}$ if its price is at or above $\underline{q} \cdot \underline{p}$ and so the lottery can only be profit enhancing as its price falls (η grows away from 0).

Lemma 1 *No consumer will strictly prefer the lottery $\underline{q}$ at a price of $\underline{q} \cdot \underline{p}$ to the fixed price options with prices $\left(\underline{p}\right)$.*

Proof. A risk neutral consumer of type (x, y) purchasing the lottery $\underline{q}$ at a price of $\underline{q} \cdot \underline{p}$ will derive utility $q_1 (x - p_1) + q_2 (y - p_2)$ which is the weighted average of the utility derived from the two fixed price options and so

$$[\text{Utility from lottery}] \leq \max \{(x - p_1), (y - p_2)\}$$

which proves the result. ■

The chapter therefore considers how profits are affected as the price of the lottery falls and consumers begin to be tempted by the lottery option.

3.1 Consumers' reaction to the lottery offer

Consumers make purchase decisions which maximise their utility. We can therefore determine which consumers respond to the introduction of the lottery offer $\underline{q}$ at a price of $\underline{q} \cdot \underline{p} - \eta$ and change their purchase behaviour by swapping to the lottery.

3.1.1 Consumers swapping from good 1 to lottery

Incentive compatibility requires good 1 purchasers to have types satisfying

$$x - p_1 > 0 \qquad x - p_1 > y - p_2$$

For such a consumer to swap to the lottery implies that

$$\begin{aligned} q_1 (x - p_1) + q_2 (y - p_2) + \eta &> x - p_1 \\ \Rightarrow y &> x - p_1 + p_2 - \frac{\eta}{q_2} \end{aligned}$$

The profit change experienced by the seller is therefore given by

$$\begin{aligned} &\int_{x=p_1}^{\infty} \int_{y=x-p_1+p_2-\frac{\eta}{q_2}}^{x-p_1+p_2} q_1 (p_1 - c_1) + q_2 (p_2 - c_2) - \eta - (p_1 - c_1)\, dF \\ = &\int_{x=p_1}^{\infty} \int_{y=x-p_1+p_2-\frac{\eta}{q_2}}^{x-p_1+p_2} \{q_2 [(p_2 - c_2) - (p_1 - c_1)] - \eta\}\, dF \qquad (3) \end{aligned}$$

Similarly consumers swapping from good 2 to the lottery results in a profit change of

$$\int_{y=p_2}^{\infty} \int_{x=y-p_2+p_1-\frac{\eta}{q_1}}^{y-p_2+p_1} \{q_1 [(p_1 - c_1) - (p_2 - c_2)] - \eta\} \, dF \tag{4}$$

3.1.2 Consumers swapping from no consumption to lottery

The consumers who were not previously being served have types satisfying

$$0 > x - p_1 \qquad 0 > y - p_2$$

For these consumers to swap to the lottery we require

$$q_1 (x - p_1) + q_2 (y - p_2) + \eta > 0$$

These three inequalities map out a triangular region in (x, y) space. The contribution to profit change is therefore given by

$$\int_{x=p_1-\frac{\eta}{q_1}}^{p_1} \int_{y=p_2-\frac{q_1}{q_2}(x-p_1)-\frac{\eta}{q_2}}^{p_2} \{q_1 (p_1 - c_1) + q_2 (p_2 - c_2) - \eta\} \, dF \tag{5}$$

3.1.3 Profit change

Combining terms (3), (4) and (5) and differentiating with respect to η gives

$$\begin{aligned}
\frac{\partial}{\partial \eta} \Delta\Pi(\eta) &= -\left\{ \begin{array}{c} \int_{x=p_1}^{\infty} \int_{y=x-p_1+p_2-\frac{\eta}{q_2}}^{x-p_1+p_2} dF + \int_{y=p_2}^{\infty} \int_{x=y-p_2+p_1-\frac{\eta}{q_1}}^{y-p_2+p_1} dF \\ + \int_{x=p_1-\frac{\eta}{q_1}}^{p_1} \int_{y=p_2-\frac{q_1}{q_2}(x-p_1)-\frac{\eta}{q_2}}^{p_2} dF \end{array} \right\} \\
&+ \frac{1}{q_2} \int_{x=p_1}^{\infty} \{q_2 [(p_2 - c_2) - (p_1 - c_1)] - \eta\} f\left(x, x - p_1 + p_2 - \frac{\eta}{q_2}\right) dx \\
&+ \frac{1}{q_1} \int_{y=p_2}^{\infty} \{q_1 [(p_1 - c_1) - (p_2 - c_2)] - \eta\} f\left(y - p_2 + p_1 - \frac{\eta}{q_1}, y\right) dy \\
&+ \frac{1}{q_2} \int_{x=p_1-\frac{\eta}{q_1}}^{p_1} \{q_1 (p_1 - c_1) + q_2 (p_2 - c_2) - \eta\} f\left(x, p_2 - \frac{q_1}{q_2}(x - p_1) - \frac{\eta}{q_2}\right) dx
\end{aligned} \tag{6}$$

At this point it would be tempting to examine whether a small increase in η above 0 can ever result in a profit improvement for the seller. Setting $\eta = 0$ in (6) above we have

$$\begin{aligned}
\left[\frac{\partial \Delta\Pi}{\partial \eta}\right]_{\eta=0} &= \frac{1}{q_2} \int_{x=p_1}^{\infty} q_2 [(p_2 - c_2) - (p_1 - c_1)] f(x, x - p_1 + p_2) \, dx \\
&\quad + \frac{1}{q_1} \int_{y=p_2}^{\infty} q_1 [(p_1 - c_1) - (p_2 - c_2)] f(y - p_2 + p_1, y) \, dy \\
&= 0
\end{aligned}$$

The rate of change in profit from the introduction of the lottery $\underline{q}$ at a price of $\underline{q} \cdot \underline{p} - \eta$ is therefore zero for small η. We cannot as yet say whether the lottery is profitable or not without looking at the second order conditions.[7]

3.1.4 Second Order Conditions on Profit Change

Differentiating (6) with respect to η and setting η to be zero and using the fact that $q_1 + q_2 = 1$ then gives

$$\begin{aligned}\left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0} &= -\frac{2}{q_2 q_1}\int_{x=p_1}^{\infty} f\left(x, x-p_1+p_2\right) dx \\ &-\frac{1}{q_2}\int_{x=p_1}^{\infty}\left[\left(p_2-c_2\right)-\left(p_1-c_1\right)\right] f_2\left(x, x-p_1+p_2\right) dx \\ &-\frac{1}{q_1}\int_{y=p_2}^{\infty}\left[\left(p_1-c_1\right)-\left(p_2-c_2\right)\right] f_1\left(y-p_2+p_1, y\right) dy \\ &+\frac{1}{q_2 q_1}\left\{q_1\left(p_1-c_1\right)+q_2\left(p_2-c_2\right)\right\} f\left(p_1, p_2\right)\end{aligned}$$

where $f_1(\cdot,\cdot)$ represents the partial derivative with respect to the first argument. This expression can be rewritten as

$$\begin{aligned}&\left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0} \\ &= \frac{1}{q_2 q_1}\left\{-2\int_{x=p_1}^{\infty} f\left(x, x-p_1+p_2\right) dx + \left\{q_1\left(p_1-c_1\right)+q_2\left(p_2-c_2\right)\right\} f\left(p_1, p_2\right)\right\} \\ &+ \frac{\left[\left(p_2-c_2\right)-\left(p_1-c_1\right)\right]}{q_2 q_1}\left\{\int_{x=p_1}^{\infty}\left\{q_2 f_1\left(x, x-p_1+p_2\right)-q_1 f_2\left(x, x-p_1+p_2\right)\right\} dx\right\}\end{aligned} \tag{7}$$

A sufficient condition for lottery $\underline{q}$, at a price of $\underline{q} \cdot \underline{p} - \eta$ with η just larger than zero, to be a profitable addition to the tioli prices $\underline{p}$ is therefore given when the right hand side of (7) is positive. This ensures that the profit change becomes positive as η increases away from 0.[8]

[7] One can prove similarly that this is actually a universal feature of lotteries as sold by a completely general multiproduct monopolist. This zero first order condition has the implication that pricing experiments aimed at exploring the profitability of lotteries must be non-local (in the sense of $\eta >> 0$) and therefore inherently riskier for the firm.

[8] This is sufficient not necessary for the lottery (q_1, q_2) to be a profitable addition to the tariff. However, one would need the profit function to not be quasi-concave in profit when lotteries are introduced for lotteries to have a chance of being profitable when (7) is negative.

4 Analysis of the Lottery Option

Section 3 has introduced a model of substitutes and noted that if the goods are priced at $\underline{p}$ then no consumer will be willing to pay more than $\underline{q} \cdot \underline{p}$ for the lottery $\underline{q}$. Condition (7) above therefore determines when the lottery $\underline{q}$ at a price of $\underline{q} \cdot \underline{p} - \eta$ will be a profitable addition to the tioli prices. However, we aim to determine whether lotteries can be more profitable than the *best* tioli prices. It is here that the substitutes model comes into its own. Determining the optimal prices for a general multiproduct monopolist is particularly difficult as noted in Thanassoulis (2001) Chapter 2 and by McAfee et al. (1989). However, in our model of substitutes, as consumers do not purchase the bundle of both goods, there are only two tioli prices to find - one in the case of symmetry. This section is therefore able to analyse when lotteries form part of the fully optimal pricing strategy.

4.1 The symmetric case

To understand better the conditions for a lottery to be profitable let us consider first the symmetric case where $c_1 = c_2 = c$, the consumer density function satisfies

$$f(x, y) = f(y, x)$$

and the optimal deterministic price is p for each good. To avoid degenerate cases we suppose that the support of the density function $f(\cdot)$ is convex in the set $\mathbb{R}^2_+ \setminus [0, c]^2$.[9] The second order conditions for a lottery to be profitable derived from (7) above then become

$$\left\{-2 \int_{x=p}^{\infty} f(x, x)\, dx + (p - c) f(p, p)\right\} > 0 \tag{8}$$

Before proceeding we define the following constants for the density f :

1. We suppose that consumer types are supported in a bounded space. In particular we suppose that $\operatorname{supp} f = \Omega \subset \left[a, a + \frac{K}{\sqrt{2}}\right]^2$ for some positive constants a and K.

2. The consumer density has a bounded derivative along the diagonal and so we define the constant g as

$$\sup_{x \in \operatorname{supp} f} f_1(x, x) = \sup_{x \in \operatorname{supp} f} f_2(x, x) := g$$

[9] The symmetry assumption along with the assumption of convexity on $\mathbb{R}^2_+ \setminus [0, c]^2$ guarantee that there exist consumers who value both goods equally and above cost. That is

$$\int_{x=c}^{\infty} f(x, x)\, dx \neq 0$$

3. The density along the diagonal is bounded above and below so we can define the constants $\{\underline{\mu}, \overline{\mu}\}$ such that $\underline{\mu} = \inf_{x \in \operatorname{supp} f} f(x,x) \leq \sup_{x \in \operatorname{supp} f} f(x,x) = \overline{\mu}$.

An argument identical to that in Armstrong (1996), Proposition 1, proves that the monopolist will always choose not to serve all consumer types so that there will be some consumers in Ω who are not making a purchase who value the two goods equally:

Lemma 2 *Either the best fixed price p is such that consumer (p,p) lies strictly inside Ω, or Ω lies across the line between (p,p) and the origin.*

Proof. This result follows by contradiction. Due to the symmetry of the problem and the convexity of Ω either the lemma holds or all consumers are being served. If all consumers were being served then the monopolist could increase the price of both goods by some small $\varepsilon > 0$. This would cause profits to grow by $O(\varepsilon)$ from those who remain buying. In return a set of consumers of order $O(\varepsilon^2)$ will stop buying corresponding to the area of those consumers remaining unserved. The profit gain exceeds the profit loss for small ε and so we would have (p,p) lying in the interior of Ω as required. ■

Analysis of equation (8) highlights two features of the consumers' tastes which lead to the presence of lotteries on the fully optimal selling strategy. These are:

1. Having few high valuation consumers. That is consumers who value both of the goods relatively highly.

   and

2. Having high margins as a result of consumers valuing the goods greatly above cost.

4.1.1 The impact of high valuation consumers

Suppose there are few of these consumers so that consumers valuing one component highly tend to value the other component less, then lotteries will be part of the fully optimal pricing strategy.[10] To prove this we note that associated with any population tastes $f(x,y)$ there is an ordered family of populations $\left\{\widetilde{f}_k(x,y)\right\}$ in which the best fixed prices, p, are unchanged but in which there are fewer high valuation consumers. Formally define $\widetilde{f}_k(x,y)$ to be the population with tastes given by $f(x,y)$ but in which all the consumers whose tastes lie in the small triangle to the North East of the line $x+y = 2\left(a + \frac{K}{\sqrt{2}}\right) - k$ are moved either South or West as depicted in Figure 2. In addition we require $k \in \left[0, 2\left(a + \frac{K}{\sqrt{2}}\right) - 2p\right)$ with the right hand end of this region labelled $\overline{k}$. A typical member of this family is depicted in Figure 2.

[10] This concept is clearly linked to the concept of negative correlation in consumers' valuations.

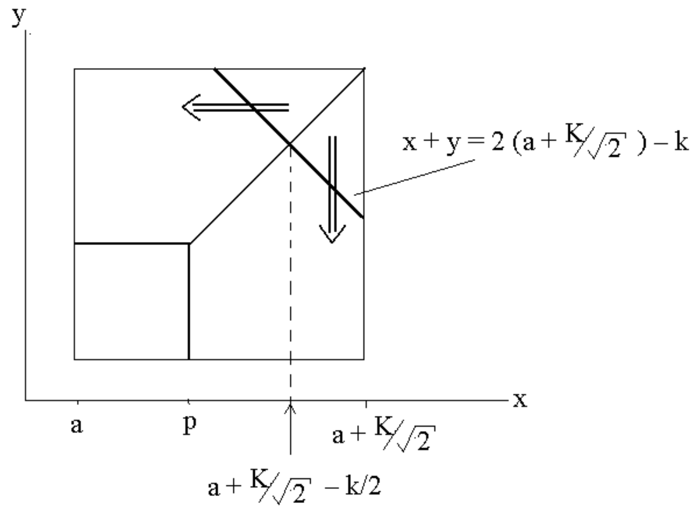


Figure 2: The distribution of types given by $\widetilde{f}_k(x,y)$ as detetmined from the population $f(x,y)$.

We note that the first order conditions for the best fixed price are the same for all members of the family $\left\{\widetilde{f}_k(x,y)\right\}$ and so the left hand side of equation (8) increases as k becomes larger.

Lemma 3 *Given a symmetric distribution of consumers' tastes, $f(x,y)$ then there exists some $k^* < \overline{k}$ such that lotteries are part of the fully optimal selling strategy for populations* $\left\{\widetilde{f}_k(x,y)\middle| k \in \left[k^*, \overline{k}\right)\right\}$

Proof. If Lemma 2 is satisfied by having $\Omega = \operatorname{supp} f$ cross the line between (p,p) and the origin then using lotteries is more profitable than the best fixed prices. To see this consider the lottery $\underline{q} = \left(\frac{1}{2}, \frac{1}{2}\right)$ priced at $\underline{p} \cdot \underline{q} - \eta$ where η is chosen so that the indifference curve between buying the lottery and not $\left(\left(\underline{x} - \underline{p}\right) \cdot \underline{q} + \eta = 0\right)$ is tangent to the North East edge of Ω. This is possible by the convexity and symmetry of Ω. This lottery will be chosen by a set of consumers who were previously not served and will be a positive contribution to profit by footnote 9. No other consumers will be affected giving the result.

If Lemma 2 is satisfied by having $(p,p) \in \Omega$ then as $k \nearrow \overline{k}$, $\int_{x=p}^{\infty} f(x,x)\,dx$ decreases monotonically towards 0 which gives the result. ■

Lemma 3 therefore illustrates the link between having few high valuation consumers who like both goods and the guaranteed optimality of lotteries. The intuition for this result is that by introducing a lottery in the manner of the model, profits are made on new consumers served but lost on previous consumers who value both goods equally and were already purchasing. By limiting the number of high type consumers we automatically reduce the magnitude of the loss and so lotteries become a more attractive proposition.

This section therefore shows that if Lemma 2 is satisfied by having (p,p) lie North East of Ω then lotteries will be part of the most profitable tariff: there are no consumers here who value both goods above their best fixed prices and so a lottery can be introduced priced low enough that some previously unserved consumers choose it.

4.1.2 The role of profit margins

The second feature of consumers' tastes which guarantees that lotteries are part of the fully optimal pricing strategy is if margins are high. In particular if consumers' valuations for the goods are far greater than cost. This will ensure that any profits made by serving a few more consumers are large and outweigh the loses made on existing business. To see that this scenario is compatible with setting the best fixed prices suppose that Lemma 2 is satisfied by having $f(p,p) \gneqq 0$. Now note from conditions 1 and 2 that $\left|\frac{d}{dx} f(x,x)\right| \leq 2g$ for all x in the support of f and so

$$\begin{aligned} \int_{x=p}^{\infty} f(x,x)\,dx &\leq \frac{1}{2}K\left(f(p,p) + 2gK + f(p,p)\right) \\ &= K\left(f(p,p) + gK\right) \end{aligned}$$

From (8) then the sufficient condition for lotteries to be profitable becomes

$$p - c > 2K\left(1 + \frac{gK}{f(p,p)}\right)$$

We can now use the fact that the optimal price p must satisfy $p \geq a$ and so deduce a condition on a for lotteries to be profitable. This is captured in the following proposition:

Proposition 4 *If the goods are symmetrically valued in the population, have equal production cost and consumer valuations satisfy conditions 1 to 3 then a sufficient condition for lotteries to be part of the optimal pricing strategy is that the lower bound of consumer valuations, a, satisfies*

$$a > c + \begin{cases} 2K\left(1 + \frac{g}{\mu}K\right) & \text{if } g \geq 0 \\ 2K\left(1 + \frac{gK}{\mu}\right) & \text{if } g < 0 \end{cases}$$

Before discussing the margins directly we note that for lotteries to be profitable we require either:

1. If the density is downward sloping along the diagonal (so that g is negative) then having consumers sufficiently diffuse along the diagonal (K large) will allow the sufficiency condition to be satisfied. This highlights the high valuation consumer effect

as, because the density is downward sloping, the number of high valuation consumers is low and there are a relatively large number of consumers with equal valuations between the goods who are indifferent between purchasing and not. Increasing the number of consumers being served through lotteries is therefore profitable.

Or

2. If the density along the diagonal is upward sloping then we need consumers to be very concentrated in their tastes (K small). This ensures that the profit gain from a small gain in consumers from a lottery is comparable with the profit loss sustained over the population along the diagonal. This is again a product of the first insight in which the number of high valuation consumers had to be limited for the sufficiency condition for lotteries to be satisfied.

We now note that Proposition 4 can always be satisfied if the margins are high. In particular if consumers value the goods sufficiently above cost.

Corollary 5 *Given any symmetric consumer density $f(x,y)$ supported on $\left[a, a+\frac{K}{\sqrt{2}}\right]^2$ and convex in $\mathbb{R}_+^2 \setminus [0,c]^2$ such that $\inf_{x \in \left[a, a+\frac{K}{\sqrt{2}}\right]} f(x,x) > 0$ with bounded derivatives then lotteries will be part of the most profitable selling strategy for symmetric goods if a is large enough.*

Proof. If Lemma 2 is satisfied by having $\Omega = \operatorname{supp} f$ cross the line between (p,p) and the origin then the proof of Lemma 3 shows that lotteries are part of the most profitable pricing strategy. Otherwise, under the conditions of the corollary, Proposition 4 is satisfied giving the result. ■

Corollary 5 therefore implies that lotteries are a robust feature of the optimal selling strategy when the goods are perfect substitutes. Under the conditions of the corollary using lotteries is profitable as they allow the proportion of the population served to be increased as compared to their not being used. This will be discussed in detail in Section 5 with the welfare implications discussed in Section 8.

4.2 General sufficient conditions for lotteries to be profitable

We now move away from symmetry and consider general sufficient conditions for lotteries to be part of the fully optimal pricing strategy. We define the following variables associated with the exogenous density function $f(\cdot,\cdot)$ and its support Ω:

1. We firstly require that the interior of the support of the density function, Ω, contains a consumer of type $\underline{p} = (p_1, p_2)$ where $\underline{p}$ is the optimal take it or leave it price. This is done by extending f by an arbitrarily small density if necessary.

2. We now suppose that the support of this density is bounded so that $\Omega \subset \left[\alpha, \alpha + \frac{K}{\sqrt{2}}\right] \times \left[\beta, \beta + \frac{K}{\sqrt{2}}\right]$ for some positive constants α, β and K.

3. The derivatives of the density function are bounded and satisfy

$$\begin{aligned}
\gamma_X &= \inf_{\{x,y\}\in \operatorname{supp} f} f_1(x,y) \leq \sup_{\{x,y\}\in \operatorname{supp} f} f_1(x,y) = \gamma^X \\
\delta_Y &= \inf_{\{x,y\}\in \operatorname{supp} f} f_2(x,y) \leq \sup_{\{x,y\}\in \operatorname{supp} f} f_2(x,y) = \delta^Y
\end{aligned}$$

Using these constants we note that $\frac{d}{dx} f(x, x - p_1 + p_2) \leq \gamma^X + \delta^Y$ and hence

$$\begin{aligned}
\int_{x=p_1}^{\infty} f(x, x - p_1 + p_2)\, dx &\leq \frac{1}{2} K \left[f(p_1, p_2) + \left(\gamma^X + \delta^Y\right) K + f(p_1, p_2) \right] \\
&= K \left(f(p_1, p_2) + \frac{1}{2}\left(\gamma^X + \delta^Y\right) K \right)
\end{aligned}$$

So from (7) we have that

$$\left[\frac{\partial^2 \Delta\Pi}{\partial \eta^2}\right]_{\eta=0} \geq \frac{1}{q_2 q_1} \left\{ \begin{array}{c} -2K\left(f(p_1,p_2) + \frac{1}{2}\left(\gamma^X + \delta^Y\right)K\right) \\ + \{q_1(p_1 - c_1) + q_2(p_2 - c_2)\} f(p_1, p_2) \\ + [(p_2 - c_2) - (p_1 - c_1)] \left\{q_2 \gamma_X - q_1 \delta^Y\right\} K \end{array} \right\}$$

Now suppose without loss of generality that the goods are labelled such that the monopolist enjoys a larger margin on good 2. That is $p_2 - c_2 \geq p_1 - c_1$. We see that if the margins on the two goods are the same then we have a condition entirely analogous to that in the symmetric case. However, in general we have the following result:

Proposition 6 *Suppose that consumer valuations satisfy assumptions 1 to 3 above. Lotteries will be more profitable than tioli prices* $\underline{p}$ *if there exists a lottery* (q_1, q_2) *with* $q_1 + q_2 = 1$ *such that*

1. *Probabilities can be chosen so that* $q_1 \delta^Y \leq q_2 \gamma_X$

   and also

2.

$$q_2(p_2 - c_2) + q_1(p_1 - c_1) > 2K\left(1 + \frac{1}{2}\frac{\left(\gamma^X + \delta^Y\right)K}{f(p_1, p_2)}\right)$$

As in the case of symmetry, condition 2 shows that having large enough margins is a prerequisite to meeting the sufficiency conditions for lotteries to be more profitable than just prices (p_1, p_2). This is because by introducing a lottery some already purchasing consumers will swap to the cheaper option and so one must ensure that the extra profits from new business outweigh the loss from those who derive the same utility from both goods and so move to the lottery. This makes clear the importance of having some consumers of type $\underline{p}$ in the population, as guaranteed by assumption 1. It is these consumers who are attracted by this lottery. If there are few of them then this lottery becomes less relevant.

Condition 1 highlights a second point in the general case. When margins on the two goods are unequal, introducing a lottery will engender a loss in profit from those that stop buying the high margin good coupled with a gain from those moving from the low margin good. We have labelled the high margin good 2. For a lottery to be profitable we need to ensure that this profit loss is not too large relative to the margin gain from those swapping from the other good to the lottery. This is ensured by making sure that the maximum rate of gain of consumers to the lottery in the Y direction $\left(\delta^Y\right)$ is at worst similar in size to the minimum rate of gain of consumers to the lottery in the X direction (γ_X). This result is captured in condition 1.

Having determined sufficient conditions in Proposition 6 for lotteries to be profitable we must consider if these conditions can ever be met if prices are set to their best tioli levels. It is here that the model of substitutes allows the insights to appear much more readily. Given Conditions 1 to 3 on the consumer density function, Proposition 6 can be made seemingly exogenous using the bounds that $p_1 \geq \alpha$ and $p_2 \geq \beta$. However, condition 1 on the density function is endogenous as it requires a consumer of type (p_1, p_2) to lie in the interior of the support of consumer tastes where (p_1, p_2) are the best tioli prices.[11] However, in the symmetric case we have seen that there are always consumers South West of (p, p) who aren't served by the best fixed prices with Corollary 5 showing that lotteries are a general feature of optimal pricing. Therefore by continuity the condition will be satisfied for a large class of asymmetric cases. This therefore guarantees that the conditions of Proposition 6 imply that lotteries form part of the most profitable selling strategy for a non-empty set of consumer distributions. The key features of these distributions are summarised together in the next section.

5 Discussion

5.1 Applicability of Lotteries

Propositions 4 and 6 determine key features of the consumer taste distribution which, if satisfied, are sufficient to guarantee that lotteries form part of the most profitable selling

[11] It is consumers with tastes immediately South West of this type who will be new consumers if the lottery is introduced.

strategy for sellers to pursue. These key criteria are:

1. Existence of target consumers

   If a lottery $\underline{q}$ is introduced awarding consumers some good with probability 1, in combination with tioli prices $\underline{p}$ then it is consumers with valuations South West of $\underline{p}$ who are the potential new consumers. These consumers therefore must exist for the sufficiency conditions to be met. Lemma 2 guarantees this is so in the symmetric case. In the general case this requirement is explicitly stipulated (Condition 1 of Section 4.2).

2. High Margins

   Initially, if the lottery $\underline{q}$ is introduced at a price of $\underline{q} \cdot \underline{p} - \eta$ for small η, only a small number of new consumers will be gained. The margins must be high to make this worth while as there will be a cost in lost profit from the introduction of the lottery.

3. Few high valuation type consumers

   For lotteries to be profitable there must be few existing consumers who would be willing to take the lottery gamble. The consumers who would take the gamble are those along the diagonal who derive the same positive utility from each good. This attention to the diagonal is clear from Propositions 4 and 6. The distribution of consumer tastes must therefore be such that there is no large concentration of mass in the North East corner of the support.

4. Asymmetric considerations

   Condition 1 of Proposition 6 reminds us that amongst existing consumers, the introduction of a lottery has asymmetric effects on profit. The seller must ensure that the loss in profit from those leaving the high margin product are at worst comparable in size to the gain in profit from those leaving the lower margin product.

These four conditions help us to build a picture of the characteristics of consumer taste which are sufficient to guarantee that lotteries are part of the most profitable selling strategy. The first condition, that concerning the existence of target consumers, deserves particular mention as it has immediate implications for the sale of goods of differing quality. If the quality differences are so large that there are no consumers indifferent between the top quality and the outside option then there can be no new consumers who would be tempted by a lottery of the two goods priced below $\underline{q} \cdot \underline{p}$. Any consumer tempted by the lottery will be an existing consumer. The lottery cannot therefore produce any new high margin gains and so its use will be only to micro adjust the current margins gained.

5.2 Implications for selling strategies

We have shown that appropriately chosen lotteries can be an important tool in a multiproduct firm's pricing. Lotteries can be introduced directly by quoting a set of probabilities and a price. However, the theory of lotteries actually captures a wide range of different pricing and haggling strategies. The two classes of examples below help explain some selling practices, theoretically equivalent to lotteries, used by many firms and suggest new insights into the selling process.

5.2.1 Claiming capacity constraints

Consider the following selling strategy for a new car salesman:

> "You can have the car in red or blue for certain at a price p. However, if you don't have a strong preference we can offer you your car for $p - \eta$. Which colour you receive will depend on which one we have left."

This selling strategy consists of exactly offering the consumers a lottery over the two substitutable goods (cars in this case). If Proposition 4 is satisfied then this strategy will be *more* profitable than just offering the consumers their preferred car at a colour of their choice.[12] This particular example has the implication for car manufacturers that painting their cars in advance of customer orders may not be as expensive as they might have thought.

A similar example of a real world situation in which such lotteries are used is provided by the tourist industry: package holiday operators will sometimes offer consumers location 1 or 2 with certainty for a certain price. If the customer has no strong preference they can pay a reduced fee to go on a waiting list and be certain to receive a holiday though at a destination which is ex ante random depending on which resort has more places remaining.

5.2.2 The importance of haggling over one good

Suppose a firm is haggling with buyers over the provision of a service from a choice of two substitutable ones. As time elapses with no agreement the buyer incurs a small inconvenience cost. The selling firm in this case can benefit from introducing lotteries into their selling strategy. This can be done, for example, by committing to the following haggling process:

[12] Those who couldn't quite stretch to the car at price p might be prepard to take it at price $p - \eta$ with a random colour. Of the existing consumers only those indifferent between the two colours would be tempted by the lottery. Under the conditions of Proposition 4 the net effect is profit enhancing.

1. In period 1 a particular buyer enters and the seller offers the buyer the two substitutable goods/services at fixed prices p.

2. The buyer can either buy one of the goods, leave, or wait for a second period. If the buyer leaves she receives a utility of 0. If the buyer waits then she incurs a small inconvenience cost (δ say).

3. If the buyer is still waiting the seller commits to lowering one of the good prices by η.[13] Which good has its price lowered is random. The probability of a particular good having its priced lowered is $\frac{1}{2}$. These prices will not be changed for this buyer in the future.

4. The buyer decides to purchase one of the goods at the new prices or not.

Clearly this haggling strategy requires the seller to be able to commit in advance. This strategy is exactly equivalent to the seller offering the lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ at a price of $p - \eta + \delta$. If Propositions 4 or 6 are satisfied then this will be a profit enhancing strategy as compared to refusing to be moved on price.

An example of an industry where this strategy would be applicable is the telecommunications industry. A telecoms firm may be offering two types of telephony service to a business customer. By waiting without a deal the business customer will be incurring some inconvenience cost (if nothing else the opportunity cost of not having struck a deal). By being able to move on price on a *random* offering the telecoms firm can enhance its profitability.

This example highlights the insight that it is important for the seller not to allow the haggling process to focus solely on the one good the buyer is most interested in. By keeping the possibility of a special offer being made on a random good the seller can enhance her profits. To see this consider a small perturbation of the above haggling strategy in which the seller allows the buyer to bargain over only one of the two services. This now becomes a one dimensional problem - there is only one relevant good for sale. Riley and Zeckhauser (1983) have shown that the seller would be best advised to not lower her price in this case and stick to take it or leave it prices. This strategy therefore foregoes all the benefits that lotteries could bestow.

5.3 How do lotteries help?

From the discussions of Section 5.1 we noted that lotteries can increase the number of consumers served. If margins are high then this effect can lead to an overall profit enhancement. However, this does not explain why it is that even at the best fixed prices

[13] This haggling strategy requires $\delta < \eta < 2\delta$ otherwise all consumers will opt for waiting or none will.

there is still scope for profits to be increased in this way. Riley and Zeckhauser (1983) have shown that indeed lotteries are not part of the most profitable pricing strategy when the seller is selling only one good - in this case all the necessary manipulations can be done through the fixed price level. To understand where Propositions 4 and 6 require the multiproduct aspect of the problem we note that in the multiproduct case the seller has fewer instruments than boundaries to set opening up the scope for lotteries to be useful. In particular we will see that lotteries allow the seller to focus on particular boundaries in a way which is impossible if she only uses the fixed prices.

When a single product monopolist is setting her optimal price she is only concerned with the boundary to be set between those that buy and those who don't. To set this one boundary she has one instrument - the price of her good. However, in the case of substitutable goods the seller has three boundaries to set: that between those who buy good 1 and nothing, that between those who buy good 2 and nothing and finally that between those who buy good 1 or good 2. To do this the seller without lotteries only has two instruments: the price of good 1 and the price of good 2. The optimal prices will therefore involve some sort of aggregation and averaging. However, with a lottery the monopolist can focus on one particular boundary and adjust the profit made on the consumers just in its vicinity.

To illustrate how the sufficient condition for lotteries being optimal focuses on one boundary we restrict ourselves to the symmetric case and examine the sufficiency condition for lotteries to be more profitable than a fixed price p given by (8) repeated as

$$-2\int_{x=p}^{\infty} f(x,x)\,dx + (p-c)\,f(p,p) > 0$$

Standard manipulations confirm that the first order condition determining the optimal fixed price is

$$-\int_{x=p}^{\infty}\int_{y=-\infty}^{x} dF + (p-c)\int_{y=-\infty}^{p} f(p,y)\,dy = 0$$

Consider changing coordinates (x,y) to (v,w) so that the x axis is rotated anticlockwise 45^o and labelled v with the y axis staying where it is and being labelled w. Given a (v,w) coordinate the v variable measures the distance travelled North East along a 45^o line beginning height w above the origin. Therefore

$$(v,w) \leftrightarrow \begin{cases} x = \frac{v}{\sqrt{2}} \\ y = w + \frac{v}{\sqrt{2}} \end{cases} \text{ or } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

The magnitude of the Jacobian of this matrix is given by the magnitude of the determinant of the above matrix which is $\frac{1}{\sqrt{2}}$. That implies that $dxdy = \frac{1}{\sqrt{2}}dvdw$. The first order

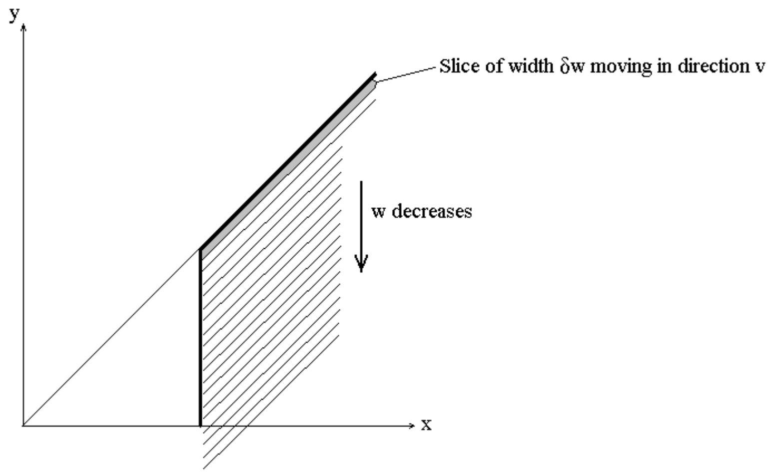


Figure 3: Figure showing the break down of the consumers who purchase a good into w and v coordinates

condition for the optimal fixed price p can therefore be written as

$$\begin{aligned} -\int_{w=-\infty}^{0}\int_{v=\sqrt{2}p}^{\infty} f\left(\frac{v}{\sqrt{2}}, w+\frac{v}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}dvdw + (p-c)\int_{w=-\infty}^{0} f(p, w+p)\,dw &= 0 \\ \int_{w=-\infty}^{0}\left\{-\frac{1}{\sqrt{2}}\int_{v=\sqrt{2}p}^{\infty} f\left(\frac{v}{\sqrt{2}}, w+\frac{v}{\sqrt{2}}\right)dv + (p-c)\,f(p, w+p)\right\}dw &= 0 \quad (9) \end{aligned}$$

We therefore see that the first order conditions for the optimal price require the averaging of a condition on the density along the diagonal slices of width δw shown in Figure 3. The lottery condition however focuses only on one slice - the one with $w = 0$. Using (9) we see that at the optimal prices

$$\begin{aligned} \int_{w=-\infty}^{0} \begin{array}{c}\text{Expression (8) aggregated across} \\ \text{all strips of width } \delta w\end{array} dw &= \\ \int_{w=-\infty}^{0} \underbrace{\left\{-\sqrt{2}\int_{v=\sqrt{2}p}^{\infty} f\left(\frac{v}{\sqrt{2}}, w+\frac{v}{\sqrt{2}}\right)dv + (p-c)\,f(p, w+p)\right\}}_{\text{Rewritten expression (8)}} dw &< 0 \end{aligned}$$

so for a lottery to be optimal locally the averaging process must be particularly restrictive as the integrand at $w = 0$ must be positive. In this case using lotteries will allow a monopolist to re-adjust her offerings at the margin profitably.

6 The general two good case

Hitherto we have established that lotteries are an important part of optimal pricing in the sale of substitutes. Under conditions of symmetry Corollary 5 has shown that given large enough margins lotteries will always be optimal. In addition Propositions 4 and 6 were able to provide specific sufficiency conditions on lotteries being more profitable than current fixed prices with the insights provided being summarised in Section 5.1. All this was achieved using the simplifying structure of the substitutes paradigm.

We are now in a position to consider the more general multiproduct monopolist of McAfee and McMillan (1988). We therefore now consider a two good monopolist facing consumers with no complementarities in demand. Consumers are still characterised by two type variables, x and y denoting the value each consumer attaches to the two individual component goods. Now, distinct from the substitutes model, we suppose that the valuation each consumer attaches to the bundle of both goods is additive and is given by $x + y$. The bundle is labelled good 3 and the two component goods 1 and 2.

One might seek to consider the profitability of offering a lottery over good 1, good 2 or the bundle. However, this cannot be done by replicating the local analysis of the previous sections. To see this note that if the best tioli prices are (p_1, p_2, p_3) with p_i corresponding to the price of good i then the consumers most tempted by lottery $\underline{q}$ are those which derive equal utility from all three goods, that is

$$x - p_1 = y - p_2 = x + y - p_3 \quad \Rightarrow \quad x = p_3 - p_2 \ , \ y = p_3 - p_1$$

These consumers will derive utility $p_3 - p_1 - p_2$ from the lottery at a price of $\underline{q} \cdot \underline{p}$. McAfee et al. (1989) suggests that if the valuations for the component goods are independently distributed then sub-additive bundling will be the best sales strategy. That is $p_3 < p_1 + p_2$. Therefore no consumer will choose to buy the lottery at a price close to $\underline{q} \cdot \underline{p}$ and so a non-local analysis would be needed. Therefore to maintain the parallels with our previous work we consider the case where the monopolist's best fixed prices involve sub-additive bundling and consider lotteries between good 1 and the bundle (good 3).

We therefore assume that the support of the density function Ω contains the consumer of type $(p_1, p_3 - p_1)$ who is indifferent between purchasing good 1 or the bundle (good 3). We seek to determine general conditions guaranteeing that the seller can enhance her profit by offering the lottery $(q_1, 0, q_3)$ such that $q_1 + q_3 = 1$ priced at $q_1 p_1 + q_3 p_3 - \eta$ for some η. This lottery signifies that the consumer will receive the whole bundle of both components with probability q_3 and failing that will receive only component 1. This latter case happens with probability $1 - q_3 = q_1$. The set of consumers affected by this lottery offer is captured in Figure 4.

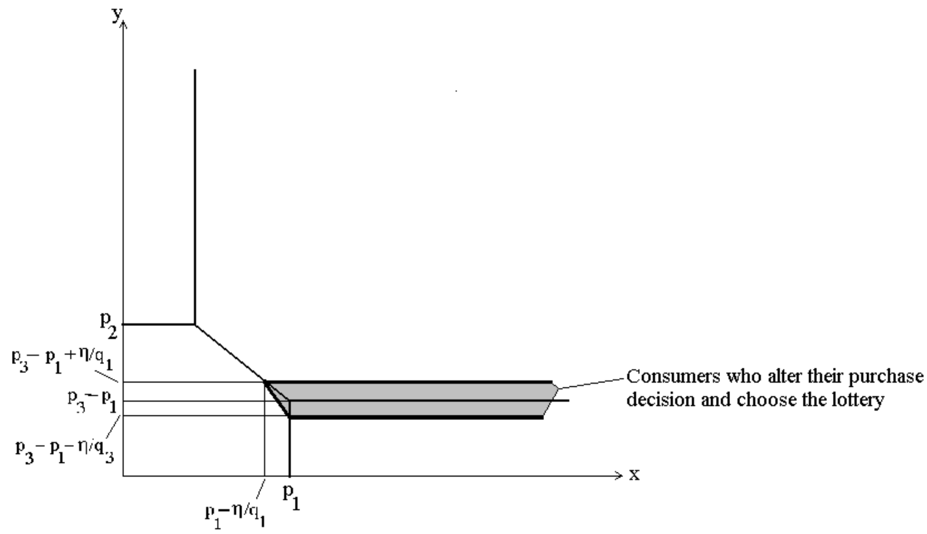


Figure 4: Figure showing the consumers affected by the introduction of the lottery offer $(q_1, 0, q_3)$

Lemma 7 *Suppose that the firm is charging optimal take it or leave it prices* $\underline{p}$ *which are not super-additive. The lottery* $(q_1, 0, q_3)$ *priced at* $q_1p_1 + q_3p_3 - \eta$ *is a profitable addition to the best take it or leave it tariff if*

$$-2\int_{x=p_1}^{\infty} f(x, p_3 - p_1)\,dx + (p_1 - c_1)\,f(p_1, p_3 - p_1) \qquad (10)$$
$$-(p_3 - p_1 - c_2)\int_{x=p_1}^{\infty} f_2(x, p_3 - p_1)\,dx \geq 0$$

This result is proved in the appendix.

The lemma is expressed in terms of the optimal take it or leave it prices (p_1, p_2, p_3). We have already seen that the Riley and Zeckhauser (1983) result doesn't extend to the case of multiproduct substitutes. If instead the no lottery result extended to general multiproduct monopolists then we would expect the lemma above to never be satisfied. The next section will show that this is not the case - the Riley and Zeckhauser (1983) result does not extend to multiple dimensions, either in the case of substitutes or generally.

6.1 A class of cases in which lotteries are beneficial

The key to satisfying the sufficiency condition of Lemma 7 is to determine a class of taste distributions such that $f(p_1, p_3 - p_1) \neq 0$ at the optimal prices. This is precisely condition 1 of Section 5.1: we must ensure that there are types in the population indifferent between

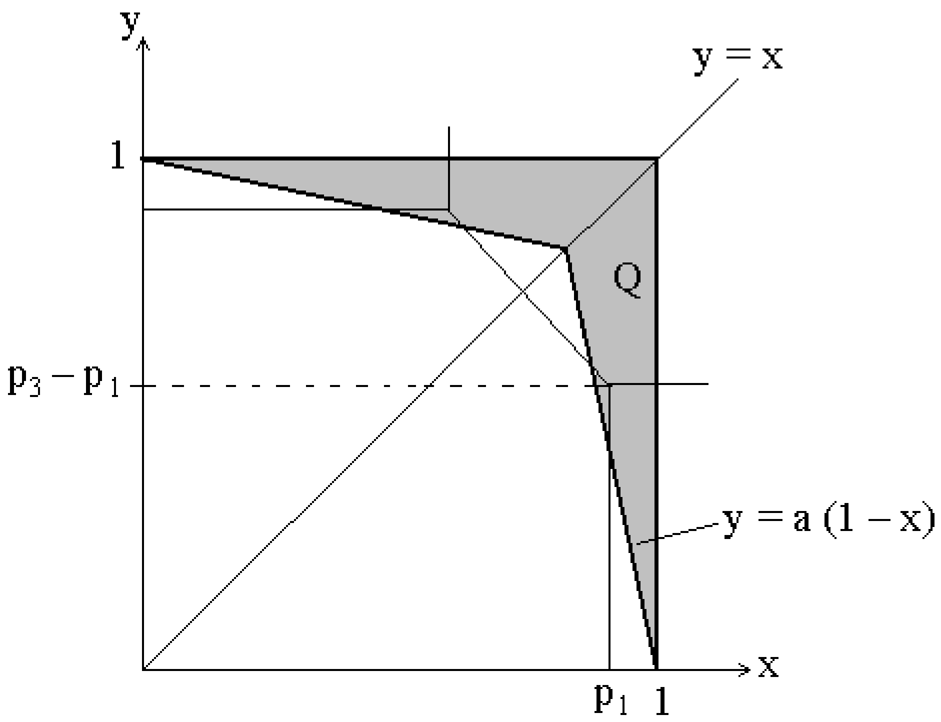


Figure 5: The population of consumers in quadrilateral Q

the lottery components and the outside option as it is consumers with tastes located South West of these who will form the new consumers attracted to the lottery offer. In the case of the substitutes paradigm such consumers were naturally present (see Lemma 2). This is not however the case in the no complementarities model of McAfee and McMillan (1988). In this case if the margins are high all consumers might well be served at the optimal prices.

With this in mind we will now provide a class of cases which satisfy the conditions of Lemma 7. Consider the symmetric arrow shaped area in Figure 5 formed close to the two outside edges of the unit square $[0,1]^2$. The inside edges of this triangle are given by the lines $y = a(1-x)$ and $y = 1 - \frac{1}{a}x$ which meet along the line $y = x$. Call this arrow quadrilateral Q and suppose that consumers are uniformly distributed on it so that the density satisfies $f(x,y) = 1 + a$. We will let a tend to infinity. In particular we suppose that $a > 1$.

Lemma 8 *At optimal prices there exist consumers in the interior of Q indifferent between* *good* 1*, the bundle and not purchasing and similarly consumers indifferent between good* 2*, the bundle and not purchasing.*

Proof. The method of proof of this lemma uses the insight behind the proof of Lemma 2. By symmetry we will have $p_1 = p_2$ at the optimal prices. If all the consumers are served then we must have the point $(p_1, p_3 - p_1)$ lying on the line $y = a(1-x)$. It is not optimal for the seller to set $p_3 = p_1 = 1$. To see this suppose that $p_3 = p_1 = 1$ and the seller raised the price of the bundle to $1 + \varepsilon$ and lowered the component price to $1 - \frac{\varepsilon}{a-1}$ to ensure all consumers are still served. The profit loss resulting from those that purchase just the component is of order ε^3 as only a small triangle of order ε^2 do so, yet the profit gain from those staying with the bundle is of order ε. For small ε, this therefore contradicts the optimality of prices $p_1 = p_3 = 1$.

But it is not optimal for all the consumers to be served. Suppose not then we have established that the point $(p_1, p_3 - p_1)$ lies on the line $y = a(1-x)$ above the point $(1, 0)$. Now suppose that both p_3 and p_1 are raised by ε. The profit gain will be of the order of ε and yet the profit loss will be of order ε^2 as only a small triangle of consumers is affected. Therefore the result is proved for small ε. ■

Lemma 8 guarantees that there are some consumers who are not purchasing but are potential targets of a lottery offer. This is one of the key lottery applicability requirements discussed in Section 5.1. It is to achieve this result that the strange quadrilateral shape Q is studied here rather than the full standard square.

To help fix ideas suppose the marginal costs c_1, c_2 are 0 and so consider what the monopolist's optimal selling strategy would be in the limit as a tends to infinity. In this case the consumers would all be uniformly distributed on the outside of the unit square. The optimal component prices would be 1 and the monopolist's profit function would be

$$\begin{aligned} \frac{\Pi}{2} &= \left[p_1(p_3 - p_1) + p_3\left[1 - (p_3 - p_1)\right]\right]_{p_1 = 1} \cdot \frac{1}{2} \\ \Rightarrow \Pi &= p_3 - 1 + p_3\left[2 - p_3\right] \end{aligned}$$

This has a maximum at $p_3 = \frac{3}{2}$ and $p_1 = 1$. This result leads to the following proposition:

Proposition 9 *If a is sufficiently large then lotteries will be a profit enhancing sales strategy to use in serving the population Q.*

Proof. We recall that if $\left[\frac{\partial^2 \Delta\Pi}{\partial \eta^2}\right]_{\eta=0} > 0$ then lowering the price of the lottery $(q_1, 0, q_3)$ just below its choke price is profit enhancing. From the proof of Lemma 7 and Lemma 8 above and using the fact that the density is flat and uniform on its support and costs are zero we note that

$$\begin{aligned}\left[\frac{\partial^2 \Delta\Pi}{\partial \eta^2}\right]_{\eta=0} &= \frac{1}{q_1 q_3}\left\{-2\left(1-p_1\right) f\left(x,y\right) + p_1 f\left(x,y\right)\right\}_{f(x,y)=1+a} \\ &= \frac{1+a}{q_1 q_3}\left\{3p_1 - 2\right\}\end{aligned}$$

where the lottery $(q_1, 0, q_3)$ has $q_1 + q_3 = 1$. Therefore if at the optimal deterministic prices $p_1 > \frac{2}{3}$ then lotteries would be profit enhancing. However, by letting a tend to infinity, p_1 tends to 1 and so $\left[\frac{\partial^2 \Delta\Pi}{\partial \eta^2}\right]_{\eta=0} > 0$ for a sufficiently large.[14] This proves the result. ■

To recap, the intuition for this result is the fact that when margins are large (produced by a large a) a firm will be very keen to try and serve as many people as possible. When consumers exist in quadrilateral Q then without using lotteries there will exist some consumers who aren't served. Lotteries allow the firm to serve some of these consumers and so increase her profits.

6.2 Implications for McAfee and McMillan (1988)

The above class of examples shows that the Riley and Zeckhauser (1983) result ruling out lotteries for the single good monopolist does not extend to the general multiproduct case. McAfee and McMillan (1988), in Section 4 of their paper, claim that the Riley and Zeckhauser (1983) result does extend to the general two good case as long as the consumer type space satisfies a regularity condition given in their work as equation (46). We can show that their result as stated is not correct: some extra restrictions need to be placed on the consumer types for the result to apply.

The McAfee and McMillan (1988) claim. Suppose that a monopolist is selling 2 component goods. The consumers have valuation (x, y) for the two goods distributed according to density $f(x, y)$ supported on $[0, b_1] \times [0, b_2]$. Suppose also that the density function satisfies the regularity condition ((46) in their work)

$$3f\left(\underline{x}\right) + \underline{x} \cdot \nabla f\left(\underline{x}\right) \geq 0$$

Then lotteries do not help the multiproduct monopolist. ■

We can use the class of examples above to find a counter example to this claim. The class of examples above have consumers uniformly distributed on a quadrilateral Q. Suppose that the rest of the support of $[0,1]^2$ is filled out *smoothly* with a vanishingly small density outside of the arrow shaped quadrilateral Q described above. If the density is sufficiently small then the optimal prices will not be substantially altered from when consumers were just in quadrilateral Q. In particular the optimal prices p_1, p_2 and p_3 will

[14] In fact $a > 2$ is sufficient as this guarantees that the best fixed component price is at least $\frac{2}{3}$.

still lie strictly in the interior of Q. The hazard rate condition in McAfee and McMillan's work will be satisfied as

$$3f\left(\underline{x}\right)+\underline{x}\cdot\nabla f\left(\underline{x}\right)=\begin{cases}0 & \text{Away from the boundary of } Q \left(\begin{array}{c}\text{both inside and outside}\\ \text{the quadrilateral}\end{array}\right)\\ >0 & \text{Near to the boundary of } Q\end{cases}$$

This is because as we move in the direction of increasing x or y the density is either flat or increasing up to the plateau Q. The proposition above then guarantees that lotteries will be profit enhancing which is a counter example to the McAfee and McMillan claim to have extended the Riley and Zeckhauser result in two dimensions. We have therefore shown that (i)lotteries can help increase profit in certain situations and (ii)the result in McAfee and McMillan (1988) requires some extra restrictions on the form of the density function which gives consumers' tastes.

7 Finding the fully optimal sales strategy

We have thus far determined that using lotteries can be part of the fully optimal pricing strategy for the multiproduct monopolist in both the substitutes paradigm of Section 3 and the no complementarities in demand paradigm of McAfee and McMillan (1988). We have yet to address how a seller can determine what the fully optimal selling strategy is. This section returns to the substitutable goods paradigm and illustrates exactly how the seller can use standard linear programming packages to determine their optimal pricing strategy. We will solve a large class of such examples fully in the next section. We therefore now return to the original model of two substitutable goods. To ease the exposition we assume that the consumers have types supported on the square $\Omega_a=[a,a+1]^2$ and are given by a symmetric density function. In addition we normalise the monopolist's unit costs to zero. These assumptions allow us to use less cumbersome notation. Relaxing these assumptions would not alter the method of solution of the problem. This class of cases includes that used as the initial motivating example of Section 2. We seek the seller's optimal sales strategy. The seller can potentially use a whole schedule of lotteries $\{\underline{q}\}$ should she wish. The lottery (q_1,q_2) represents the option through which the consumer receives good 1 with probability q_1 or good 2 with probability q_2 with $q_1+q_2\leq 1$.[15] The consumers are all risk neutral. A consumer's ex ante expected utility if the price of lottery $\underline{q}$ is $p\left(\underline{q}\right)$ is therefore given by $q_1x+q_2y-p\left(\underline{q}\right)$. The surplus, $v\left(\cdot\right)$, of a consumer of type

[15] We now include all possible lotteries including those in which a consumer receives nothing in some instances.

$\underline{x} = (x, y)$ can be derived from the incentive compatibility constraints and is given by

$$v(\underline{x}) = \max_{\underline{q}} \left\{ \underline{q} \cdot \underline{x} - p\left(\underline{q}\right) \right\}$$

Due to the symmetry of the problem the monopolist's optimal selling strategy will be symmetric. Rochet and Choné (1998) have shown that we can suppose that the seller selects a consumer surplus allocation function. From this we can determine what lotteries are sold and at what price.[16] Rochet (1987) has shown that for the surplus $v(\cdot)$ to be implementable we require v convex and $\nabla v(\underline{x}) = \underline{q}(\underline{x})$ almost everywhere by the envelope theorem.

The firm's profit is given by

$$\begin{aligned} \pi &= \int_{\Omega_a} \left\{ p\left(\underline{q}(\underline{x})\right) - \underbrace{\underline{q}(\underline{x}) \cdot \underline{c}}_{=0} \right\} f(\underline{x})\, d\underline{x} \\ \Rightarrow \quad \pi(v(\cdot)) &= \int_{\Omega_a} \{ \underline{x} \cdot \nabla v(\underline{x}) - v(\underline{x}) \} f(\underline{x})\, d\underline{x} \end{aligned}$$

where we have used the fact that

$$p\left(\underline{q}(\underline{x})\right) = \underline{q}(\underline{x}) \cdot \underline{x} - v(\underline{x})$$

from the incentive compatibility condition.

The divergence theorem guarantees that

$$\int_{\Omega} \operatorname{div} \underline{F} dV = \int_{\partial\Omega} \underline{F} \cdot \underline{n} dS$$

where $\underline{n}$ is the outward pointing unit normal and $\underline{F}$ is a vector field. In this case note that

$$\operatorname{div}\left[v(\underline{x}) f(\underline{x}) \underline{x}\right] = \underline{x} \cdot \{ f(\underline{x}) \nabla v(\underline{x}) + v(\underline{x}) \nabla f(\underline{x}) \} + 2v(\underline{x}) f(\underline{x})$$

[16] Given an implementable consumer surplus allocation function $v(x, y)$ a consumer of type $\underline{x}$ receives the lottery

$$\nabla v(\underline{x})$$

in return for a payment of

$$\underline{x} \cdot \nabla v(\underline{x}) - v(\underline{x})$$

We therefore know how much each consumer pays and what lottery she decides to purchase. We therefore have the full tariff structure as a function of the lottery chosen.

as $\underline{x} \in \mathbb{R}_+^2$. Using the divergence theorem we have that

$$\begin{aligned}
\pi(v(\cdot)) &= \int_{\Omega_a} \{\operatorname{div}[v(\underline{x}) f(\underline{x}) \underline{x}] - \underline{x} \cdot v(\underline{x}) \nabla f(\underline{x}) - 2v(\underline{x}) f(\underline{x})\} d\underline{x} - \int_{\Omega_a} v(\underline{x}) f(\underline{x}) d\underline{x} \\
&= \int_{\partial\Omega_a} v(\underline{x}) f(\underline{x}) \underline{x} \cdot \underline{n} dS - 3\int_{\Omega_a} v(\underline{x}) f(\underline{x}) d\underline{x} - \int_{\Omega_a} v(\underline{x}) \underline{x} \cdot \nabla f(\underline{x}) d\underline{x}
\end{aligned}$$

Going clockwise let the four corners of the square Ω_a be labelled A at $(a, a+1)$, through to D at (a, a). We then see that

$$\underline{x} \cdot \underline{n} = \begin{cases} a+1 & \text{along line } \overline{AB} \\ a+1 & \text{along line } \overline{BC} \\ -a & \text{along line } \overline{CD} \\ -a & \text{along line } \overline{DA} \end{cases}$$

So we have

$$\begin{aligned}
\pi(v) &= (a+1)\int_{\overline{AB} \cup \overline{BC}} v(\underline{x}) f(\underline{x}) d\underline{x} - a\int_{\overline{CD} \cup \overline{DA}} v(\underline{x}) f(\underline{x}) d\underline{x} \\
&\quad - \int_{\Omega_a} v(\underline{x}) \{3f(\underline{x}) + \underline{x} \cdot \nabla f(\underline{x})\} d\underline{x}
\end{aligned}$$

We finally note that $v(a, a) = 0$ as we established (Lemma 2) that at the optimum some consumers will always be left unserved.

Combining these insights we have the following problem

Problem 10 *Let* $v(\cdot) : \mathbb{R}_+^2 \longmapsto \mathbb{R}$ *and* $\Omega_a = [a, a+1]^2$

$$\max_{v(\cdot)} \left\{ \begin{array}{c} (a+1)\int_{\overline{AB} \cup \overline{BC}} v(\underline{x}) f(\underline{x}) d\underline{x} - a\int_{\overline{CD} \cup \overline{DA}} v(\underline{x}) f(\underline{x}) d\underline{x} \\ -\int_{\Omega_a} v(\underline{x}) \{3f(\underline{x}) + \underline{x} \cdot \nabla f(\underline{x})\} d\underline{x} \end{array} \right\}$$

subject to

$$v \textit{ symmetric } (v(x, y) = v(y, x))$$

$$v(a, a) = 0 \tag{11}$$

$$v(x, y) \geq 0 \tag{12}$$

$$0 \leq \nabla v \tag{13}$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \leq 1 \tag{14}$$

$$v \textit{ convex } \Rightarrow \begin{cases} \frac{\partial^2 v}{\partial x^2} \geq 0 \\ \frac{\partial^2 v}{\partial y^2} \geq 0 \\ \left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^2 v}{\partial y^2}\right) \geq \left(\frac{\partial^2 v}{\partial x \partial y}\right)^2 \end{cases} \tag{15}$$

This is a complicated problem in which the seller hopes to find an optimal function $v(\cdot)$ to maximise her objective function subject to a large number of constraints. Rochet and Choné (1998) show that analytical results are difficult to come by in this situation.

7.1 The finite differencing technique

7.1.1 The objective function

We can solve Problem 10 using a finite differencing technique. We can approximate the consumer surplus function $v(x,y)$ on the square Ω_a by its value at every point on a fine $N \times N$ grid. As the problem is symmetric we need only consider the lower triangle of $[a,a+1]^2$ using a grid which runs from $v_{0,0}$ at the point (a,a) through to $v_{N,N}$ at the point $(a+1,a+1)$. We therefore have the variables $(\{v_{i,j} | 0 \leq i \leq N, 0 \leq j \leq i\})$.[17]

Given N we are covering the lower half triangle of $[a,a+1]^2$ by a grid of points separated by a distance of $h = \frac{1}{N}$. Each point $v_{i,j}$ represents the height of a characteristic function resembling an $h \times h$ square tower block centred on $(a+ih, a+jh)$ with area of cross-section h^2. Recalling that the square Ω_a is labelled as A at $(a, a+1)$ through to D at (a,a) and using the symmetry we can rewrite the objective function as:

$$2\left\{\begin{array}{l} (a+1)\int_{\overline{CB}} v(\underline{x}) f(\underline{x})\, d\underline{x} - a\int_{\overline{DC}} v(\underline{x}) f(\underline{x})\, d\underline{x} \\ -\int_{\text{LowerTriangle}(\Omega_a)} v(\underline{x}) \{3f(\underline{x}) + \underline{x}\cdot\nabla f(\underline{x})\}\, d\underline{x} \end{array}\right\}$$

We now use the parameter $H_{i,j}$ to represent the value of $\{3f(\underline{x}) + \underline{x}\cdot\nabla f(\underline{x})\}$ at the point $(a+ih, a+jh)$. We note that $H_{i,j}$ is not a variable to be calculated but rather a parameter that can be found in advance as the seller knows the distribution of consumer types f. The terms of the objective function can then be given by:

$$\begin{aligned} &\int_{\text{LowerTriangle}(\Omega_a)} v(\underline{x}) \{3f(\underline{x}) + \underline{x}\cdot\nabla f(\underline{x})\}\, d\underline{x} \\ = \;& h^2 \left\{\begin{array}{c} \sum_{\substack{j\leq i \\ i\neq j \\ j\neq 0 \\ i\neq N}} v_{i,j}H_{i,j} + \frac{1}{2}\left(\sum_{i=1}^{N-1} v_{i,0}H_{i,0} + \sum_{j=1}^{N-1} v_{N,j}H_{N,j} + \sum_{i=1}^{N-1} v_{i,i}H_{i,i}\right) \\ +\frac{1}{8}v_{N,N}H_{N,N} + \frac{1}{4}v_{N,0}H_{N,0} \end{array}\right\} \end{aligned}$$

where we have used the fact that $v(a,a) = v_{0,0} = 0$. Letting $f_{i,j}$ denote the value of $f(\cdot)$ at the point $(a+ih, a+jh)$ the first two terms of the objective function become:

$$\int_{\overline{CB}} v(\underline{x}) f(\underline{x})\, d\underline{x} = h\left\{\sum_{j=1}^{N-1} v_{N,j}f_{N,j} + \frac{1}{2}v_{N,0}f_{N,0} + \frac{1}{2}v_{N,N}f_{N,N}\right\}$$

$$\int_{\overline{DC}} v(\underline{x}) f(\underline{x})\, d\underline{x} = h\left\{\sum_{i=1}^{N-1} v_{i,0}f_{i,0} + \frac{1}{2}v_{N,0}f_{N,0}\right\}$$

Summing these we have the maximisation objective function of Problem 10.

[17] An $N \times N$ grid in the case of symmetry will result in

$$\sum_{r=1}^{N+1} r = \frac{1}{2}(N+1)(N+2)$$

free variables. (So if $N = 20$ then we will have 231 variables).

7.1.2 The constraints

Constraints (11) and (12) are easily handled by

$$\begin{aligned} v_{0,0} &= 0 \\ v_{i,j} &\geq 0 \text{ for } \{v_{i,j} | \, 0 \leq i \leq N \text{ and } 0 \leq j \leq i\} \end{aligned}$$

For the derivative constraints (13) and (14), we use the second order accurate approximation

$$\left[\frac{\partial v}{\partial x}\right]_{i,j} \approx \frac{v_{i+1,j} - v_{i-1,j}}{2h}$$

Harnessing the symmetry of the problem we can therefore use the following expressions in the optimisation problem:

$$\left[\frac{\partial v}{\partial x}\right]_{i,j} = \begin{cases} \frac{1}{2h}\left(v_{i+1,j} - v_{i-1,j}\right) & 1 \leq i \leq N-1 \text{ and } 0 \leq j \leq i-1 \\ \frac{1}{2h}\left(v_{i+1,i} - v_{i,i-1}\right) & 1 \leq i \leq N-1 \text{ and } j = i \\ \frac{1}{h}\left(v_{N,j} - v_{N-1,j}\right) & i = N \text{ and } 0 \leq j \leq N-1 \\ \quad \text{not defined} & i = N \text{ and } j = N \end{cases}$$

Along the far edge $\left(\overline{BC}\right)$ given by $i = N$ we have used the one sided derivative running from $i = N - 1$ through to $i = N$. Along the main diagonal we have used the fact that $v_{i,j} = v_{j,i}$ by symmetry. We note that constraints (13) and (14) will be linear in the $\{v_{i,j}\}$.

We finally now turn to the convexity constraints (15). We can approximate the second derivatives $\frac{\partial^2 v}{\partial x^2}$ by determining the rate of change of the expression for $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ given above. This gives

$$\left[\frac{\partial^2 v}{\partial x^2}\right]_{i,j} \approx \frac{1}{h^2}\left(v_{i+1,j} + v_{i-1,j} - 2v_{i,j}\right)$$

The constraints $\left[\frac{\partial^2 v}{\partial x^2}\right]_{i,j} \geq 0$ and $\left[\frac{\partial^2 v}{\partial y^2}\right]_{i,j} \geq 0$ will therefore be linear in the $v_{i,j}$. However the final convexity constraint is not linear and as such has two effects:

1. Nonlinear constraints cause the optimisation problem to be much harder to solve. The algorithms for the solution of such a problem are therefore slower for any given fineness of the $N \times N$ grid.

2. The coarseness of the difference expressions used above can become problematic in a non-linear expression for the convexity constraint. This problem can be mitigated by choosing a very fine grid. This however has the effect of aggravating 1.

The solution in the case of substitutable goods is to require the consumer surplus function $v(x,y)$ to satisfy a much weaker condition than convexity or even quasi-convexity. One must then check that the final solution does satisfy the full convexity constraint. We aim to rule out surplus functions $v(\cdot)$ which assign low surplus near to the edge $\overline{CD}$ and high surplus towards the diagonal. Such solutions will not be convex when v is extended to cover the whole of the square Ω_a. We therefore note that quasi-convexity of v implies that the consumer surplus iso contours have gradient in the lower triangle that lies in $(-\infty, -1]$. This condition can be captured in the simple constraint:

$$v_{i-1,j+1} \leq v_{i,j} \tag{16}$$

to be applied at interior points of the lower triangle of Ω_a.

These constraints are therefore all linear as is the objective function. Standard linear programming routines can therefore be used to solve for the optimal selling strategy.

7.2 Worked example – the uniform case

We explicitly solve the case in which consumers are uniformly distributed on the square $\Omega_a = [a, a+1]^2$. In this case $f_{i,j} = 1$ and $\nabla f \equiv 0$ and so we have $H_{i,j} = 3$ in the objective function determined above. Bringing terms together we can write the sellers problem as

Problem 11 *Removing a factor of* $2h$ *the objective function becomes*

$$\max_{\{v_{i,j}\}} \quad \begin{array}{l} \left(\sum_{j=1}^{N-1} v_{N,j}\right)\left(a+1-\frac{3}{2}h\right) - 3h \left(\sum_{\substack{j\leq i \\ i\neq j \\ j\neq 0 \\ i\neq N}} v_{i,j} + \frac{1}{2}\sum_{i=1}^{N-1} v_{i,i} \right) \\ -\left(\sum_{i=1}^{N-1} v_{i,0}\right)\left(a+\frac{3}{2}h\right) + v_{N,0}\left(\frac{1}{2}-\frac{3}{4}h\right) + v_{N,N}\left(\frac{a}{2}+\frac{1}{2}-\frac{3}{8}h\right) \end{array} \tag{17}$$

subject to the constraints given above.

This problem was solved at a number of different parameter values (that is choices of a and N) using an optimiser called BPMPD available from the NEOS server at: http://www-neos.mcs.anl.gov/. The routine is written by C. Mészaros of the Mta Sztaki in Budapest, Hungary. The linear programming problem must be submitted in AMPL format. A typical such submission used to solve this problem is available in Appendix B.

The output data from programs, such as that in Appendix B, gives the optimal values of the variables, $\{v_{i,j}\}$. These represent the amount of surplus a consumer i,j receives from the seller's optimal selling strategy. The data displays the derivatives $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ and $\left[\frac{\partial v}{\partial y}\right]_{i,j}$ calculated as finite differences. We recall that these show the lottery which the consumers receive through the optimal selling strategy. Finally the data displays a variable called 'probreceive' which gives $\left[\frac{\partial v}{\partial x}\right]_{i,j} + \left[\frac{\partial v}{\partial y}\right]_{i,j}$. This variable is used to determine if any lotteries are used which give the consumer a positive probability of receiving nothing in return for their money.

The following results stand out from the output files:

1. The resultant values for $v_{i,j}$ satisfy the full convexity constraints. We can see this by plotting the resultant consumer surplus function and checking that it is convex (bowl shaped). This surplus function has been drawn in Figure 6 for the case $a = 5$ and $N = 20$.

2. The derivatives $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ and $\left[\frac{\partial v}{\partial y}\right]_{i,j}$ are almost all given by 0.5 and 1. This suggests that the lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the take it or leave it prices form the lions share of the optimal tariff.

3. The sum of the derivatives $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ and $\left[\frac{\partial v}{\partial y}\right]_{i,j}$ for almost everyone add to 1. The exceptions occur only at the very edge of the sample where boundary effects in our finite differencing approximation are to blame. This therefore shows that at the optimal selling strategy no consumer pays money for a positive probability of receiving nothing.

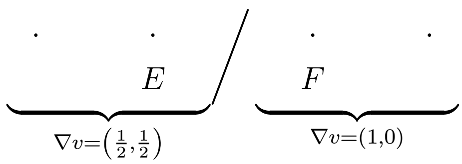
4. The table of $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ values has 0.5 and 1 as the main entries apart from a line of entries of value 0.676777 followed by 0.926777 (in the case of $a = 10$). Note that the difference between these two values is *exactly* 0.25. This will prove crucial. At first glance this suggests that the optimal selling strategy is subtly more complicated than just using the lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ with the take it or leave it prices. We will see below that this interpretation is misleading.

Proposition 12 *Suppose consumers are uniformly distributed on a square* $\Omega_a = [a, a+1]^2$ *and the seller has two symmetric substitutable goods to sell with marginal costs normalised to 0. The fully optimal selling strategy is to use take it or leave it prices in combination with the lottery* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *only.*

Numerical Proof. This proposition is substantiated through a large number of numerical optimisations. The typical such results are discussed in the four points given above. The most important point is to note (4) which describes that the $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ variable has two lines of entries which differ from 0, $\frac{1}{2}$ and 1. We first need to show that this is compatible with the optimal surplus allocation function being piecewise flat and containing only a couple of distinct pieces. To see this consider the following depiction of 4 grid

Figure 6: The consumer surplus remaining after the optimal selling strategy is used for consumers uniformly distributed on $[5,6]^2$.

points spanning the boundary between the part of $v(\cdot)$ with gradient $(1,0)$ and the part with gradient $\left(\frac{1}{2},\frac{1}{2}\right)$. The distance between E and F is h. We define the proportion of the distance between E and F lying to the left of the boundary as α.



Now recall that the finite difference approximation for $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ is given by

$$\frac{v_{i+1,j}-v_{i-1,j}}{2h}$$

we therefore have:

1. If $v_{i,j}$ is at the point E then

$$\begin{aligned} v_{i+1,j} &= v+\frac{1}{2}\alpha h+(1-\alpha)h \\ v_{i-1,j} &= v-\frac{1}{2}h \\ &\Rightarrow \left[\frac{\partial v}{\partial x}\right]_{\text{at } E} = \frac{\frac{1}{2}\alpha h+(1-\alpha)h+\frac{1}{2}h}{2h}=\frac{3}{4}-\frac{\alpha}{4} \end{aligned}$$

2. If $v_{i,j}$ is at the point F then

$$\begin{aligned} v_{i+1,j} &= v+h \\ v_{i-1,j} &= v-(1-\alpha)h-\frac{1}{2}\alpha h \\ \Rightarrow \left[\frac{\partial v}{\partial x}\right]_{\text{at } F} &= \frac{h+(1-\alpha)h+\frac{1}{2}\alpha h}{2h}=1-\frac{\alpha}{4} \end{aligned}$$

The gradient with respect to x on either side of E and F will give the values 0.5 and 1 as expected. We now note that

$$\left[\frac{\partial v}{\partial x}\right]_{\text{at } F}-\left[\frac{\partial v}{\partial x}\right]_{\text{at } E}=0.25 \text{ exactly}$$

We therefore see that the line of parallel figures (0.676777 followed by 0.926777 in the case of $a=10$) is entirely expected if the consumer surplus function satisfies the conditions of the proposition.

The same effect of the finite differencing approximation is expected and seen at all the other boundary points.

The proof is completed by ensuring that $\left[\frac{\partial v}{\partial x}\right]_{i,j}$ and $\left[\frac{\partial v}{\partial y}\right]_{i,j}$ only take the values $\frac{1}{2}$, 1 and 0 and sum to 1, except where the finite differencing technique causes two distinct values to be averaged across a boundary. This process is repeated for a large number of different parameter values a. I have run this experiment for $a \in \{0,1,2,3,5,10,20,50\}$ and have found that the Proposition holds in all of these cases. ■

We note that the above proof is numerical and so does not constitute an analytical proof. Such proofs would be hard to come by due to the large number of constraints active on candidate surplus functions $v(\cdot)$.

7.3 The scale of possible profit gain from lotteries – Numerical examples

7.3.1 The uniform case

Proposition 12 has established that when consumers are uniformly distributed on $[a,a+1]^2$ and the seller has two substitutable goods to sell with marginal costs normalised to 0, then the fully optimal selling strategy is achieved by using the lottery $\left(\frac{1}{2},\frac{1}{2}\right)$ in combination with the take it or leave it prices. This section will determine what the scope for profit gain is from such a strategy as compared to not using lotteries at all.

Lemma 13 *When the seller of two substitutable goods does not use lotteries then her profit function at a price of p for each good is given by*

$$\pi^{no\ lot}(p)=p\cdot 2\cdot\frac{1}{2}(a+1-p)(1+p-a)$$

with the optimal price being given by

$$p^{no\ lot} = \frac{2a + \sqrt{a^2+3}}{3}$$

This lemma follows as the problem is symmetric and the profit function can be found by an application of the area of a trapezium rule.

Now consider the optimal prices if the firm offers the deterministic goods at a price p and also a lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ at a price $p - \eta$. This lottery awards a consumer the first good with probability $\frac{1}{2}$ or the second good with probability $\frac{1}{2}$. There is no chance the consumer receives nothing or both goods together. This case constitutes the fully optimal selling strategy for the monopolist as shown by Proposition 12.

Lemma 14 *The fully optimal prices for the monopolist selling two substitutable goods to consumers uniformly distributed on* $[a, a+1]^2$ *are component prices* p *and the lottery* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *priced at* $p - \eta$ *where*

1. *If* $a > 1.25$ *then* $\eta = \frac{1}{6}$ *and* $p = \eta + \frac{2a+\sqrt{a^2+\frac{3}{2}}}{3}$.
2. *If* $1 < a \leq 1.25$ *then* $\eta = 2p - \frac{4}{3}(a+1)$ *and*

$$p = \frac{1}{21}\left(2(7a+8) - \sqrt{4(7a+8)^2 - 7(a+1)(29a+35)}\right)$$

3. *If* $a \leq 1$ *then* $\eta = 0$, *that is lotteries are not part of the optimal selling strategy.*

The proof of this result is given in the appendix. The profit attained from using the lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ and not can be calculated from the profit equations in the appendix and compared. The percentage profit gain is then calculated. This is drawn in Figure 7. In this case the profit gain is very modest, at best of the order of a single percentage point.

7.3.2 An asymmetric example

Until now all the concrete numerical examples in which lotteries have proved profitable have been symmetric with profit gains resulting from the enlargement of the firm's customer base. In addition the profit gains have all been very modest. The following simple example shows us that lotteries can in fact be very profitable and offer much higher percentage profit gains than the best fixed prices. In addition this example shows that the number of consumers served need not be increased for lotteries to be profitable.

Consider a discrete market with 2 consumers who have additive valuations for the two goods, x and y. Suppose that the firm has no unit costs in production, that one of the consumers is at $(1, 3)$ and the other at $(2, 1)$. This market is drawn in Figure 8.

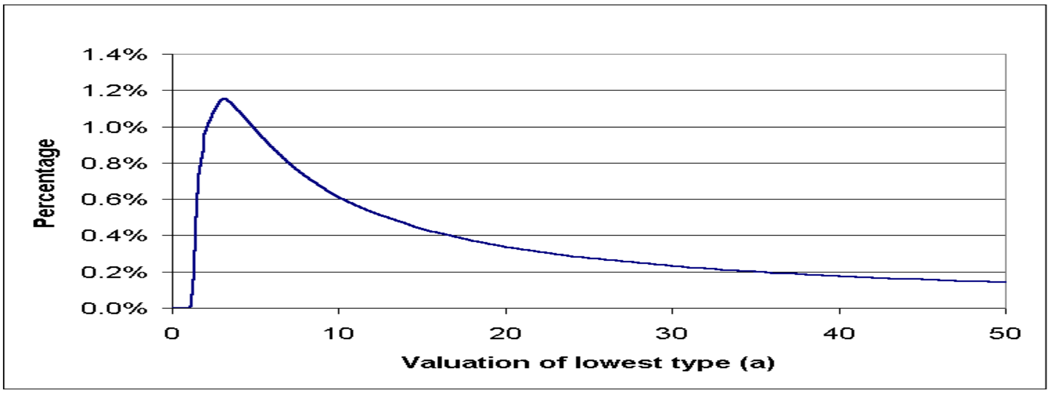


Figure 7: Graph showing the percentage gain in profit from the optimal selling strategy using the lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$ as compared to fixed prices when serving consumers uniformly distributed on $[a, a+1]^2$

The optimal deterministic tariff in this case has consumer $(2, 1)$ receiving good 1 with probability 1 at a price of 2, and consumer $(1, 3)$ receiving both goods with probability 1 at a price of 4. This provides the seller with a profit of 6.

The optimal tariff is however stochastic with the type $(1, 3)$ consumer's bundle unchanged and the $(2, 1)$ consumer receiving either good 1 only or both goods with equal probability for a price of $2\frac{1}{2}$. This selling strategy now provides the seller with a profit of 6.5. This is a gain of over 8% even though no new consumers were served.

8 The Welfare Implications of Lotteries

Thus far this chapter has been about firm profitability. We have been able to construct a method of introducing a lottery $\underline{q}$ at a price $\underline{q} \cdot \underline{p} - \eta$ which allows us to use a local derivative approach to ascertain how the lottery affects firm profit. This chapter has therefore been able to construct sufficiency conditions which guarantee that adding a lottery option will be profit enhancing. In particular Proposition 4 gives conditions on consumer density in the substitutes paradigm which guarantee that lotteries form part of the fully optimal pricing strategy thus proving that the Riley and Zeckhauser (1983) no lottery result from the sale of 1 good does not extend to the multiproduct context.

Our new approach to lotteries can now be applied to an analysis of the effects of lotteries on welfare. We have established that firms will often be most profitable if they use lotteries. Is this to be encouraged or discouraged? Would welfare be enhanced if lotteries/haggling was not allowed and fixed prices had to be posted and adhered to?

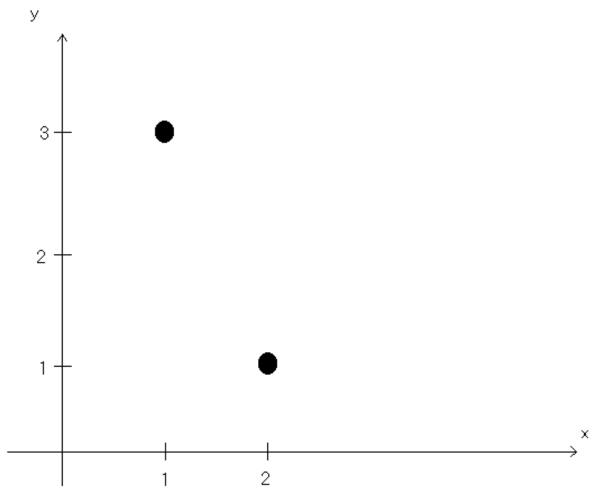


Figure 8: A discrete market in which lotteries provide a substantial profit gain

Proposition 15 *If the introduction of lottery $\underline{q}$ to fixed prices $\underline{p}$ at a price of $\underline{q} \cdot \underline{p} - \eta$ is profit enhancing for η close to 0 in either the substitutes paradigm or the no complementarities paradigm of McAfee and McMillan (1988) then it is also welfare enhancing.*

The proof of this result is established by showing that the expression for the rate of change of welfare with respect to the parameter η is equal to the rate of change of profit with respect to η plus a positive constant. Therefore if profits increase so must welfare.

This result parallels the intuition provided by Varian (1989) in the slightly different context of third degree price discrimination. Varian (1989) notices that when considering third degree price discrimination a necessary condition for welfare to increase is that output rises as a result of the discrimination. Section 5.1 notices that a direct way for lotteries to be beneficial is if there exist new consumers who can be gained by introducing a lottery offer. Indeed in the case of symmetry in the substitutes paradigm this is the only way the sufficiency conditions given can be met. Proposition 15 confirms that this output enlargement effect is sufficient to guarantee that welfare is increased.

Proposition 15 therefore suggests that a central planner need not worry: when introducing lotteries is in the firm's interests then a small change introducing a lottery tariff is also welfare enhancing. However, this does not necessarily mean that as the monopolist moves to the fully optimal lottery tariff welfare will be improved as compared to maintaining fixed prices. The introduction of the lottery allows the monopolist to raise the price of the goods provided with certainty. As consumers swap from their preferred good to a lottery their valuations for the purchased good are reduced, though more consumers

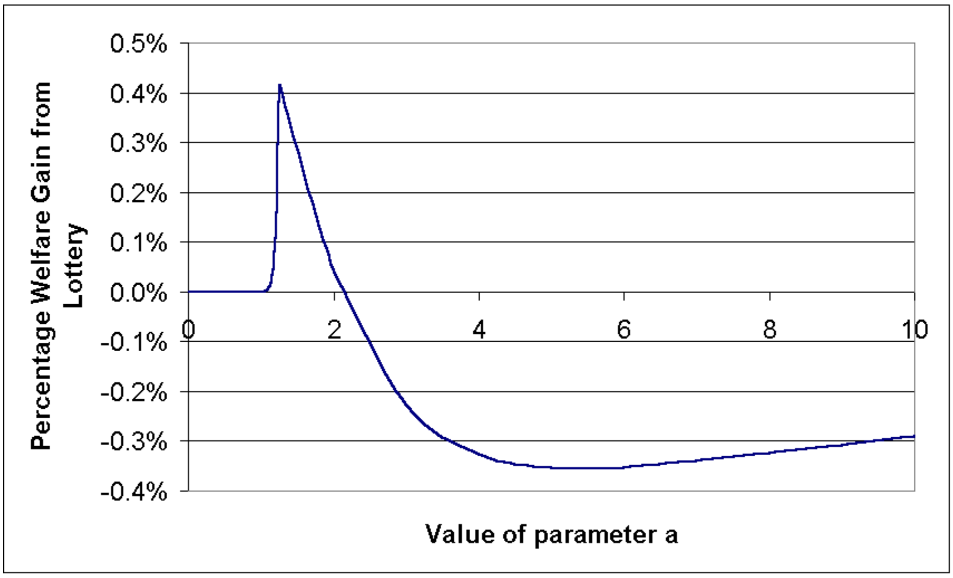


Figure 9: The welfare gain in moving from the optimal deterministic tariff to the fully optimal lottery tariff. The consumers are uniformly distributed on $\Omega_a = [a, a+1]^2$ and see the goods as substitutes.

are served. The first effect has a counterbalancing effect on the welfare calculation.

Proposition 16 *The fully optimal tariff need not be a welfare improvement on the best fixed prices.*

To prove this result we consider the case of consumers distributed uniformly on the square $[a, a+1]^2$ choosing between two symmetric substitutes produced at zero marginal cost. The globally optimal tariff structure was determined for this population in Section 7.3.1.

The proof of Proposition 16 is contained in the appendix. The calculations in the proof result in us being able to calculate the percentage welfare gain in moving from not using a lottery to using the fully optimal lottery tariff. This is plotted in Figure 9.[18]

Not withstanding Proposition 15, moving to the fully optimal lottery tariff from having not used lotteries can be detrimental to welfare. Proposition 15 only showed that small deviations to introduce lotteries were welfare enhancing. The Proposition above shows that the rise in the prices of the goods delivered with certainty can result in the overall effect on welfare being negative even if the effect on profits is positive.

[18]The kink in the graph is due to the movement from an interior to a corner solution for the optimal lottery price at the point $a = 1.25$ as documented in Lemma 14.

9 Lotteries in Competitive Markets

This chapter has shown that lotteries can help a multiproduct *monopolist* increase her profit. However, we will see that they are of no use in strongly competitive one-stop shop environments. Consider a model consisting of two multiproduct firms with N different product bundles in direct competition across a Hotelling line of length one. Consumers have valuations $\underline{x} \in \mathbb{R}_+^N$ for the products with the density of types independent of the location on the Hotelling line and equal to $f(\underline{x})$. Suppose that consumers take part in one-stop shopping (that is go to only one supplier for their products) and are uniformly distributed along the Hotelling line. Consumers have a transport cost proportional to λ where λ is small to model strong competition. The consumers' valuation is private information but both firms are assumed to know the distribution of valuations $f(\cdot)$ in the population.

We suppose that all consumers value at least one of the goods strictly above cost. The firms offer consumers the whole schedule of lotteries $\left\{ \underline{q} \middle| \sum_i q_i \leq 1 \right\}$ where q_i represents the probability that the consumer will receive good i. This one-stop shop model of competition is a good paradigm for industries such as mobile telephone calls and supermarkets. This model is analysed in considerable detail in Thanassoulis (2001) Chapter 4. Armstrong and Vickers (2001), Proposition 6 have shown that if the transport cost λ is sufficiently small then there exists a symmetric equilibrium in which all consumers are served with the prices being

$$t\left(\underline{q}\right) = \lambda + \underline{c} \cdot \underline{q}$$

where $\underline{c}$ is a vector of unit costs. This implies that the take it or leave it (tioli) equilibrium price of product i is $\lambda + c_i$. However, we noted in Lemma 1 that if the fixed prices were given by the vector $\underline{p}$ no consumer would be willing to pay more than $\underline{p} \cdot \underline{q}$ for the lottery $\underline{q}$. Hence:

$$\begin{array}{c}\text{Maximum willingness to}\\ \text{pay for lottery } \underline{q}\end{array} = \sum_{\text{products } i} q_i \left(\text{tioli price of good } i\right) \begin{array}[t]{cl} = & \sum_i q_i \left(\lambda + c_i\right) \\ = & \lambda \sum_i q_i + \underline{c} \cdot \underline{q} \\ \leq & t\left(\underline{q}\right) \\ = & \begin{array}{c}\text{Equilibrium price}\\ \text{of lottery } \underline{q}\end{array} \end{array}$$

The price of all the lotteries will therefore be priced at or above their choke price. This implies that at best only a set of consumers of zero measure purchase them so that they have no role to play in the competitive equilibrium.

This brief section therefore highlights that even though lotteries can be useful to multiproduct monopolists they are not of use to multiproduct firms in strong competition in one-stop shop industries. This is because strong competition has driven the prices of the goods so close to cost that consumers find the equilibrium lottery prices unappealing.

10 Conclusion

This chapter has shown that lotteries are a profitable tool for the multiproduct monopolist generally. The result is particularly robust when the goods are perfect substitutes but extends in certain situations even if the goods have no complementarities between them. This confirms that the Riley and Zeckhauser (1983) no lottery result does not extend to more than one good. This result provides two insights into optimal selling strategy. Firstly, we see that sellers have a justification in using capacity constraints (actual or alleged) to randomise over what good is delivered. This helps to explain the tour operators of footnote 1 and suggests that painting new cars before they're bought might not be a bad idea. Secondly we have shown that it can be important for sellers to haggle with consumers over more than one good/service. Failure to do so forgoes all the benefits that lotteries can bestow.

This chapter has assumed two key features which tilted the tables against lotteries and yet they were still found to be important pricing tools. Firstly the consumers were risk neutral and secondly the seller had no commitment or credibility problems. If consumers are risk averse, Riley and Zeckhauser (1983) note that a seller can do better than take it or leave it prices even in the one good case. If consumers are made to bid for their product with the probability of receiving it being higher the higher the bid, risk averse consumers will bid almost all of their valuation for their preferred product yielding the seller almost all the consumer surplus. With regard to commitment, Wang (1998) considers a risk averse principal who can't commit over time, bargaining with an informed agent who is one of two types. To preserve as much bargaining power as possible, the principal offers a menu of contracts which has every type of consumer accepting something. The intuition for this is that as the low type accepts the high type cannot afford to reject and bargain for better. Doing so would reveal her type and allow the principal to extract full rents. Analogously, if a multiproduct monopolist has no commitment power then she must offer a menu of contracts which have the lowest types accepting, otherwise the high valuation types can pool with the low types in rejecting. The requirement to serve all consumers is accomplished by offering the low types lotteries over whether or not they receive the good. The high valuation types will not want to risk not receiving the good.

These considerations combine to suggest that lotteries are a robust feature of a monopolist's optimal selling strategy.

A Proofs of technical results

Proof of Lemma 7. We first examine the change in consumer behaviour as a result of the lottery being offered. This is captured in Figure 4 on page 25.

1. Those consumers indifferent between good 1 and the lottery will have types satisfying

$$\begin{aligned} q_1(x-p_1)+q_3(x+y-p_3)+\eta &= x-p_1 \\ &\Rightarrow y = p_3 - p_1 - \frac{\eta}{q_3} \end{aligned}$$

   Recalling that those consumers who were previously indifferent between good 1 and the bundle have types such that $y = p_3 - p_1$ we see that the gain in profit from those that swap from good 1 to the lottery is given by

$$\int_{x=p_1}^{\infty}\int_{y=p_3-p_1-\frac{\eta}{q_3}}^{p_3-p_1} q_3\left((p_3-c_1-c_2)-(p_1-c_1)\right)-\eta dF$$

2. Those consumers indifferent between purchasing the bundle and the lottery will have types satisfying

$$\begin{aligned} q_1(x-p_1)+q_3(x+y-p_3)+\eta &= x+y-p_3 \\ &\Rightarrow y = p_3 - p_1 + \frac{\eta}{q_1} \end{aligned}$$

   The gain in profit from those that swap from the bundle consumption to the lottery is therefore given by

$$\int_{y=p_3-p_1}^{p_3-p_1+\frac{\eta}{q_1}}\int_{x=p_3-y}^{\infty} q_1\left((p_1-c_1)-(p_3-c_1-c_2)\right)-\eta dF$$

3. Finally those consumers who are now indifferent between the lottery and the outside option will have types satisfying

$$\begin{aligned} q_1(x-p_1)+q_3(x+y-p_3)+\eta &= 0 \\ &\Rightarrow x + q_3 y = q_3 p_3 + q_1 p_1 - \eta \end{aligned}$$

   referring to Figure 4 we see that the gain in profit from those consumers that swap from not purchasing to the lottery is given by

$$\int_{x=p_1-\frac{\eta}{q_1}}^{p_1}\int_{y=-\frac{x}{q_3}+p_3+\frac{q_1p_1}{q_3}-\frac{\eta}{q_3}}^{p_3-x} q_1(p_1-c_1)+q_3(p_3-c_1-c_2)-\eta dF$$

Combining these three expressions gives $\Delta\Pi(\eta)$, the change in profit as a result of the lottery addition to the tariff. Differentiating this expression with respect to η gives

$$\begin{aligned}
&\frac{\partial \Delta\Pi}{\partial \eta} \\
= & -\int_{y=p_3-p_1-\frac{\eta}{q_3}}^{p_3-p_1+\frac{\eta}{q_1}} \int_{x=q_3p_3+q_1p_1-q_3y-\eta}^{\infty} dF \\
&+\frac{1}{q_3}\left(\int_{x=p_1}^{\infty} \left[q_3\left(p_3-p_1-c_2\right)-\eta\right] f\left(x, p_3-p_1-\frac{\eta}{q_3}\right) dx\right) \\
&+\frac{1}{q_1}\left(\int_{x=p_1-\frac{\eta}{q_1}}^{\infty} \left[-q_1\left(p_3-p_1-c_2\right)-\eta\right] f\left(x, p_3-p_1+\frac{\eta}{q_1}\right) dx\right) \\
&+\frac{1}{q_3}\left(\int_{x=p_1-\frac{\eta}{q_1}}^{p_1} \left[q_1\left(p_1-c_1\right)+q_3\left(p_3-c_1-c_2\right)-\eta\right] f\left(x, -\frac{x}{q_3}+p_3+\frac{q_1p_1}{q_3}-\frac{\eta}{q_3}\right) dx\right)
\end{aligned}$$

We note that this expression vanishes when $\eta = 0$ implying that this lottery has no first order effect on profit. Differentiating again with respect to η and setting $\eta = 0$ gives

$$\left[\frac{\partial^2 \Delta\Pi}{\partial \eta^2}\right]_{\eta=0} = \frac{1}{q_1q_3}\left\{\begin{array}{c} -2\int_{x=p_1}^{\infty} f\left(x, p_3-p_1\right) dx + \left(p_1-c_1\right) f\left(p_1, p_3-p_1\right) \\ -\left(p_3-p_1-c_2\right)\int_{x=p_1}^{\infty} f_2\left(x, p_3-p_1\right) dx \end{array}\right\} \tag{18}$$

This is exactly the expression given in the statement of Lemma 7. If the right hand side of this expression is positive then lotteries will be profit enhancing as η increases from 0 as required.

■

Proof of Lemma 14. We consider this problem as a number of distinct cases.

Suppose that the optimal η is so large that no consumer in Ω_a is indifferent between a good with certainty and not buying at all. That is $2\eta \geq p-a$.

In this case the profit of the firm is given by

$$\begin{aligned}
\Pi(p,\eta) &= \begin{array}{c} 2p\frac{1}{2}\left(a+1-a-2\eta\right)\left(a+1-2\eta-a\right) \\ +\left(p-\eta\right)\left[1-\left(a+1-a-2\eta\right)^2-\frac{1}{2}\left(2\left(p-\eta\right)-2a\right)^2\right] \end{array} \\
&= \eta\left(1-2\eta\right)^2+\left(p-\eta\right)\left[1-2\left(p-\eta-a\right)^2\right] \qquad (19)
\end{aligned}$$

Differentiating this with respect to p and η gives

$$\frac{\partial \Pi}{\partial p} = 1-2\left(p-\eta-a\right)\left(3p-3\eta-a\right) \tag{20}$$

$$\frac{\partial \Pi}{\partial \eta} = \left(1-2\eta\right)\left(1-6\eta\right)-1+2\left(p-\eta-a\right)\left(3p-3\eta-a\right) \tag{21}$$

Setting (20) = 0 in (21) gives

$$\begin{aligned}(1-2\eta)(1-6\eta) &= 0 \\ &\Rightarrow \eta = \frac{1}{2} \text{ or } \frac{1}{6}\end{aligned}$$

Brief investigation confirms that the maximum is at $\eta = \frac{1}{6}$.

Now in (20) let $P = p - \eta$ giving

$$\begin{aligned}1-2(P-a)(3P-a) &= 0 \\ &\Rightarrow P = \frac{4a \pm \sqrt{4a^2+6}}{6}\end{aligned}$$

The maximum is found again at

$$p = \eta + \frac{2a + \sqrt{a^2 + \frac{3}{2}}}{3}$$

We finally note that this formula can only apply when $2\eta \geq p - a$. Basic manipulations confirm that this is true if and only if $a \geq 1.25$.

Now suppose that η is sufficiently small that there exist consumers indifferent between the take it or leave it option and not buying at all. That is $2\eta < p - a$.

In this case the profit of the firm is given by

$$\begin{aligned}\Pi(p,\eta) &= 2p(a+1-p)\tfrac{1}{2}(1-2\eta+p-2\eta-a) \\ &\quad + (p-\eta)\left(\tfrac{1}{2}(2\eta)^2 + 2\eta(a+1-p)2\right) \\ &= -p^3 + 2ap^2 + p\left(6\eta^2 + 1 - a^2\right) - 4\eta^2(a+1) - 2\eta^3 \qquad (22)\end{aligned}$$

Differentiating this expression with respect to p and η gives

$$\frac{\partial \Pi}{\partial \eta} = 2\eta\left[6p - 4(a+1) - 3\eta\right] \qquad (23)$$

$$\frac{\partial \Pi}{\partial p} = -3p^2 + 4ap + 6\eta^2 + 1 - a^2 \qquad (24)$$

We know from above that if $a \geq 1.25$ then η will be so big as to make this profit function incorrect. To determine when lotteries will be used at all we consider the second derivative of profit with respect to η

$$\frac{\partial^2 \Pi}{\partial \eta^2} = 4\left[3p - 2a - 2 - 3\eta\right]$$

If it is optimal not to use a lottery ($\eta = 0$) we know that the best fixed price is $p = \frac{2a+\sqrt{a^2+3}}{3}$. We therefore deduce that

$$\left[\frac{\partial^2 \Pi}{\partial \eta^2}\right]_{\eta=0}\Bigg|_{p=\frac{2a+\sqrt{a^2+3}}{3}} > 0 \text{ if and only if } a^2 > 1$$

Therefore if $a \leq 1$ lotteries will not help the seller.

Therefore suppose that $1 < a < 1.25$. In this case

$$\frac{\partial \Pi}{\partial \eta} = 0 \Rightarrow \eta = 2p - \frac{4}{3}(a+1) \text{ for the optimum}$$

Setting (24) $= 0$ and using the expression for η given above we have

$$3p^2 - 4ap + (a+1)(a-1) - 6\left\{4p^2 + \frac{16}{9}(a+1)^2 - \frac{16p}{3}(a+1)\right\} = 0$$

which implies that for the optimal we have

$$p = \frac{2(7a+8) - \sqrt{4(7a+8)^2 - 7(a+1)(29a+35)}}{21}$$

This completes the proof of the lemma. ■

Proof of Proposition 15. We prove the proposition in two parts by firstly considering the substitutes paradigm of Section 3 and secondly the no complementarities model of McAfee and McMillan (1988).

Firstly, we aim to determine the effect of introducing a lottery offer (q_1, q_2) with $q_1 + q_2 = 1$ to the optimal deterministic tariff for substitutes on the population's welfare. We note that there are three types of consumer affected by introducing the lottery: those swapping from good 1 to the lottery, those swapping from good 2 and those who previously didn't make a purchase. Each individual consumer contributes the difference between their willingness to pay and the cost of manufacture to welfare. We therefore have:

1. The change in welfare arising from a consumer swapping from good 1 to the lottery is given by

$$\begin{aligned} & [q_1(x-c_1) + q_2(y-c_2)] - [x-c_1] \\ = \; & q_2[(y-c_2) - (x-c_1)] \end{aligned}$$

2. The change in welfare arising from a consumer swapping from good 2 to the lottery is given by

$$q_1[(x-c_1) - (y-c_2)]$$

3. The change in welfare arising from a consumer swapping from consuming nothing to purchasing the lottery is given by

$$q_1(x-c_1)+q_2(y-c_2)$$

Combining these three effects with the analysis of consumers' reaction to the lottery offer in Section 3.1 we can establish the change in welfare for the population arising from the introduction of the lottery (q_1, q_2) priced at a price of $q_1p_1+q_2p_2-\eta$ as

$$\begin{aligned}
\Delta W(\eta) &= \int_{x=p_1}^{\infty}\int_{y=x-p_1+p_2-\frac{\eta}{q_2}}^{x-p_1+p_2} q_2\left[(y-c_2)-(x-c_1)\right]dF \\
&+\int_{y=p_2}^{\infty}\int_{x=y-p_2+p_1-\frac{\eta}{q_1}}^{y-p_2+p_1} q_1\left[(x-c_1)-(y-c_2)\right]dF \\
&+\int_{x=p_1-\frac{\eta}{q_1}}^{p_1}\int_{y=p_2-\frac{q_1}{q_2}(x-p_1)-\frac{\eta}{q_2}}^{p_2} q_1(x-c_1)+q_2(y-c_2)\,dF
\end{aligned}$$

Differentiating this expression with respect to η gives

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\Delta W(\eta) \qquad (25)\\
=\ &\int_{x=p_1}^{\infty}\left\{(p_2-c_2)-(p_1-c_1)-\frac{\eta}{q_2}\right\}f\left(x,x-p_1+p_2-\frac{\eta}{q_2}\right)dx \\
&+\int_{y=p_2}^{\infty}\left\{(p_1-c_1)-(p_2-c_2)-\frac{\eta}{q_1}\right\}f\left(y-p_2+p_1-\frac{\eta}{q_1},y\right)dy \\
&+\int_{x=p_1-\frac{\eta}{q_1}}^{p_1}\left\{\frac{q_1}{q_2}(p_1-c_1)+(p_2-c_2)-\frac{\eta}{q_2}\right\}f\left(x,p_2-\frac{q_1}{q_2}(x-p_1)-\frac{\eta}{q_2}\right)dx
\end{aligned}$$

We now note that $\left[\frac{\partial}{\partial\eta}\Delta W(\eta)\right]_{\eta=0}$ vanishes and so the first order effect on welfare from introducing the lottery (q_1, q_2) at a price of $q_1p_1+q_2p_2-\eta$ is zero. To determine the welfare effects of introducing the lottery we must determine the second order conditions. Differentiating (25) with respect to η and setting η equal to 0 then gives

$$\begin{aligned}
&\left[\frac{\partial^2\Delta W}{\partial\eta^2}\right]_{\eta=0} \qquad (26)\\
=\ &\frac{1}{q_1q_2}\left\{-\int_{x=p_1}^{\infty}f(x,x-p_1+p_2)\,dx+\{q_1(p_1-c_1)+q_2(p_2-c_2)\}f(p_1,p_2)\right\} \\
&+\frac{(p_2-c_2)-(p_1-c_1)}{q_1q_2}\left\{\int_{x=p_1}^{\infty}\{q_2f_1(x,x-p_1+p_2)-q_1f_2(x,x-p_1+p_2)\}\right\}
\end{aligned}$$

The local introduction of a lottery is welfare enhancing if and only if (26) is positive. However, comparing (26) with the expression for the local change in profit, $\left[\frac{\partial^2\Delta\Pi}{\partial\eta^2}\right]_{\eta=0}$

contained in (7) gives

$$\left[\frac{\partial^2 \Delta W}{\partial \eta^2}\right]_{\eta=0} = \underbrace{\frac{1}{q_1 q_2}\int_{x=p_1}^{\infty} f\left(x, x-p_1+p_2\right)dx}_{\geq 0} + \left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0}$$

We therefore note that if introducing a lottery to the optimal deterministic tariff is locally profit enhancing then by definition $\left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0}$ will be positive. This in turn guarantees that welfare is enhanced by the introduction of the lottery.

We now turn to the no complementarities model of McAfee and McMillan (1988). We consider the seller introducing the lottery $(q_1, 0, q_3)$ with $q_1 + q_3 = 1$ to the best fixed prices. The welfare changes in this case are given by:

1. The change in welfare arising from a consumer swapping from good 1 to the lottery is given by

$$\begin{aligned} & \left[q_1\left(x-c_1\right)+q_3\left(x+y-c_1-c_2\right)\right]-\left[x-c_1\right] \\ = \; & q_3\left(y-c_2\right) \end{aligned}$$

2. The change in welfare arising from a consumer swapping from the bundle to the lottery is given by

$$-q_1\left(y-c_2\right)$$

3. The change in welfare arising from a consumer swapping from consuming nothing to purchasing the lottery is given by

$$\left(x-c_1\right)+q_3\left(y-c_2\right)$$

Combining these three effects with the analysis of consumer behaviour in response to the lottery contained in the proof of Lemma 7, we can establish the change in welfare for the population arising from the introduction of the lottery (q_1, q_3) priced at a price of $q_1 p_1 + q_3 p_3 - \eta$ as

$$\begin{aligned} \Delta W(\eta) = \; & \int_{x=p_1}^{\infty}\int_{y=p_3-p_1-\frac{\eta}{q_3}}^{p_3-p_1} q_3\left(y-c_2\right)dF \\ & -\int_{y=p_3-p_1}^{p_3-p_1+\frac{\eta}{q_1}}\int_{x=p_3-y}^{\infty} q_1\left(y-c_2\right)dF \\ & +\int_{x=p_1-\frac{\eta}{q_1}}^{p_1}\int_{y=-\frac{x}{q_3}+p_3+\frac{q_1 p_1}{q_3}-\frac{\eta}{q_3}}^{p_3-x} \left(x-c_1\right)+q_3\left(y-c_2\right)dF \end{aligned}$$

Differentiating this expression with respect to η gives

$$
\begin{aligned}
&\frac{\partial \Delta W}{\partial \eta} \\
= &\int_{x=p_1}^{\infty}\left(p_3-p_1-c_2-\frac{\eta}{q_3}\right) f\left(x, p_3-p_1-\frac{\eta}{q_3}\right) dx \\
&-\int_{x=p_1-\frac{\eta}{q_1}}^{\infty}\left(p_3-p_1-c_2+\frac{\eta}{q_1}\right) f\left(x, p_3-p_1+\frac{\eta}{q_1}\right) dx \\
&+\frac{1}{q_3}\int_{x=p_1-\frac{\eta}{q_1}}^{p_1}\left\{-c_1+q_3\left(p_3+\frac{q_1 p_1}{q_3}-\frac{\eta}{q_3}-c_2\right)\right\} f\left(x,-\frac{x}{q_3}+p_3+\frac{q_1 p_1}{q_3}-\frac{\eta}{q_3}\right) dx
\end{aligned}
$$

Noting that this expression vanishes when $\eta=0$ we again differentiate with respect to η to find

$$
\begin{aligned}
\left[\frac{\partial^2 \Delta W}{\partial \eta^2}\right]_{\eta=0} = &\int_{x=p_1}^{\infty}-\frac{1}{q_3} f\left(x, p_3-p_1\right)-\frac{1}{q_3}\left(p_3-p_1-c_2\right) f_2\left(x, p_3-p_1\right) dx \\
&-\int_{x=p_1}^{\infty}\frac{1}{q_1} f\left(x, p_3-p_1\right)+\frac{1}{q_1}\left(p_3-p_1-c_2\right) f_2\left(x, p_3-p_1\right) dx \\
&-\frac{1}{q_1}\left(p_3-p_1-c_2\right) f\left(p_1, p_3-p_1\right) \\
&+\frac{1}{q_1 q_3}\left\{-c_1+q_3\left(p_3+\frac{q_1 p_1}{q_3}-c_2\right)\right\} f\left(p_1, p_3-p_1\right)
\end{aligned}
$$

Combining terms and using the fact that $q_1+q_3=1$ gives

$$
\left[\frac{\partial^2 \Delta W}{\partial \eta^2}\right]_{\eta=0}=\frac{1}{q_1 q_3}\left\{\begin{array}{c}-\int_{x=p_1}^{\infty} f\left(x, p_3-p_1\right) dx+\left(p_1-c_1\right) f\left(p_1, p_3-p_1\right) \\ -\left(p_3-p_1-c_2\right) \int_{x=p_1}^{\infty} f_2\left(x, p_3-p_1\right) dx\end{array}\right\} \tag{27}
$$

Comparing (27) to the expression (18) which gives $\left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0}$ in the general two good case we see that

$$
\left[\frac{\partial^2 \Delta W}{\partial \eta^2}\right]_{\eta=0}=\underbrace{\frac{1}{q_1 q_3}\int_{x=p_1}^{\infty} f\left(x, p_3-p_1\right) dx}_{\geq 0}+\left[\frac{\partial^2 \Delta \Pi}{\partial \eta^2}\right]_{\eta=0}
$$

We therefore have an analogous result to that in the substitutes paradigm.

This therefore proves the proposition. ■

Proof of Proposition 16. We consider the case of consumers distributed uniformly on the square $[a, a+1]^2$ choosing between two symmetric substitutes which the firm produces at zero marginal cost. The globally optimal tariff structure was determined for this population in Section 7.3.1. We can therefore calculate the welfare provided when using the fully optimal lottery pricing strategy and when using the best fixed prices only.

No lottery used If no lottery is offered and the goods are sold at a price p with the population uniform on $\Omega_a = [a, a+1]^2$ we have welfare

$$\begin{aligned} W(a) &= 2\int_{x=p}^{a+1}\int_{y=a}^{x} x\,dydx = 2\int_{x=p}^{a+1} x(x-a)\,dx = 2\left[\frac{x^3}{3}-\frac{ax^2}{2}\right]_p^{a+1} \\ &= \frac{1}{3}\left\{\left[-a^3+3a+2\right]+p^2(3a-2p)\right\} \end{aligned} \tag{28}$$

From Lemma 14 this is the case if $a \leq 1$.

Lottery used an interior solution to profit maximisation From Lemma 14 if $1 < a < 1.25$ the optimal tariff has a lottery used which ensures that there exist consumers in Ω_a indifferent between purchasing the good delivered with certainty or the outside option. In this case we have welfare

$$\begin{aligned} W(a) &= 2\left\{\begin{array}{c} \int_{x=p}^{a+1}\int_{y=a}^{x-2\eta} x\,dydx + \int_{x=p-\eta}^{p}\int_{y=2(p-\eta)-x}^{x}\frac{1}{2}(x+y)\,dydx \\ +\int_{x=p}^{a+1}\int_{y=x-2\eta}^{x}\frac{1}{2}(x+y)\,dydx \end{array}\right\} \\ &= 2\left\{\begin{array}{c} \int_{x=p}^{a+1} x(x-2\eta-a)\,dx + \int_{x=p-\eta}^{p} x(x-(p-\eta)) + \left[\frac{y^2}{4}\right]_{2(p-\eta)-x}^{x} dx \\ +\int_{x=p}^{a+1}\eta x + \left[\frac{y^2}{4}\right]_{x-2\eta}^{x} dx \end{array}\right\} \\ &= 2\left\{\int_{x=p}^{a+1} x^2 - x(a+2\eta)\,dx + \int_{x=p-\eta}^{p} x^2-(p-\eta)^2\,dx + \int_{x=p}^{a+1} 2\eta x-\eta^2 dx\right\} \\ &= a-2\eta^2+\frac{2}{3}-2a\eta^2+p^2a+4p\eta^2-\frac{a^3}{3}-\frac{4}{3}\eta^3-\frac{2}{3}p^3 \end{aligned} \tag{29}$$

Lottery used a corner solution to the profit maximisation problem We finally note from Lemma 14 that if $a \geq 1.25$ then the optimal tariff ensures that there are no consumers in Ω_a indifferent between purchasing the good delivered with certainty or the outside option. In this case we have welfare

$$\begin{aligned} W(a) &= 2\left\{\begin{array}{c} \int_{x=a+2\eta}^{a+1}\int_{y=a}^{x-2\eta} x\,dydx + \int_{x=p-\eta}^{2(p-\eta)-a}\int_{y=2(p-\eta)-x}^{x}\frac{1}{2}(x+y)\,dydx \\ +\int_{x=2(p-\eta)-a}^{a+2\eta}\int_{y=a}^{x}\frac{1}{2}(x+y)\,dydx + \int_{x=a+2\eta}^{a+1}\int_{y=x-2\eta}^{x}\frac{1}{2}(x+y)\,dydx \end{array}\right\} \\ &= 2\left\{\begin{array}{c} \int_{x=a+2\eta}^{a+1} x(x-2\eta-a)\,dx + \int_{x=p-\eta}^{2(p-\eta)-a} x(x-(p-\eta)) + \left[\frac{y^2}{4}\right]_{2(p-\eta)-x}^{x} dx \\ +\int_{x=2(p-\eta)-a}^{a+2\eta}\frac{x}{2}(x-a) + \left[\frac{y^2}{4}\right]_{a}^{x} dx + \int_{x=a+2\eta}^{a+1}\eta x + \left[\frac{y^2}{4}\right]_{x-2\eta}^{x} dx \end{array}\right\} \\ &= 2\left\{\begin{array}{c} \int_{x=a+2\eta}^{a+1} x^2 - x(a+2\eta)\,dx + \int_{x=p-\eta}^{2(p-\eta)-a} x^2-(p-\eta)^2\,dx \\ \int_{x=2(p-\eta)-a}^{a+2\eta}\frac{3x^2}{4}-\frac{ax}{2}-\frac{a^2}{4}dx + \int_{x=a+2\eta}^{a+1} 2\eta x-\eta^2 dx \end{array}\right\} \\ &= a-4ap\eta-2\eta^2+\frac{2}{3}+2a\eta^2+4p^2\eta+2p^2a-4p\eta^2-\frac{2}{3}a^3+4\eta^3-\frac{4}{3}p^3 \end{aligned} \tag{30}$$

We can therefore calculate the effect of lotteries on welfare. Lemmas 13 and 14 give the optimal prices if no lottery is used and those if a lottery is used for all values of the parameter a. Expressions (28), (29) and (30) allow us to determine the welfare of the populations as a function of the parameter a when lotteries are not used and when they are under the fully optimal tariff. These results are combined and the percentage welfare gain in moving from not using a lottery to using the fully optimal lottery tariff is plotted in Figure 9 on page 41. This shows that welfare can be both increased and decreased when lotteries/haggling are allowed. This proves the proposition. ∎

B Linear Programming code

Section 7.2 was solved using an AMPL program to determine the optimal selling strategy of a seller facing consumers uniformly distributed on the square $[a, a+1]^2$. A typical such program follows.

```
# Objective:  consumer surplus
# Constraints:  incentive compatibility and therefore non-linear
param a := 20; #the bottom left hand point of the grid
param N := 50; #the number of grid points along the bottom row (starting
from 0)
param h := 1/N; #the distance between grid points
#######
#Now the variables
#######

var v {i in 0..N, j in 0..i};
##dv/dx using symmetry
var dvdx {i in 1..N, j in 0..i} =
 if i in 1..N-1 and j in 0..i-1 then ( (v[i+1,j]-v[i-1,j])/(2*h) )
 else if i in 1..N-1 and j=i then ( (v[i+1,i]-v[i,i-1])/(2*h) )
 else if i=N and j in 0..N-1 then ( (v[N,j]-v[N-1,j])/h )
 else if i=N and j=N then ((v[N,N]-v[N,N-1])/h)
#but won't use the N,N one
 ;
##dv/dv using symmetry
var dvdy {i in 1..N, j in 0..i} =
 if i in 2..N and j in 1..i-1 then ( (v[i,j+1]-v[i,j-1])/(2*h) )
 else if i in 1..N-1 and j=i then ( (v[i+1,i]-v[i,i-1])/(2*h) )
 else if i in 1..N and j=0 then ((v[i,j+1]-v[i,j])/h)
 else if i=N and j=N then ((v[N,N]-v[N,N-1])/h)
```

```
 ;
##only use second order conditions in the interior
##d2v/dx2
var d2vdx2 {i in 1..N-1, j in 0..i-1} = (v[i+1,j]+v[i-1,j]-2*v[i,j])/(h^2)
;
##no values at 0,0 or i=N or along middle edge
##d2v/dy2
var d2vdy2 {i in 2..N, j in 1..i-1} = (v[i,j+1]+v[i,j-1]-2*v[i,j])/(h^2)
;
##no values at N,N or j=0 or along middle edge
##d2v/dxdy
var d2vdxdy {i in 3..N-1, j in 1..i-2} =
 (v[i+1,j+1]-v[i+1,j-1]-(v[i-1,j+1]-v[i-1,j-1]))/(4*h^2);
##no values at i=N
##sum for printing out
var probreceive {i in 1..N-1, j in 0..i} = dvdx[i,j]+dvdy[i,j];
maximize integral:
 h*( (a+1-1.5*h) * sum {j in 1..N-1} (v[N,j])
 - 3*h*( (sum{i in 2..N-1}( sum{j in 1..i-1}(v[i,j]) ) )
 + (0.5*sum{i in 1..N-1}(v[i,i]) ) )
 - (a+1.5*h)*sum{i in 1..N-1}(v[i,0]) + (0.5-3*h/4)*v[N,0]
 + (a/2+0.5- 3*h/8)*v[N,N] );
subject to pos{i in 0..N, j in 0..i}:
 v[i,j]>=0;
subject to bound:  v[0,0]=0;
subject to dx{i in 1..N, j in 0..i}:
 dvdx[i,j] >= 0;

subject to dy{i in 1..N, j in 0..i}:
 dvdy[i,j] >= 0;
subject to aprob{i in 1..N-1, j in 0..i}:
 (dvdx[i,j]+dvdy[i,j]) <= 1;
subject to bprob{j in 0..N-1}:
 (dvdx[N,j]+dvdy[N,j]) <= 1;
subject to d2x{i in 1..N-1, j in 0..i-1}:
 d2vdx2[i,j] >=0;
subject to d2y{i in 2..N, j in 1..i-1}:
 d2vdy2[i,j] >=0;
```

```
subject to relax{i in 2..N, j in 0..i-2}:
 v[i-1,j+1]-v[i,j] <= 0;
solve;
display integral;
display v;
display dvdx;
display dvdy;
display probreceive;
```

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