Business Cycle Asymmetries: Characterisation and Testing based on Markov-Switching Autoregressions

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Abstract

We propose testing for business cycle first-moment asymmetries in Markov-switching autoregressive (MS-AR) models. We derive the parametric restrictions on MS-AR models that rule out types of asymmetries such as deepness, steepness, and sharpness, and set out a testing procedure based on Wald statistics which have standard asymptotics. For a two-regime model, such as that popularised by Hamilton (1989), we show that deepness implies sharpness (and vice versa) while the process is always non-steep. We illustrate with two and three-state MS-AR models of US GNP growth, and models of US investment and consumption growth. Our findings are compared with those obtained from standard non-parametric tests, which are unable to distinguish between first-moment asymmetries and heteroscedasticity.

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Key words: Business cycle asymmetries, Markov-switching models, Deepness, Steepness, Sharpness, non-parametric tests.

1 Introduction

> '*the most violent declines exceed the most considerable advances [...] Business contractions appear to be a briefer and more violent process than business expansions*'
> Mitchell (1927, p. 290)

There has been much interest in whether macroeconomic variables behave differently over the phases of the business cycle. Sichel (1993, p. 224) defines an asymmetric cycle as 'one in which some phase of the cycle is different from the mirror image of the opposite phase'. McQueen and Thorley (1993, pp. 342 – 343) and Sichel (1993, pp. 225 – 226) discuss the importance, from both theoretical and empirical viewpoints, of establishing whether there are asymmetries in the business cycle. The finding of asymmetry is compatible with a number of business cycle models, but would rule out linear models with symmetric errors. Sichel notes that models of asymmetric price adjustment can generate

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deepness (defined below), and references De Long and Summers (1988) and Ball and Mankiw (1994). More recently, the 'output-gap' literature (see, e.g., Laxton, Meredith and Rose, 1995 and Clark, Laxton and Rose, 1996) suggests that more of the adjustment in response to a negative demand shock falls on output than prices, compared to the response to positive shocks. Finally, Sichel references models where entry to an industry is more costly than exit, which could result in steepness asymmetries.

A number of types of asymmetry have been discussed in the literature. Primary ones are those of steepness, deepness, and sharpness (or turning point asymmetry) (henceforth, SDS), which are typically tested for using separate non-parametric tests. Other types of asymmetries have been explored in parametric models, such as asymmetric persistence to shocks (see Beaudry and Koop, 1993 and Hess and Iwata, 1997a), and business cycle duration dependence (see, e.g., Sichel, 1991, Diebold, Rudebusch and Sichel, 1993, Filardo, 1994 and Filardo and Gordon, 1998). Our aim in this paper is to analyse the conditions under which the Markov-switching autoregressive model (MS-AR) class is capable of generating SDS asymmetries. The conditions are expressed as restrictions on the parameters of the MS-AR model which, if they hold, would rule out a particular type of asymmetry. We then derive tests of these restrictions based on estimated MS-AR models, providing parametric tests as alternatives of the non-parametric tests typically used in the literature. Our tests are able to detect asymmetries in the propagation mechanisms of shocks, or first-moment asymmetries, while non-parametric tests are unable to discriminate between first-moment asymmetries and asymmetries in the shocks. Since Hamilton (1989) the MS-AR model class has been extensively used in the empirical macroeconomics literature to analyse business cycle phenomena, and good estimation and inferential procedures are available, making it an obvious choice for the development of parametric tests of asymmetry.

The basic MS-AR model at the centre of our analysis is the following. A stationary time series $\{x_t\}$ is assumed to have been generated by an AR(p) with M Markov-switching regimes in the Mean of the process, which we label an MSM(M)-AR(p) process:

$$x_t - \mu(s_t) = \sum_{k=1}^{p} \alpha_k \left(x_{t-k} - \mu(s_{t-k})\right) + u_t, \quad u_t | s_t \sim \mathsf{NID}\left(0, \sigma^2\right). \tag{1}$$

We can order the regimes by the magnitude of μ such that $\mu_1 < \ldots < \mu_M$. The Markov chain is ergodic, irreducible, and there does not exist an absorbing state, i.e., $\bar{\xi}_m \in (0,1)$ for all $m = 1, \ldots, M$, where $\bar{\xi}_m$ is the ergodic or unconditional probability of regime m. The transition probabilities are time-invariant:

$$p_{ij} = prob(s_{t+1} = j | s_t = i), \quad \sum_{j=1}^{M} p_{ij} = 1 \quad \forall i, j \in \{1, \ldots, M\}, \tag{2}$$

so that the probability of a switch between regimes i and j does not depend on how long the process has been in regime i.[1]

The MS-AR framework can be readily extended to multivariate settings (see e.g., Ravn and Sola, 1995, Diebold and Rudebusch, 1996, Hamilton and Lin, 1996, Krolzig, 1997, Krolzig and Sensier, 2000, and Krolzig and Toro, 1998), which is a distinct advantage given that business cycles were originally viewed as consisting of co-movements in many economic variables (see, e.g., Burns and Mitchell, 1946). In equation (1) x_t can be a vector of variables. The extension of our tests to multivariate settings, and models which include regime-dependent heteroscedasticity, and switches in intercepts, are discussed in section 4.

[1] Diebold *et al.* (1993), Filardo (1994) and Filardo and Gordon (1998) extend the Hamilton (1989) model to allow for time-varying transition probabilities. Generally there appears to be positive duration dependence in contractions in the US post-War period, so that the probability of moving out of recession increases with the duration of recession. Non-constant transition probabilities would complicate the derivation of the SDS tests.

As well as constructing tests of asymmetries, a related goal is to establish precisely which types of asymmetries MS-AR models are in principle capable of generating, given the widespread popularity of these models in applied research, and because some of the literature appears confused on this point[2]. Recently Hess and Iwata (1997b) have investigated by simulation whether empirically-estimated models, including MS-AR models, are able to replicate the 'fundamental business cycle features' of observed durations and amplitudes of contractions and expansions. Our focus is different, since we ask whether in principle MS-AR models can generate certain asymmetric features, in addition to whether empirically estimated models possess these features.

Finally, because measures of economic activity exhibit secular increases, the asymmetries relate to the de-trended log of output. For example, Speight and McMillan (1998) consider the de-trended component (x_t) of the variable y_t, where $x_t = y_t - \tau_t$. τ_t is a non-stationary trend component, and x_t is stationary, possibly consisting of cycle and noise components. Throughout the paper, x_t refers to the detrended series. We assume the non-stationarity can be removed by differencing, i.e., $x_t = \Delta y_t$. Trend elimination by differencing is natural in our setup, because the MS-AR model is typically estimated on the first difference of the log of output.[3] However, none of the propositions on asymmetries in MS-AR models that follow, nor the testing procedures, require this method of de-trending, and remain valid whichever method is used. All we require is that a MS-AR model can be estimated for the de-trended series, howsoever obtained. The sensitivity of the findings of asymmetries to the method of trend elimination requires further research, and Gordon (1997) shows that in general the model of the short-run fluctuations in output may depend on the treatment of the trend component.

The plan of the paper is as follows. In section 2 we review the literature on testing for business cycle asymmetries and show how (the absence of) SDS asymmetries can be mapped into parameter restrictions of MS-AR models, paying particular attention to the empirically relevant two- and three-regime models. Then, section 3 derives the Wald tests of SDS hypotheses. Wald tests obviate the necessity of estimating the restricted (null) MS-AR model and are attractive for that reason. Section 4 shows how the basic testing approach can be extended in a number of directions. Section 5 uses Monte Carlo simulations to investigate the small sample properties of the tests, their performance in the presence of heteroscedasticity, and their robustness to model mis-specification. Section 6 sets out the empirical illustrations. Section 7 concludes.

2 Business cycle asymmetries

2.1 A brief review of the literature on business cycle asymmetries

2.1.1 Steepness and deepness

Sichel (1993) distinguishes two types of business cycle asymmetry: 'steepness' and 'deepness'. The former relates to whether contractions are steeper (or less steep) than expansions, the latter to whether the amplitude of troughs exceeds (or is shallower than) that of peaks. The top left panel of figure 1 depicts a schematic business cycle that is non-deep and non-steep. The second panel in the first column shows deepness of troughs (but non-steepness), the third panel steepness of expansions (but non-deepness) and the last panel shows both properties.

[2]For example, Sichel (1993, p. 232, footnote 19) states that the Hamilton (1989) two-state model implies steepness in US GNP. In fact, steepness (as defined formally below) can not arise in such a model.

[3]An exception is Lam (1990) who allows for a general autoregressive process in the level of the log of output, rather than imposing a unit root.

A number of ways of testing for steepness and deepness have been proposed in the literature. Neftci (1984) proposed a test of whether there are longer runs of increases than decreases in a series, indicating that the length of expansions exceeds that of contractions, so that contractions are necessarily steeper than expansions. He defines an indicator variable $I_t = 1$ if $x_t > 0$ (expansion) and $I_t = -1$ if $x_t \leq 0$ (recession). Suppose I_t can be represented by a second-order Markov process, then $p_{22} > p_{11}$ (where $p_{22} = prob[I_t = 1 \mid I_{t-1} = 1, I_{t-2} = 1]$ and $p_{11} = prob[I_t = -1 \mid I_{t-1} = -1, I_{t-2} = -1]$) implies a form of cylical asymmetry because the length of expansions exceeds that of contractions. A possible problem with this procedure is its sensitivity to noise. If increases (decreases) are inadvertently measured as decreases (increases), then the counts of transitions from which the estimates of the transition probabilities are derived will be affected. Using this approach, Neftci found evidence of steepness in post-War US unemployment during contractions. By way of contrast, Falk (1986) failed to find evidence of steepness in other US quarterly macroeconomic series using Neftci's procedure, and Sichel (1989) suggested an error in Neftci's work and indicated that the procedure might fail to find steepness when in fact it is present. Rothman (1991) finds evidence of asymmetry in the quarterly unemployment rate series using a modified version of Neftci's test, and Sichel (1989) finds strong evidence of asymmetry in annual unemployment, for which measurement error is presumably less important. Luukkonen and Teräsvirta (1991) note that self-exciting threshold autoregressive models (see, e.g., Tong and Lim, 1980, Tong, 1995) and smooth transition autoregressive model (see, e.g., Luukkonen, Saikkonen and Teräsvirta, 1988, Teräsvirta and Anderson, 1992) may imply cyclical asymmetry in this sense, in that the probabilities of remaining in the regimes, once entered, may not be equal due to different dynamic structures.

Sichel (1993) suggests a test of deepness based on the coefficient of skewness calculated for the detrended series $\{x_t\}$. Deepness of contractions will show up as negative skewness, since it implies that the average deviation of observations below the mean will exceed that of observations above the mean. Steepness (of expansions) implies positive skewness in the first difference of the detrended series, $\{\Delta x_t\}$: increases should be larger, though less frequent, than decreases. Figure 1 illustrates. The second column depicts the histograms (and densities) for $\{x_t\}$ corresponding to the schematic business cycles in the first column. The densities are symmetric for the first and third rows because the business cycles are non-deep: those in the second and fourth rows exhibit negative skewness because of the deepness of contractions. The third and fourth columns depict the times series of $\{\Delta x_t\}$ and their histograms (and densities) for the schematic business cycles in the first column. The densities are symmetric for the first and second rows because the business cycles are non-steep: those in the third and fourth rows exhibit positive skewness because of the steepness of expansions.

On the basis of these tests, deepness is found to characterise quarterly post-War US unemployment and industrial production, with weaker evidence for GNP, while only unemployment (of the three) appears to exhibit steepness. We also report these non-parametric tests based on the coefficients of skewness (hereafter NP tests) in our empirical work.[4] In addition, the concepts of non-deepness and non-steepness are used to construct parametric tests based on the MS-AR model in (1), which we turn to after discussing the third notion of asymmetry.

[4]These are calculated as in Sichel (1993, p. 227–8). That is, an asymptotically heteroscedasticity and serial correlation consistent standard error is calculated for the coefficent of skewness using the Newey and West (1987) procedure. This is done because the detrended series will not be i.i.d.

2.1.2 Sharpness

Sharpness or turning point (TP) asymmetry, as introduced by McQueen and Thorley (1993), would result if, e.g., troughs were 'sharp' and peaks more 'rounded'. They present two tests. The first is based on the magnitude of growth rate changes around NBER-dated peaks and troughs. The mean absolute changes are calculated for peaks and troughs separately, and the test for asymmetry is based on rejecting the null of the population mean changes in the variable at peaks and troughs being equal. McQueen and Thorley (1993) find the null of equal turning point sharpness can be rejected for both the unemployment rate and industrial production. Their second testing procedure is based on a second-order three state Markov chain. They distinguish between contraction, moderate, and high (recovery) states. The hypothesis in Hicks (1950), that troughs are sharper than peaks, corresponds to the probability of jumping from the contraction to high growth state exceeding the probability of jumping directly from high growth to contraction. 'Complete' TP symmetry requires that these switches are equally likely, and in addition, that switches to moderate growth from contraction, and from high growth, are equally likely, and that movements to high growth and contraction, from moderate growth, are also equally likely. They again find evidence of sharpness asymmetry for post-War unemployment and industrial production, but the susceptibility of the test to noise is evident when they consider pre-War industrial production and post-War agricultural unemployment: in both cases quarterly volatility in the series interrupts runs of ones and threes, reducing the number of sharp TPs and the power of the test. Their second approach can be implemented directly in a MS-AR model.

2.2 Formal definition of asymmetries

For clarity, we formally define the concepts of steepness, deepness and sharpness (SDS).

Definition 1. *Deepness. Sichel (1993). The process $\{x_t\}$ is said to be* **non-deep (non-tall)** *iff x_t is not skewed:*

$$\mathsf{E}\left[(x_t - \bar{\mu})^3\right] = 0.$$

Analogously we can define steepness as skewness of the differences:

Definition 2. *Steepness. Sichel (1993). The process $\{x_t\}$ is said to be* **non-steep** *iff Δx_t is not skewed:*

$$\mathsf{E}\left[\Delta x_t^3\right] = 0.$$

The business cycle literature indicates the possibility of negative skewness of x_t and Δx_t — thus steep and deep contractions. The opposite case is of **tall** ($\mathsf{E}[(x_t - \mu)^3] > 0$) and **steep (**$\Delta x_t$ positively skewed) expansions.

Definition 3. *Sharpness. McQueen and Thorley (1993). The process $\{x_t\}$ is said to be* **non-sharp** *iff the transition probabilities to and from the two outer regimes are identical:*

$$p_{m1} = p_{mM} \text{ and } p_{1m} = p_{Mm}, \text{ for all } m \neq 1, M; \text{ and } p_{1M} = p_{M1}.$$

In a two-regime model, for example, non-sharpness implies that $p_{12} = p_{21}$. In a three-regime model, it requires $p_{13} = p_{31}$ and in addition $p_{12} = p_{32}$ and $p_{21} = p_{23}$. When $M = 4$ the following restrictions on the matrix of transition probabilities are required to hold for non-sharpness:

$$P = \begin{bmatrix} 1-a-b-c & a & b & c \\ d & * & * & d \\ e & * & * & e \\ c & a & b & 1-a-b-c \end{bmatrix} \qquad (3)$$

2.3 Asymmetries in MS-AR processes

We now present the restrictions on the parameter space of the MSM-AR model that correspond to the concepts of asymmetry. Proofs of these propositions are confined to an appendix. While the restrictions implied by sharpness follow immediately, testing for deepness and steepness is less obvious.

According to definition 1, deepness implies skewness. Using the properties of the MS-AR defined in equations (1) and (2), the following necessary and sufficient moment condition results:

Proposition 1. *An MSM(M)-AR(p) process is* **non-deep** *iff*

$$\sum_{m=1}^{M}\bar{\xi}_m\mu_m^{*3} = \sum_{m=1}^{M-1}\bar{\xi}_m\mu_m^{*3} + \left(1-\sum_{m=1}^{M-1}\bar{\xi}_m\right)\mu_M^{*3} = 0 \tag{4}$$

with $\mu_m^* = \mu_m - \mu_x = \sum_{i\neq m}(\mu_m-\mu_i)\bar{\xi}_i$, where $\bar{\xi}_m$ is the unconditional probability of regime m, and μ_x is the unconditional mean of x_t.

The expression (4) is a complicated third-order polynomial in the regime-dependent parameters of the process $\mu_1,\dots,\mu_M$ and the unconditional regime probabilities $\bar{\xi}_1,\dots,\bar{\xi}_{M-1}$, which are non-linear functions of the transition parameters p_{ij}. Equation (4) is derived from the condition that the kth moment of μ_t (with $k=3$) equals zero, where μ_t is the Markov chain component of the process (see the appendix):

$$\begin{aligned}
\mathsf{E}\left[\mu_t^k\right] &= \sum_{m=1}^{M}\bar{\xi}_m\left(\mu_m-\sum_{i=1}^{M}\bar{\xi}_i\mu_i\right)^k = \sum_{m=1}^{M}\bar{\xi}_m\left[\mu_m-\mu_M-\sum_{i=1}^{M-1}(\mu_i-\mu_M)\,\bar{\xi}_i\right]^k \\
&= \sum_{m=1}^{M-1}\bar{\xi}_m\left[(\mu_m-\mu_M)-\sum_{i=1}^{M-1}(\mu_i-\mu_M)\,\bar{\xi}_i\right]^k + \left(1-\sum_{m=1}^{M-1}\bar{\xi}_m\right)\left[-\sum_{i=1}^{M-1}(\mu_i-\mu_M)\,\bar{\xi}_i\right]^k
\end{aligned}$$

For $M=2$ the problem becomes more tractable analytically.

Example 1. *Consider the case of the two regime MSM(2)-AR(p) process. Invoking proposition 1, the skewness of the Markov chain is given by:*

$$\mathsf{E}\left[\mu_t^3\right] = \sum_{m=1}^{2}\bar{\xi}_m\mu_m^{*3} = \bar{\xi}_1\mu_1^{*3} + (1-\bar{\xi}_1)\mu_2^{*3}$$

where $\bar{\xi}_1 = p_{21}/(p_{12}+p_{21})$ is the unconditional probability of regime one, $\mu_1^* = \mu_1-\mu_x = (1-\bar{\xi}_1)(\mu_1-\mu_2)$ and $\mu_2^* = \mu_2-\mu_x = (-\bar{\xi}_1)(\mu_1-\mu_2)$. Substituting for $\bar{\xi}_1$, μ_1^* and μ_2^* we obtain:

$$\mathsf{E}\left[\mu_t^3\right] = \bar{\xi}_1(1-\bar{\xi}_1)\left[1-2\bar{\xi}_1\right](\mu_1-\mu_2)^3 .$$

As the Markov-switching model implies that $\mu_1\neq\mu_2$ and $\bar{\xi}_1\in(0,1)$, it is apparent that non-deepness, $\mathsf{E}[\mu_t^3]=0$, requires that $\bar{\xi}_1=0.5$. Hence the matrix of transition probabilities must be symmetric, $p_{12}=p_{21}$. This also implies that the regime-conditional means μ_1 and μ_2 are equidistant to the unconditional mean μ_x.

Hence, in the case of two regimes non-deepness can be tested by testing the hypothesis $p_{12}=p_{21}$. This is equivalent to the test of non-sharpness. For processes with $M>2$ we propose to test for non-deepness based on the μ_m^* conditional on μ_x and the $\bar{\xi}_m$.

Example 2. *Consider now an MSM(3)-AR(p) process. Again, by invoking proposition 1, the skewness of the Markov chain μ_t is given by:*

$$\mathsf{E}\left[\mu_t^3\right] = \sum_{m=1}^{3}\bar{\xi}_m\mu_m^{*3} = \bar{\xi}_1\mu_1^{*3} + \bar{\xi}_2\mu_2^{*3} + (1-\bar{\xi}_1-\bar{\xi}_2)\mu_3^{*3}$$

where $\mu_m^* = \mu_m - \mu_x = \mu_m - \bar{\xi}_1\mu_1 - \bar{\xi}_2\mu_2 - (1-\bar{\xi}_1-\bar{\xi}_2)\mu_3 = \sum_{i\neq m}\bar{\xi}_i(\mu_m - \mu_i)$. *Thus:*

$$\mathsf{E}\left[\mu_t^3\right] = \sum_{m=1}^{3}\bar{\xi}_m\left[\sum_{i\neq m}\bar{\xi}_i(\mu_m-\mu_i)\right]^3.$$

Non-deepness, $\mathsf{E}\left[\mu_t^3\right] = 0$*, requires that:*

$$\mu_3^{*3} = \frac{\bar{\xi}_1}{(1-\bar{\xi}_1-\bar{\xi}_2)}\mu_1^{*3} + \frac{\bar{\xi}_2}{(1-\bar{\xi}_1-\bar{\xi}_2)}\mu_2^{*3}.$$

We now derive conditions for the presence of steepness which is based on the skewness of the differenced series.

Proposition 2. *An MSM(M)-AR(p) process is* **non-steep** *if the size of the jumps,* $\mu_j - \mu_i$*, satisfies the following condition:*

$$\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left(\bar{\xi}_i p_{ij} - \bar{\xi}_j p_{ji}\right)\left[\mu_j - \mu_i\right]^3 = 0. \tag{5}$$

Symmetry of the matrix of transition parameters (which is stronger than the definition of sharpness) is sufficient but not necessary for non-steepness. A proof of this proposition appears in the appendix.

In contrast to deepness, the condition for steepness depends not only on the ergodic probabilities, $\bar{\xi}_j$, but also directly on the transition parameters.

Example 3. *In an MSM(2)-AR(p) process, condition (5) gives:*

$$\mathsf{E}\left[\Delta\mu_t^3\right] = \left(\bar{\xi}_1 p_{12} - \bar{\xi}_2 p_{21}\right)\left[\mu_2-\mu_1\right]^3 = 0.$$

Example 4. *For an MSM(3)-AR(p) process we get:*

$$\begin{aligned}\mathsf{E}\left[\Delta\mu_t^3\right] &= \sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\bar{\xi}_i p_{ij} - \bar{\xi}_j p_{ji}\right)\left[\mu_j-\mu_i\right]^3 \\ &= \left(\bar{\xi}_1 p_{12} - \bar{\xi}_2 p_{21}\right)\left[\mu_2-\mu_1\right]^3 + \left(\bar{\xi}_1 p_{13} - \bar{\xi}_3 p_{31}\right)\left[\mu_3-\mu_1\right]^3 + \left(\bar{\xi}_2 p_{23} - \bar{\xi}_3 p_{32}\right)\left[\mu_3-\mu_2\right]^3\end{aligned}$$

While this is a complicated expression, the concept of steepness can be made operational by using the sufficient condition, that is, the symmetry of the matrix of transition parameters, which implies non-steepness. As noted above, this is stronger than the property of non-sharpness.

We close this section with a corollary characterizing the two-regime MS-AR model, which shows the impossibility of the MS(2)-AR exhibiting steepness, and the equivalence of the concepts of deepness and sharpness.

Corollary 1. *A two-regime Markov-switching model is always non-steep. Non-sharpness implies non-deepness and vice versa.*

Non-steepness is evident from **Example 3**. Since $\bar{\xi}_1/\bar{\xi}_2 = p_{21}/p_{12}$, we have that $\bar{\xi}_1 p_{12} - \bar{\xi}_2 p_{21} = 0$ and hence $\mathsf{E}[\Delta\mu_t^3] = 0$. Further, non-sharpness (symmetric transition probabilities, $p_{12} = p_{21}$) implies non-deepness, $\mathsf{E}[(x_t - \mu_x)^3] = 0$, and *vice versa*. In particular, both concepts imply that the regime-conditional means μ_1 and μ_2 are equidistant to the unconditional mean μ_x.

3 Parametric tests based on the MS-AR model

Testing the MS-AR model against a linear null (or three regimes versus two) is complicated due to the presence of unidentified nuisance parameters under the null of linearity (that is, the transition probabilities) and because the scores associated with parameters of interest under the alternative may be identically zero under the null. These issues have been looked at by a number of authors (see e.g., Hansen, 1992, 1996), but are not of direct interest to us here, because the number of regimes remains unchanged under all three asymmetry hypotheses, so that standard asymptotics can be invoked. But we note that in practice they may complicate the identification of the appropriate model on which to carry out the asymmetry tests.

Wald tests of the asymmetry hypotheses are computationally attractive, since the model does not have to be estimated under the null. In general terms, we consider Wald (W) tests of the hypothesis:

$$H_0 : \phi(\lambda) = \mathbf{0} \quad \text{vs.} \quad H_1 : \phi(\lambda) \neq \mathbf{0},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^r$ is a continuously differentiable function with rank r, $r = \operatorname{rk}\left(\frac{\partial \phi(\lambda)}{\partial \lambda'}\right) \leq \dim \lambda$. As the p_{ij} are restricted to the $[0,1]$ interval, the tests are formulated on the logits $\pi_{ij} = \log\left(\frac{p_{ij}}{1-p_{ij}}\right)$ which avoids problems if one or more of the p_{ij} is close to the border. It is worth noting that if $\frac{1}{T}\left(\tilde{\pi}_{ij} - \pi_{ij}\right) \xrightarrow{d} \mathsf{N}(0, \sigma^2_{\pi_{ij}})$, then $\frac{1}{T}\left(\tilde{p}_{ij} - p_{ij}\right) \xrightarrow{d} \mathsf{N}(0, p_{ij}^2(1-p_{ij})^2\sigma^2_{\pi_{ij}})$ as $p_{ij} = \dfrac{e^{\pi_{ij}}}{1+e^{\pi_{ij}}}$. If one of the transition parameters is estimated to lie on the border, $p_{ij} \in \{0,1\}$, then the parameter is taken as being fixed and eliminated from the parameter vector λ.

Let $\tilde{\lambda}$ denote the unconstrained MLE of $\lambda = (\mu_1, \ldots, \mu_M; \alpha_1, \ldots, \alpha_p, \sigma^2; \boldsymbol{\pi})$, and $\hat{\lambda}$ the restricted MLE under the null. Then the Wald test statistic W is based on the unconstrained estimator $\tilde{\lambda}$, which is asymptotically normal:

$$\sqrt{T}\left(\tilde{\lambda} - \lambda\right) \xrightarrow{d} \mathsf{N}\left(\mathbf{0}, \Sigma_{\tilde{\lambda}}\right),$$

where, for the MLE, $\Sigma_{\tilde{\lambda}} = \Im_a^{-1}$ is the inverse of the asymptotic information matrix. This can be calculated numerically. It follows that $\phi(\tilde{\lambda})$ is also normal for large samples:

$$\sqrt{T}[\phi(\tilde{\lambda}) - \phi(\lambda)] \xrightarrow{d} \mathsf{N}\left(\mathbf{0}, \left.\frac{\partial \phi(\lambda)}{\partial \lambda'}\right|_{\tilde{\lambda}} \Sigma_{\tilde{\lambda}} \left.\frac{\partial \phi(\lambda)'}{\partial \lambda}\right|_{\tilde{\lambda}}\right).$$

Thus, if $H_0 : \phi(\lambda) = \mathbf{0}$ is true and the variance–covariance matrix is invertible,

$$T\phi(\tilde{\lambda})' \left[\left.\frac{\partial \phi(\lambda)}{\partial \lambda'}\right|_{\tilde{\lambda}} \tilde{\Sigma}_{\tilde{\lambda}} \left.\frac{\partial \phi(\lambda)'}{\partial \lambda}\right|_{\tilde{\lambda}}\right]^{-1} \phi(\tilde{\lambda}) \xrightarrow{d} \chi^2(r),$$

where $\tilde{\Sigma}_{\tilde{\lambda}}$ is a consistent estimator of $\Sigma_{\tilde{\lambda}}$.

3.1 Deepness

The Wald test for the null of non-deepness is based on:

$$\phi_D(\boldsymbol{\lambda}) = \phi_D(\boldsymbol{\pi}, \boldsymbol{\mu}; \cdot) := \sum_{m=1}^{M} \bar{\xi}_m (\mu_m - \mu_x)^3$$

where $\bar{\xi}_m(\boldsymbol{\pi})$ is the ergodic probability of regime m and $\mu_x(\boldsymbol{\pi}, \boldsymbol{\mu}) = \sum_{m=1}^{M} \bar{\xi}_m \mu_m$ is the unconditional mean of x_t. As $\frac{\partial \phi}{\partial \lambda_i} = 0$ for $\lambda_i \in \{\alpha_1, \ldots, \alpha_p, \sigma^2\}$ the Wald test statistic for non-deepness is given by:

$$T\phi_D(\boldsymbol{\lambda})' \left[\begin{bmatrix} \frac{\partial \phi}{\partial \boldsymbol{\mu}'} & \frac{\partial \phi}{\partial \boldsymbol{\pi}'} \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_{\boldsymbol{\mu}} & \tilde{\Sigma}_{\boldsymbol{\mu}\boldsymbol{\pi}} \\ \tilde{\Sigma}_{\boldsymbol{\pi}\boldsymbol{\mu}} & \tilde{\Sigma}_{\boldsymbol{\pi}} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial \boldsymbol{\mu}} \\ \frac{\partial \phi}{\partial \boldsymbol{\pi}} \end{bmatrix}\right]^{-1} \phi_D(\boldsymbol{\lambda}), \tag{6}$$

Example 5. *Thus for $M = 2$, the null of non-deepness is based on*

$$\begin{aligned}\phi(\pi_{12}, \pi_{21}, \mu_1, \mu_2) &= \phi(\xi_1(\pi_{12}, \pi_{21}), \mu_1, \mu_2) \\ &= \xi_1 \left(\mu_1 - \xi_1 \mu_1 - (1-\xi_1)\mu_2\right)^3 + (1-\xi_1)\left(\mu_2 - \xi_1 \mu_1 - (1-\xi_1)\mu_2\right)^3\end{aligned}$$

Differentiating w.r.t. ξ_1, μ_1, μ_2 gives:

$$\begin{aligned}\frac{\partial\phi}{\partial\xi_1} &= -(\mu_2-\mu_1)^3 (1 - 6\,\xi_1(1-\xi_1)) \\ \frac{\partial\phi}{\partial\mu_1} &= -3\,\xi_1\,(\mu_2-\mu_1)^2 (2\,\xi_1 - 1)(1-\xi_1) \\ \frac{\partial\phi}{\partial\mu_2} &= 3\,\xi_1\,(\mu_2-\mu_1)^2 (2\,\xi_1 - 1)(1-\xi_1)\end{aligned}$$

Using that $\xi_1 = \frac{p_{21}}{p_{12}+p_{21}}$ and $\pi_{ij} = \log(p_{ij}) - \log(1-p_{ij})$, we have

$$\begin{aligned}\frac{\partial\phi}{\partial\pi_{21}} &= \frac{\partial\phi}{\partial\xi_1}\frac{\partial\xi_1}{\partial\pi_{21}} = -(\mu_2-\mu_1)^3 (1 - 6\,\xi_1(1-\xi_1))\,\xi_1\,(1-\xi_1)(1-p_{21}) \\ \frac{\partial\phi}{\partial\pi_{12}} &= \frac{\partial\phi}{\partial\xi_1}\frac{\partial\xi_1}{\partial\pi_{12}} = (\mu_2-\mu_1)^3 (1 - 6\,\xi_1(1-\xi_1))\,\xi_1\,(1-\xi_1)\,(1-p_{12})\end{aligned}$$

Because this test is difficult to implement for $M > 2$, for models with more than two regimes we use a version of the deepness test with $\bar{\xi}_m$ and μ_x taken as fixed. This Wald test for the null of non-deepness is based on:

$$\phi_{D2}(\lambda) = \phi_{D2}(\boldsymbol{\mu};\cdot) := \sum_{m=1}^{M} \bar{\xi}_m(\mu_m - \mu_x)^3 \tag{7}$$

where $\frac{\partial\phi}{\partial\lambda_i} = 3\bar{\xi}_m(\mu_m-\mu_x)^2$ for $\lambda_i = \mu_m$, $m = 1,\ldots,M$, and $\frac{\partial\phi}{\partial\lambda_i} = 0$ for $\lambda_i \in [\alpha_1,\ldots,\alpha_p,\sigma^2;\boldsymbol{\pi}]$.

Example 6. *Thus for $M = 3$, the null of non-deepness is tested by $\phi_{D2}(\lambda) = 0$ so has the form:*

$$T\left[\sum_{m=1}^{3}\bar{\xi}_m(\widetilde{\mu}_m - \mu_x)^3\right]^2 \times$$
$$\left[\left[\begin{array}{ccc} 3\bar{\xi}_1(\widetilde{\mu}_1-\mu_x)^2 & 3\bar{\xi}_2(\widetilde{\mu}_2-\mu_x)^2 & 3\bar{\xi}_3(\widetilde{\mu}_3-\mu_x)^2\end{array}\right]\tilde{\Sigma}_{\tilde{\lambda}_D}\left[\begin{array}{c}3\bar{\xi}_1(\widetilde{\mu}_1-\mu_x)^2 \\ 3\bar{\xi}_2(\widetilde{\mu}_2-\mu_x)^2 \\ 3\bar{\xi}_3(\widetilde{\mu}_3-\mu_x)^2\end{array}\right]\right]^{-1} \xrightarrow{d} \chi^2(1),$$

where $\tilde{\lambda}_D = [\widetilde{\mu}_1\ \widetilde{\mu}_2\ \widetilde{\mu}_3]'$.

3.2 Steepness

A Wald test for the null of non-steepness can be based on:

$$\phi_S(\lambda) = \phi_S(\boldsymbol{\mu};\cdot) := \sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left(\bar{\xi}_i p_{ij} - \bar{\xi}_j p_{ji}\right)[\mu_j - \mu_i]^3$$

where the $\bar{\xi}_m$, p_{ij} and μ_y again are taken as fixed. Thus the test only concerns the vector of mean parameters:

$$\nabla\boldsymbol{\mu} = \begin{bmatrix} \mu_2 - \mu_1 \\ \vdots \\ \mu_M - \mu_1 \\ \vdots \\ \mu_M - \mu_{M-1} \end{bmatrix} = Q\boldsymbol{\mu}, \text{ with } Q = \frac{\partial \nabla\boldsymbol{\mu}}{\partial \boldsymbol{\mu}'} = \begin{bmatrix} -1 & 1 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 1 \\ 0 & -1 & 1 & \\ & -1 & 0 & 1 \\ & & \ddots & \\ 0 & & -1 & 1 \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \cdots & \mu_M \end{bmatrix}'.$$

Thus $\frac{\partial\phi}{\partial\boldsymbol{\mu}'} = \frac{\partial\phi}{\partial\nabla\boldsymbol{\mu}'}\frac{\partial\nabla\boldsymbol{\mu}}{\partial\mu'}$ with $\frac{\partial\phi}{\partial\nabla\boldsymbol{\mu}_m} = 3\left(\bar{\xi}_i p_{ij} - \bar{\xi}_j p_{ji}\right)\left[\mu_j - \mu_i\right]^2$ and $\frac{\partial\phi}{\partial\lambda_i} = 0$ otherwise. The Wald test statistic has the form:

$$\phi(\tilde{\lambda})'\left[\frac{\partial\phi}{\partial\nabla\boldsymbol{\mu}'}Q\left(\tfrac{1}{T}\tilde{\Sigma}_{\tilde{\mu}}\right)Q'\frac{\partial\phi'}{\partial\nabla\boldsymbol{\mu}}\right]^{-1}\phi(\tilde{\lambda}) \xrightarrow{d} \chi^2(1).$$

In the case of a three-state Markov chain, for example:

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and } \frac{\partial\phi'}{\partial\nabla\boldsymbol{\mu}} = \begin{bmatrix} 3\left(\bar{\xi}_1 p_{12} - \bar{\xi}_2 p_{21}\right)\left[\mu_2 - \mu_1\right]^2 \\ 3\left(\bar{\xi}_1 p_{13} - \bar{\xi}_3 p_{31}\right)\left[\mu_3 - \mu_1\right]^2 \\ 3\left(\bar{\xi}_2 p_{23} - \bar{\xi}_3 p_{32}\right)\left[\mu_3 - \mu_2\right]^2 \end{bmatrix}.$$

3.3 Sharpness

The null of non-sharpness can be expressed as:

$$\phi_{TP}(\lambda) = \phi_{TP}(\boldsymbol{\pi};\cdot) := \Phi\boldsymbol{\pi},$$

where the matrix Φ is defined such that $p_{m1} = p_{mM}$ and $p_{1m} = p_{Mm}$, for all $m \neq 1, M$, and $p_{1M} = p_{M1}$. Let the π_{ij} be collected to the matrix Π:

$$\Pi = \begin{bmatrix} \pi_{11} & \cdots & \pi_{M1} \\ \vdots & \ddots & \vdots \\ \pi_{1M} & \cdots & \pi_{MM} \end{bmatrix}$$

the matrix of logit transition probabilities. Then the vector $\boldsymbol{\pi}$ is given by $\mathrm{vecd}(\Pi)$, defined as $\mathrm{vec}(\Pi)$ with the diagonal elements π_{ij} excluded. In the case of a three-state Markov chain, for example, we have that:

$$\boldsymbol{\pi} = (\pi_{12}, \pi_{13}, \pi_{21}, \pi_{23}, \pi_{31}, \pi_{32})' \quad \text{and} \quad \Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}.$$

For linear restrictions the relevant Wald statistic can be expressed as:

$$W_{TP} = T(\Phi\tilde{\lambda} - \varphi)'\left[\Phi\ \tilde{\Sigma}_{\tilde{\lambda}}\ \Phi'\right]^{-1}(\Phi\tilde{\lambda} - \varphi).$$

Thus under the null of symmetric transition probabilities the Wald statistic has the form:

$$W_{TP} = \tilde{\pi}'\Phi'\left[\Phi\left(\tfrac{1}{T}\tilde{\Sigma}_{\tilde{\pi}}\right)\Phi'\right]^{-1}\Phi\tilde{\pi}.$$

4 Extensions to testing framework

In this section we outline three extensions to the basic framework for testing for asymmetries in MS-AR models. We deal with models in which the intercept, rather than the mean, switches between regimes; models which display regime-dependent heteroscedasticity; and consider multivariate settings.

4.1 Switching intercepts

The MSI(M)-AR(p) model is characterised by switching in the **I***ntercept*, rather than the **M***ean* (MSM-AR):

$$x_t = \mu(s_t) + \sum_{j=1}^{p} \alpha_j x_{t-j} + u_t, \tag{8}$$

where $u_t \sim \mathsf{NID}(0, \sigma^2)$ and $s_t \in \{1, \ldots, M\}$ is generated by a Markov chain.

As before for the MSM-AR process, the MSI(M)-AR(p) process can be written as the sum of two independent processes:

$$x_t - \mu_x = \mu_t + z_t \tag{9}$$

where $\mu_x = \alpha^{-1}(1) \sum_{m=1}^{M} \bar{\xi}_m \mu_m$, $\alpha(L) = 1 - \alpha_1 L - \ldots - \alpha_p L^p$, so $\alpha(1) = 1 - \sum_{i=1}^{p} \alpha_i$, and $\mathsf{E}[\mu_t - \mu_x] = 0$. $\{z_t\}$ is a gaussian process, $\alpha(L) z_t = u_t$, and $\mathsf{E}[z_t] = 0$, so that μ_t represents the contribution of the Markov chain, and $\mathsf{E}[\mu_t] = 0$. To derive an expression for μ_t, first rewrite (8) as:

$$\alpha(L)(x_t - \mu_x) = \nu_t + u_t. \tag{10}$$

where ν_t is defined as

$$\nu_t = \mu(s_t) - \bar{\mu} = \sum_{m=1}^{M} \mu_m \left(\xi_{mt} - \bar{\xi}_m\right),$$

and $\bar{\mu} = \alpha^{-1}(1) \mu_x$. In the case of a two-regime model we have that $\nu_t = (\mu_1 - \mu_2)\zeta_t$ with $\zeta_t = \xi_{1t} - \bar{\xi}_1$, which equals $1 - \bar{\xi}_1$ if the regime is 1 and $-\bar{\xi}_1$ otherwise. As (9) has to be equivalent to (8), the following expression for μ_t is obtained:

$$\mu_t = \alpha^{-1}(L) \nu_t. \tag{11}$$

Thus in contrast to the MSM-AR model, considered so far, where a shift in regime causes a once-and-for-all jump in the level of the observed time series, the MSI-AR model implies a smooth transition in the level of the process after a shift in regime.

Tests for asymmetries in MSI(M)-AR(p) models can be based on ν_t, which can be seen to be equivalent to the μ_t in MSM(M)-AR(p) models. Wald tests for deepness and steepness can be easily constructed by applying the procedures developed in section 3 to parametric tests for the skewness of ν_t and $\Delta\nu_t$, respectively.

A potential problem arises when the roots of $\alpha(L)$ are close to the unit circle, and in the extreme, for the first-order polynomial, $\alpha(L) = 1 - \alpha L$, $\alpha = 1$. Then $\nu_t = \Delta\mu_t$, and testing ν_t for deepness leads to conclusions for the deepness of Δx_t (rather than x_t). In other words applying the conditions derived for deepness in the MSM-AR model to ν_t provides a test of steepness of the MSI-AR model. In our examples, the roots of $\alpha(L)$ are a long way from unity as we are modelling first differences, and these exhibit little dependence relative to models in levels. Furthermore, the extreme case of a unit root implies the data have not been differenced a sufficient number of times prior to modelling.

4.2 Regime-dependent heteroscedasticity

In the original Hamilton model the variance of the disturbance term does not depend on the regime. However, regime-dependent heteroscedasticity is often manifest when the model is applied to financial data, and perhaps, albeit to a lesser extent, to macroeconomic data[5], so that the assumption that $u_t|s_t \sim \mathsf{NID}\left(0, \sigma^2\right)$ in (1) may need to be replaced by $u_t|s_t \sim \mathsf{NID}\left(0, \sigma^2\left(s_t\right)\right)$, where, in a 2-regime model, for example, $\sigma^2\left(s_t\right) = \sigma_1^2$ when $s_t = 1$ and $\sigma^2\left(s_t\right) = \sigma_2^2$ when $s_t = 2$. In such a model, asymmetries in the observed variable can arise either from asymmetries in the model's propagation mechanism, or from asymmetries in the innovations. Failure to allow for heteroscedasticity in the MS-AR model when it is present in the data may affect the properties of the SDS tests, as shown in section 5. Luukkonen and Teräsvirta (1991) test for cyclical asymmetry by testing whether a linear autoregressive model can be rejected in favour or a smooth transition auotoregression, and are concerned that asymmetry may result simply because of regime heteroscedasticity. Consequently, they also test for autoregressive conditional heteroscedasticity (ARCH). The SDS tests can be calculated within a model which explicitly allows for heteroscedasticity.

Our tests are designed to detect asymmetries in the model's propagation mechanism, while the non-parametric (NP) tests are unable to discriminate between the two sources of asymmetry.

4.3 Multivariate models

The tests we have outlined apply equally to a vector process with a single state variable. Models of this sort arise when the variables share a common cyclical component, as in the MS-AR model of post-war US employment and output of Krolzig and Toro (1998), or the dynamic-factor model with regime switching of Diebold and Rudebusch (1996). Note that the test of sharpness is intrinsically a system-based test as it evaluates the transition probabilies of the common latent regime variable. In contrast, the tests for deepness and steepness (for $M > 2$) focus on variable-specific asymmetries. It is therefore possible to test each variable for asymmetry.

In some instances it may be appropriate to allow more than one state variable, as in the bivariate model of stock returns and output growth of Hamilton and Lin (1996), where each variable responds to a specific state variable. The procedures developed in section 3 can then be applied in the same way, using the regime means, and ergodic and transition probabilities, relevant for each pairing of variable and state variable.

5 Properties of testing procedures

In this section, we explore by Monte Carlo (*i*) the size and power properties of our parametric tests relative to non-parametric tests, (*ii*) the impact on our testing procedures of ignoring regime-dependent heteroscedasticity, and (*iii*) their properties under model mis-specification, by which we mean applying the MS-AR model-based tests when the process was generated by an alternative model.

5.1 Size and power

Table 1 reports the empirical sizes and powers of the tests from a Monte Carlo (based on 1000 replications) in for two data generating processes based on:

[5] As an example, Koop, Pesaran and Potter (1996) find evidence of regime-dependent error variances in a non-linear model of output growth and unemployment rate changes.

$$x_t = \mu(s_t) + \epsilon_t, \quad \text{where } \epsilon_t \sim \mathsf{NID}(0, \sigma^2) \quad \text{and} \;\; s_t \in \{1, 2\}. \tag{12}$$

For the first process, labelled 'Symmetric MSM' in the table, $\mu_1 = -\mu_2 = -1.5$, $\sigma^2 = 1$, and $p_{11} = p_{22} = 0.85$, so that from the propositions stated in section 2.3, it is apparent that the values of $\mu(s_t)$ and $\bar{\xi}_1$ satisfy the conditions for μ_t, and thus x_t, to exhibit non-deepness (non-steepness is a property of the model, and non-deepness implies non-sharpness in this model). The parameter values for the second process, labelled 'Asymmetric MSM', are the same except that $p_{11} = 0.65$, and the conditions for non-deepness do not hold.

Because the data generating process consists of two-regimes, there are no entries for our steepness test (CK:Steepness) in table 1 – such processes can not exhibit steepness. The non-parametric test rejection frequencies for steepness (NP:Steepness) are close to the nominal sizes for all the data generating processes. The test for sharpness has size close to nominal even for $T = 100$, and power of nearly 60% at a 5% size. The parametric deepness test (CK:Deepness) is correctly-sized asymptotically and only a little too large for $T = 100$. Moreover, it has good power for the asymmetric process for $T = 100$.

5.2 Robustness under heteroscedasticity

Our tests are designed to detect asymmetries in the Markov chain component, μ_t, which we have termed first-moment asymmetry, while the non-parametric tests would be expected to reject the null of asymmetry in the presence of regime-dependent variances of the shocks ('heteroscedasticity'). The data generating processes chosen to explore these issues is a simple extension of (12) to allow for heteroscedasticity:

$$x_t = \mu(s_t) + \epsilon_t, \quad \text{where } \epsilon_t \sim \mathsf{NID}(0, \sigma^2(s_t)) \quad \text{and } s_t \in \{1, 2\}. \tag{13}$$

To illustrate, each panel of figure 2 plots the density of x_t generated by (13), and the density conditional on being in a regime. In the first panel, $\mu_1 = -\mu_2 = -1.5$, $\sigma_1^2 = \sigma_2^2 = 1$, and $prob(s_t = 1) = 0.5$. From the propositions stated in section 2.3, it is apparent that the values of $\mu(s_t)$ and $\bar{\xi}_1$ satisfy the conditions for μ_t, and thus x_t, to exhibit non-deepness (non-steepness is a property of the model). The density exhibits skewness in the top right panel because the condition for non-deepness is not satisfied by $\mu_1 = -\mu_2 = -1.5$ and $prob(s_t = 1) = 0.3$. In the bottom left panel the condition for non-deepness is satisfied, because $\mu_1 = -\mu_2 = -1.5$ and $prob(s_t = 1) = 0.5$, but nonetheless heteroscedasticity, $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$, induces skewness in x_t. The final panel is akin to the top right but now with heteroscedastic errors. In the bottom left panel, then, the contribution of the Markov process is symmetric but the unequal variances result in asymmetry in the marginal distribution of x_t.

Panel *A* of table 2 reports the properties of the testing procedures for heteroskedastic MS-AR processes, when the estimated model allows for regime-dependent error variances. The non-parametric test rejection frequencies for steepness (NP:Steepness) are close to the nominal sizes for all the data generating processes (DGPs), so the presence of heteroscedasticity does not inflate the size of the test. The test for sharpness has size close to nominal even for $T = 100$. The power approximately halves when the DGP is heteroscedastic (compare the 'Asymmetric MSMH' columns in panel *A* of table 2 with the 'Asymmetric MSMH' columns of table 1), but as the entries for $T = 1000$ confirm, this is a small-sample effect. The parametric deepness test (CK:Deepness) is correctly-sized asymptotically and only a little too large for $T = 100$. Moreover, it has good power for the asymmetric DGP for $T = 100$. By contrast, the non-parametric (NP:Deepness) test is less powerful, has a size that approaches one asymptotically for the symmetric MSMH, and a power approximately equal to size for a 5% test for the asymmetric MSMH when $T = 1000$.

The second aspect the Monte Carlo explores is the effect of using homoscedastic models when the DGP is heteroscedastic. Panel *B* of table 2 shows that the sizes of the parametric sharpness and deepness tests are a little inflated for $T = 100$, and are 25 to 30% for a nominal size of 5% for the large sample. Finally, the failure to model the heteroscedasticity reduces the power of the asymmetry tests for the DGP considered.

That regime-dependent variances can affect the skewness of the observed variables empirically is apparent from our results for the MSIH(3)-AR(4) model for 1948 – 90 (see table 6). The observed growth rates (x_t) display negative skewness (deepness of contractions) but the NonDeeepness test, though not significant, indicates positive skewness. In these models the third regime (high-growth) is partly associated with the Korea boom in 1951-1952, and induces positive skewness of the hidden Markov chain. However, the variance is much higher in regime 1 (recession), so that the observed variable is overall negatively skewed (but not significantly).

5.3 Robustness under model mis-specification

In the literature alternative regime-switching models have been proposed to explain business cycle phenomena. Because these models may in some instances provide a better characterisation of the data than the MS-AR model, a relevant question is how well the parametric tests perform when the model on which they are based (the MS-AR model) is not the model that generated the data. In this section we report Monte Carlo simulations designed to investigate the properties of the proposed testing procedures under model mis-specification. Results are recorded in table 3 for a self-exciting threshold autoregressive (SETAR) DGP and a smooth transition autoregressive (STAR) DGP.

In both models, the regime-generating process is not assumed to be exogenous, but is directly linked to a transition variable. For the SETAR model[6], the transition variable is a lag of the endogenous variable, say, x_{t-d}:

$$x_t = \left(\mu_1 + \sum_{i=1}^{p} \alpha_{1i} x_{t-i}\right)\left(1 - \mathsf{I}\left(x_{t-d}; c\right)\right) + \left(\mu_2 + \sum_{i=1}^{p} \alpha_{2i} x_{t-i}\right)\mathsf{I}\left(x_{t-d}; c\right) + \varepsilon_t \qquad (14)$$

where $\varepsilon_t \sim \mathsf{IID}(0, \sigma^2)$, $\mathsf{I}\left(x_{t-d}; c\right) = 1$ if $x_{t-d} > c$, and zero otherwise, and c is the threshold at which the switching between regimes occurs.

In the STAR model[7] the weight attached to the regimes depends on the realization of an exogenous or lagged endogenous variables z_t, so that the transition between regimes is 'smooth':

$$x_t = \left(\mu_1 + \sum_{i=1}^{p} \alpha_{1i} x_{t-i}\right)\left(1 - G\left(z_t; \gamma, c\right)\right) + \left(\mu_2 + \sum_{i=1}^{p} \alpha_{2i} x_{t-i}\right) G\left(z_t; \gamma, c\right) + \varepsilon_t \qquad (15)$$

where $\varepsilon_t \sim \mathsf{IID}(0, \sigma^2)$, and the transition function $G\left(z_t; \gamma, c\right)$ is a continuous function, usually bounded between 0 and 1. We consider the LSTAR model, where the transition function is given by:

$$G\left(z_t; \gamma, c\right) = \frac{1}{1 + \exp\left\{-\gamma(z_t - c)\right\}}.$$

γ is the smoothness parameter, and the transition variable z_t is taken to be the lagged endogenous variable ($z_t = x_{t-d}$). For $\gamma > 0$, as $z_t \to -\infty$, $G\left(\cdot\right) \to 0$, and for $z_t \to \infty$, $G\left(\cdot\right) \to 1$.

[6] A number of authors have estimated SETAR models for US output growth, including Tiao and Tsay (1994) and Potter (1995): Tong (1995) provides a statistical analysis of the model.

[7] STAR models have been popularised by the work of, e.g., Luukkonen *et al.* (1988) and Teräsvirta and Anderson (1992).

To simplify matters, in our simulations we set $\alpha_{1i} = \alpha_{2i} = 0$ for all i, $d = 1$, $\sigma^2 = 1$, $\mu_1 = -\mu_2 = -1.5$, and vary c to give symmetric and asymmetric models.

The results in table 3 demonstrate the excellent performance of the Markov-switching model-based sharpness test. Even for the small sample size of $T = 100$ it has approximately the correct size under model mis-specification. It also has good power in small samples for both the asymmetric SETAR and LSTAR DGPs. In contrast, the parametric and non-parametric non-deepness tests work less well in small samples, but for large samples the parametric non-deepness tests (CK:Deepness) is only slightly oversized, and clearly better behaved than the non-parametric one.

Overall, the Monte Carlo results support the use of the proposed tests for business cycle asymmetries: the tests (i) behave as expected for correctly specified Markov-switching models, they are (ii) robust against skewness due to heteroscedasticity and (iii) the sharpness test has been found to be a very reliable test for business cycle asymmetries even if the underlying DGP is different from the empirical model. The tests proposed in this paper enhance the role of the Markov-switching model as a flexible tool for empirical modelling.

6 Empirical illustrations

The SDS tests are illustrated on a number of data set. Specifically, we apply the parametric tests discussed in section 3 to the MS(2)-AR(4) model of output growth of Hamilton (1989) for his original sample period of 1953-1984, to the MS(3)-AR(4) model of Clements and Krolzig (1998) on more recent data, and to a variety of models of US investment and consumption growth. In each case, the outcomes of the tests for asymmetries are compared with non-parametric tests of skewness.

6.1 The Hamilton (1989) model of US output growth

MS-AR models have been used in contemporary empirical macroeconomics to capture certain features of the business cycle, but the formal testing of asymmetries has been largely confined to non-parametric approaches. The seminal paper by Hamilton (1989) fit a fourth-order autoregression ($p = 4$) to the quarterly percentage change in US real GNP, x_t, from 1953 to 1984:

$$x_t - \mu(s_t) = \alpha_1 (x_{t-1} - \mu(s_{t-1})) + \ldots + \alpha_4 (x_{t-p} - \mu(s_{t-4})) + \epsilon_t, \tag{16}$$

where $\epsilon_t \sim \mathsf{NID}(0, \sigma^2)$ and the conditional mean $\mu(s_t)$ switches between two states, 'expansion' and 'contraction':

$$\mu(s_t) = \begin{cases} \mu_1 < 0 & \text{if } s_t = 1 \text{ ('contraction' or 'recession')} \\ \mu_2 > 0 & \text{if } s_t = 2 \text{ ('expansion' or 'boom')} \end{cases}$$

with the variance of the disturbance term, $\sigma^2(s_t) = \sigma^2$, assumed the same in both regimes. This is an MS*M*ean model, with the autoregressive parameters and disturbances independent of the state s_t.

The maximization of the likelihood function of an MS-AR model entails an iterative estimation technique to obtain estimates of the parameters of the autoregression and the transition probabilities governing the Markov chain of the unobserved states: see Hamilton (1990) for an Expectation Maximization (EM) algorithm for this class of model, and Krolzig (1997) for an overview of alternative numerical techniques for the maximum likelihood estimation these of models.

The results of testing for asymmetry based on the original Hamilton model and data set (1952:2 – 1984:4) are recorded in Table 4. The non-parametric test for skewness indicates significant negative skewness in output growth (i.e., deepness of contractions) at the 5% level. Our parametric test of non-deepness also indicates negative skewness, but is only significant at the 20% level. There is evidence of sharpness at the 10% level with the probability of switching from contraction to expansion exceeding the probability of movement in the reverse direction.

6.2 The 3-regime heteroscedastic model of US output growth

Sichel (1994) argues that post-War business cycles typically consist of three phases: contraction, followed by high-growth recovery, and then a period of moderate growth. To capture this in a parametric model, we consider the three-state MS-AR model of Clements and Krolzig (1998)[8], where there is a shifting intercept term and a heteroscedastic error term (denoted as an MSIH(3)-AR(4) model — where the H flags the heteroscedastic error term, and 3 and 4 refer to the number of regimes and autoregressive lags, respectively):

$$x_t = \mu(s_t) + \sum_{k=1}^{4} \alpha_k x_{t-k} + \epsilon_t, \tag{17}$$

where $\epsilon_t \sim \mathsf{NID}(\sigma^2(s_t))$ and $s_t \in \{1, 2, 3\}$ is generated by a Markov chain.

Figure 3 and table 5 (reproduced from Clements and Krolzig, 1998) summarize the business-cycle characteristics of this model. The figure depicts the filtered and smoothed probabilities of the 'high growth' regime 3 and the contractionary regime 1 (the middle regime 2 probabilities are not shown). The expansion and contraction episodes produced by the three-regime model correspond fairly closely to the NBER classifications of business-cycle turning points. In contrast to the two-regime model, all three regimes are reasonably persistent.

While Hess and Iwata (1997b) find their three-state MS-AR model estimated for 1949-92 fails to generate contractions of sufficient duration or depth, their estimated p_{11} is only 0.1267, while the lowest value in the MSIH models recorded in table 5 is over 0.78, which directly translates into a longer duration of the recession regime, and so we conjecture that the MSIH model may not have this shortcoming.

The tests for asymmetries in MSIH(3)-AR(4) models are recorded in tables 6 and 7 for various historical periods. For the first sample period (1948 –90), the NP skewness and model-based tests both indicate steepness of expansions, with the MS-AR model test permitting rejection of the null at the 1% level. Moreover, there is clear evidence of asymmetric turning points (or sharpness), which results from a rejection of $p_{21} = p_{23}$, because moving from moderate to low growth is more likely than moving from moderate to high growth. The three-state model permits rejection of the non-sharpness hypotheses at a higher confidence level than the two-state model.

For the later sample period (shown in table 7), the MS-model test continues to reject non-steepness at the 5% level, in contrast to the NP test that now flags deepness of recesssions rather than steepness of expansions. The major change in inference using the parametric tests is that there is no evidence of sharpness in the later period.

6.3 Models of US investment and consumption growth

To further illustrate the method of testing for asymmetries, we apply the tests to US investment and consumption growth using a number of MS models. These models contain either two or three regimes, and either allow the error variance to depend upon the regime or restrict it to be heteroscedastic. In all cases we consider models without lags, so that MSI and MSM models are equivalent.

[8]A number of authors, including Boldin (1996) and Clements and Krolzig (1998), have found that the 2-regime MS-AR model does not yield a particularly good representation of the business cycle when fitted to periods outside that in Hamilton (1989). For example, Clements and Krolzig (1998) find an average duration of contraction (1) of 2–3 quarters for the period 1947–90, and of less than 2 quarters for 1959–96.

The first four panels of figure 4 depict the recession regime probabilities for investment growth (DI). The allocation of observations to the recession regime is more dependent on whether or not the errors are allowed to be heteroscadastic than on whether there are two or three regimes. The main difference between the two and three regime models with heteroscedastic errors is that there is some evidence of a recession in the investment series around 1990 in the former but not in the latter. Table 8 shows that the model-based steepness tests (CK NonSteepness) reject the null at the 5% level in both the homoscedastic and heteroscedastic three-regime models, and indicate steepness of expansions, while the non-parametric tests suggest 'tallness' of expansions.

The last four panels in figure 4 give the recession probabilities for consumption growth. Here the estimates of the 'recession' regime for the two and three regime heteroscedastic models are quite different, and from table 8 the homoscedastic model indicates tallness and steepness of expansions, in line with the non-parametric tests, while both features are absent in the heteroscedastic model.

These examples suggest a number of points. The results of testing for asymmetries based on parametric models may be sensitive to the model specification employed. Specifically, it is likely to matter whether the model allows for heteroscedastic errors. The findings here confirm the Monte Carlo results in section 5.2. The regime categorization in models that allow heteroscedastic disturbances will reflect shifts in both the mean and the variance of the series, and so will not necessarily coincide with that in homoscedastic models if, for example, the shifts in mean and variance are not in line. Thus, it is important to adequately capture the business-cycle features of the series: we argued in section 6.2, following Sichel (1994), that for modelling US output growth a three-regime model with heteroscedastic errors appears to be required. In the case of the investment and consumption series, a closer examination of the individual models would reveal which is the most appropriate: the results in table 8 are simply illustrative.

7 Conclusions

We have set out the parametric restrictions on MS-AR models for the series generated by those models to exhibit neither deepness, steepness or sharpness business-cycle asymmetries. For the popular two-state model first proposed by Hamilton (1989) we have shown that deepness implies sharpness and vice versa, and that the model (at least with gaussian disturbances) can not generate steepness. For three-state models, which arguably afford a better characterisation of the business cycle, the three concepts are distinct. We have shown how the parameter restrictions can be applied as Wald tests, and to illustrate, report the results of testing for asymmetries in Hamilton's original model of US output growth, and in two and three-state models US investment and consumption growth. The tests detect first-moment asymmetries, and are not affected by regime-dependent heteroscedasticity, provided this is modelled.

A comparison of the empirical results for our tests with the non-parametric outcomes suggests our tests have reasonable power to detect asymmetries. This was confirmed by a Monte Carlo study which showed that our tests have good size and power properties, and perform well relative to the non-parametric tests. The latter are adversely affected by regime-dependent heteroscedasticity, and can give misleading inferences concerning first-moment asymmetries. Moreover, our tests work reasonably well when the data are generated from other classes of regime-switching models.

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8 Appendix

Proposition 1. *An MSM(M)-AR(p) process is* **non-deep** *iff:*

$$\sum_{m=1}^{M}\bar{\xi}_m\mu_m^{*3}=\sum_{m=1}^{M-1}\bar{\xi}_m\mu_m^{*3}+\left(1-\sum_{m=1}^{M-1}\bar{\xi}_m\right)\mu_M^{*3}=0 \tag{18}$$

with $\mu_m^*=\mu_m-\mu_x=\sum_{i\neq m}(\mu_m-\mu_i)\bar{\xi}_i$ *and where* $\bar{\xi}_m$ *is the unconditional probability of regime* m*.*

Proof. MSM(M)-AR(p) processes can be rewritten as the sum of two independent processes: $x_t-\mu_x=\mu_t+z_t$. μ_x is the unconditional mean of x_t:

$$\mu_x=\mathsf{E}[x_t]=\sum_{m=1}^{M}\bar{\xi}_m\mu_m,$$

and both z_t and μ_t are zero mean, $\mathsf{E}[\mu_t]=\mathsf{E}[z_t]=0$. The process $z_t=\sum_{j=1}^{p}\alpha_j z_{t-j}+u_t$ is gaussian and hence symmetric. The component, μ_t, is potentially asymmetric, and represents the contribution of the Markov chain:

$$\mu_t=\sum_{m=1}^{M}\xi_{mt}(\mu_m-\mu_x)=\sum_{m=1}^{M}\xi_{mt}\mu_m^*=\mu_M^*+\sum_{m=1}^{M-1}\xi_{mt}(\mu_m^*-\mu_M^*)$$

with $\mu_m^*=\mu_m-\mu_x$ and $\xi_{mt}=1$ if the regime is m at period t, and is 0 otherwise.

Thus the k-th moment of μ_t is given by:

$$\mathsf{E}\left[\mu_t^k\right]=\sum_{m=1}^{M}\bar{\xi}_m(\mu_m-\mu_x)^k=\sum_{m=1}^{M}\bar{\xi}_m\mu_m^{*k},$$

where $\overline{\xi}_m=\mathsf{E}[\xi_{mt}]$, and $\mathsf{E}\left[\xi_{mt}^k\right]=\overline{\xi}_m\ \forall k$.

Using the adding-up restriction, $\sum_{m=1}^{M}\bar{\xi}_m=1$, we have:

$$\mathsf{E}\left[\mu_t^k\right]=\sum_{m=1}^{M-1}\bar{\xi}_m\mu_m^{*k}+\left(1-\sum_{m=1}^{M-1}\bar{\xi}_m\right)\mu_M^{*k}=\mu_M^{*k}+\sum_{m=1}^{M-1}\bar{\xi}_m\left(\mu_m^{*k}-\mu_M^{*k}\right).$$

■

Proposition 2. *An MSM(M)-AR(p) process is* **non-steep** *if the size of the jumps,* $\mu_j-\mu_i$*, satisfies the following condition:*

$$\sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left(\bar{\xi}_i p_{ij}-\bar{\xi}_j p_{ji}\right)[\mu_j-\mu_i]^3=0. \tag{19}$$

Proof. Write $\mu_t=\mathbf{M}\xi_t$, where $\mathbf{M}=[\mu_1\cdots\mu_M]$ and $\xi_t=[\xi_{1t}\cdots\xi_{Mt}]'$. $\xi_{mt}=1$ if the period t regime is m, and zero otherwise. Then $\Delta\mu_t=\mu_t-\mu_{t-1}=\mathbf{M}\boldsymbol{\Delta}\xi_t=\mathbf{M}\xi_t-\mathbf{M}\xi_{t-1}$. Clearly, $\mathsf{E}[\Delta\mu_t]=0$. We now introduce $\nabla\mathbf{M}=[\mathbf{M}'\otimes 1_M-1_M\otimes\mathbf{M}']'$ and $\xi_t^{(2)}=\xi_t\otimes\xi_{t-1}$, such that:

$$\Delta\mu_t=\nabla\mathbf{M}\xi_t^{(2)}=\sum_{i=1}^{M}\sum_{j=1}^{M}\xi_{i,t-1}\xi_{j,t}[\mu_j-\mu_i].$$

Using that $\mu_j-\mu_i=0$ for $i=j$, we can simplify to:

$$\Delta\mu_t=\sum_{i=1}^{M}\sum_{j\neq i}\xi_{i,t-1}\xi_{j,t}[\mu_j-\mu_i].$$

The third moment is then given by:

$$\begin{aligned}
\mathsf{E}\left[\Delta\mu_t^3\right] &= \sum_{i=1}^{M}\sum_{j\neq i}\bar{\xi}_i p_{ij}\left[\mu_j-\mu_i\right]^3 \\
&= \sum_{i=1}^{M-1}\sum_{j=i+1}^{M}\left\{\left(\bar{\xi}_i p_{ij}-\bar{\xi}_j p_{ji}\right)\left[\mu_j-\mu_i\right]^3\right\},
\end{aligned}$$

where the last line uses $[\mu_j - \mu_i]^3 = -[\mu_i - \mu_j]^3$.

Symmetry of the matrix of transition parameters (which is stronger than the definition of sharpness) is sufficient for non-steepness as it implies that, for all $i, j = 1, \ldots, M$, we have that $\bar{\xi}_i p_{ij} - \bar{\xi}_j p_{ji} = 0$.

■

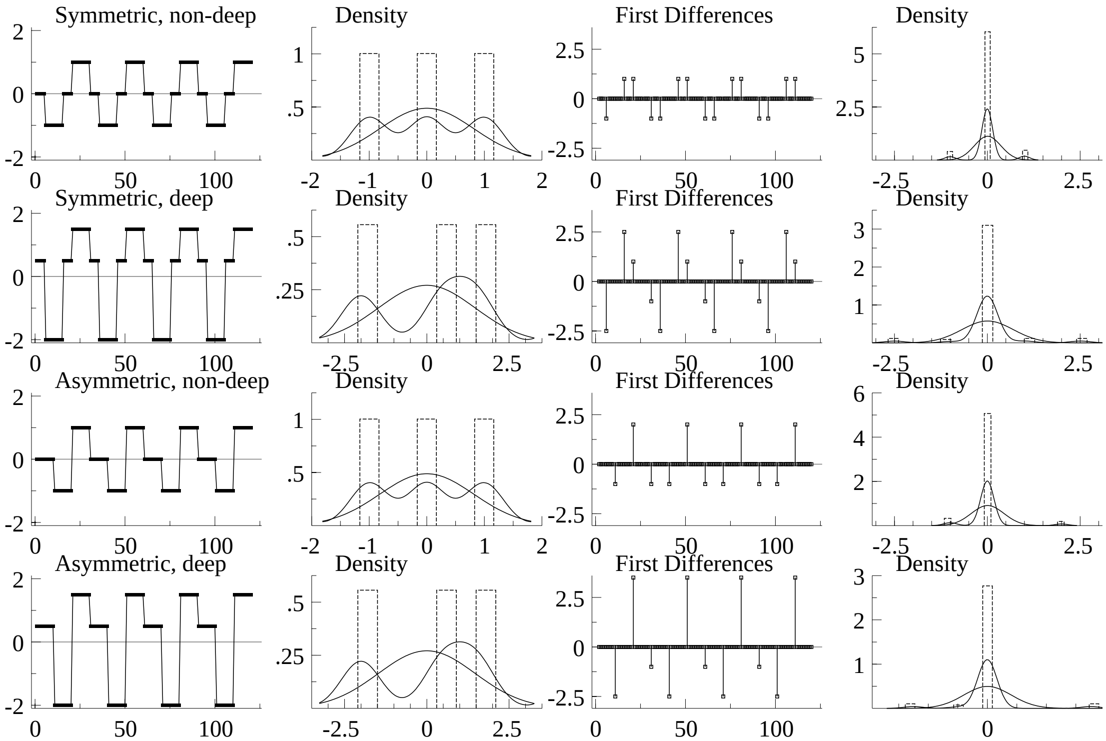

Figure 1 Schematic of Business Cycle Asymmetries. The figures in each row depict: (*i*) time paths for detrended output x_t, (*ii*) histograms and densities for x_t, (*iii*) time paths for Δx_t, (*iv*) histograms and densities for Δx_t. Gaussian curves are super-imposed on the densities. The top row corresponds to the non-deep and non-steep case. Deepness of contractions (row 2) shows up in negative skewness in x_t, and steepness of expansions (row 3) in positive skewness of Δx_t. Row 4 shows the two together.

Table 1 Empirical Size and Power.

Homoscedastic MS-AR DGPs							
		Symmetric MSM			Asymmetric MSM		
	α	0.10	0.05	0.01	0.10	0.05	0.01
$T = 100$	CK:Sharpness	0.107	0.055	0.012	0.696	0.585	0.280
	NP:Deepness	0.162	0.098	0.029	0.587	0.440	0.195
	CK:Deepness	0.212	0.153	0.084	0.762	0.691	0.525
	NP:Steepness	0.138	0.081	0.020	0.112	0.061	0.021
	CK:Steepness	0	0	0	0	0	0
$T = 1000$	CK:Sharpness	0.087	0.045	0.010	1.000	1.000	1.000
	NP:Deepness	0.154	0.088	0.027	1.000	0.999	0.999
	CK:Deepness	0.098	0.051	0.016	1.000	1.000	1.000
	NP:Steepness	0.167	0.100	0.030	0.116	0.063	0.015
	CK:Steepness	0	0	0	0	0	0

Design of the Monte Carlo:

Symmetric MSM : $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $p_{11} = p_{22} = 0.85$;

Asymmetric MSM : $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $p_{11} = 0.65$, $p_{22} = 0.85$.

According to the propositions in section 2.3 the processes have the following properties:

Symmetric MSM : non-sharp, non-deep, non-steep.

Asymmetric MSM : non-steep.

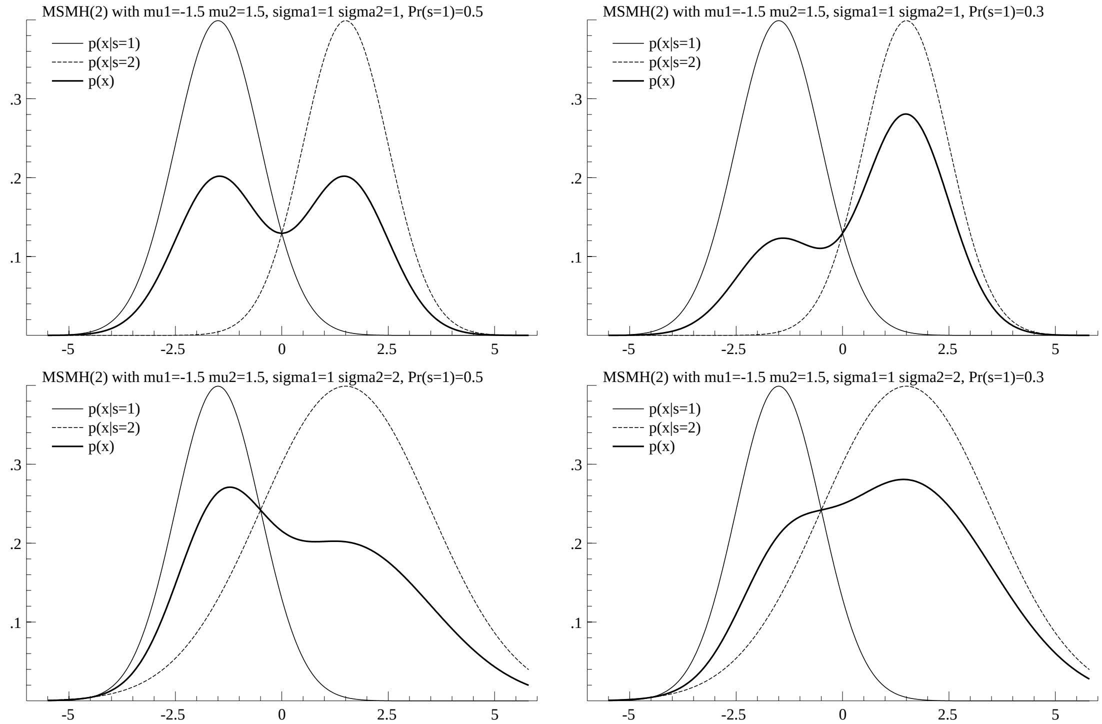

Figure 2 Asymmetries due to regime-dependent heteroscedasticity. The figure depicts densities (conditional on the regime, and unconditional) of x_t constructed for two-regime MS-AR models. In the top left panel, the MS-AR model satisfies the conditions for a symmetric propagation mechanism, and the process is homoscedastic. The unconditional density (the solid line) is symmetric. The top right panel is drawn for a homoscedastic process with an asymmetric propagation mechanism. The unconditional density is skewed. The process in the bottom left panel has a symmetric propagation mechanism but has regime-dependent heteroscedasticity. The unconditional density is skewed. The bottom right has asymmetric innovations and an asymmetric propagation mechanism.

Table 2 Empirical Size and Power: Effects of Heteroscedasticity.

Heteroskedastic MS-AR DGPs													
		A. MS-AR models match the MS-AR DGPs						B. Homoscedastic MS-AR models					
		Symmetric MSMH			Asymmetric MSMH			Symmetric MSMH			Asymmetric MSMH		
	α	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
$T = 100$	CK:Sharpness	0.099	0.051	0.013	0.411	0.271	0.099	0.150	0.082	0.028	0.238	0.136	0.021
	NP:Deepness	0.361	0.249	0.115	0.043	0.020	0.002	0.361	0.249	0.115	0.043	0.020	0.002
	CK:Deepness	0.173	0.120	0.059	0.493	0.417	0.255	0.222	0.171	0.093	0.273	0.184	0.044
	NP:Steepness	0.119	0.072	0.024	0.096	0.044	0.014	0.119	0.072	0.024	0.096	0.044	0.014
	CK:Steepness	0	0	0	0	0	0	0	0	0	0	0	0
$T = 1000$	CK:Sharpness	0.073	0.044	0.010	1.000	1.000	0.995	0.394	0.263	0.091	0.983	0.965	0.879
	NP:Deepness	0.986	0.974	0.914	0.112	0.053	0.006	0.986	0.974	0.914	0.112	0.053	0.006
	CK:Deepness	0.078	0.056	0.013	1.000	1.000	0.997	0.408	0.294	0.118	0.984	0.967	0.893
	NP:Steepness	0.141	0.085	0.017	0.082	0.043	0.006	0.141	0.085	0.017	0.082	0.043	0.006
	CK:Steepness	0	0	0	0	0	0	0	0	0	0	0	0

Design of the Monte Carlo:

Symmetric MSMH: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = 1, \sigma_2 = 2$, $p_{11} = p_{22} = 0.85$;

Asymmetric MSMH: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = 1, \sigma_2 = 2$, $p_{11} = 0.65$, $p_{22} = 0.85$.

According to the propositions in section 2.3 the processes have the following properties:

Symmetric MSMH: non-sharp, non-deep, non-steep.

Asymmetric MSMH: non-steep.

Table 3 Empirical Size and Power: SETAR and LSTAR DGPs .

MS-AR model applied to data from SETAR and LSTAR models													
		Symmetric SETAR			Asymmetric SETAR			Symmetric LSTAR			Asymmetric LSTAR		
	α	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
$T = 100$	CK:Sharpness	0.116	0.053	0.010	0.529	0.429	0.161	0.114	0.048	0.010	0.505	0.404	0.158
	NP:Deepness	0.398	0.313	0.169	0.485	0.428	0.314	0.390	0.296	0.158	0.383	0.313	0.211
	CK:Deepness	0.306	0.261	0.193	0.212	0.137	0.074	0.324	0.272	0.192	0.192	0.113	0.047
	NP:Steepness	0.057	0.025	0.007	0.062	0.027	0.005	0.067	0.035	0.008	0.081	0.038	0.008
	CK:Steepness	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$T = 1000$	CK:Sharpness	0.077	0.034	0.004	0.999	0.998	0.998	0.062	0.026	0.003	0.993	0.992	0.992
	NP:Deepness	0.410	0.333	0.186	0.997	0.993	0.987	0.384	0.295	0.183	0.972	0.958	0.935
	CK:Deepness	0.113	0.071	0.020	0.974	0.926	0.644	0.096	0.054	0.017	0.943	0.854	0.475
	NP:Steepness	0.073	0.033	0.010	0.105	0.046	0.010	0.089	0.045	0.016	0.287	0.160	0.045
	CK:Steepness	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Design of the Monte Carlo:

Symmetric SETAR: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $d = 1$, $c = 0$.

Asymmetric SETAR: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $d = 1$, $c = -0.75$.

Symmetric LSTAR: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $d = 1$, $c = 0$, $\gamma = 10$.

Asymmetric LSTAR: $\mu_1 = -1.5$, $\mu_2 = 1.5$, $\sigma_1 = \sigma_2 = 1$, $d = 1$, $c = -0.75$, $\gamma = 10$.

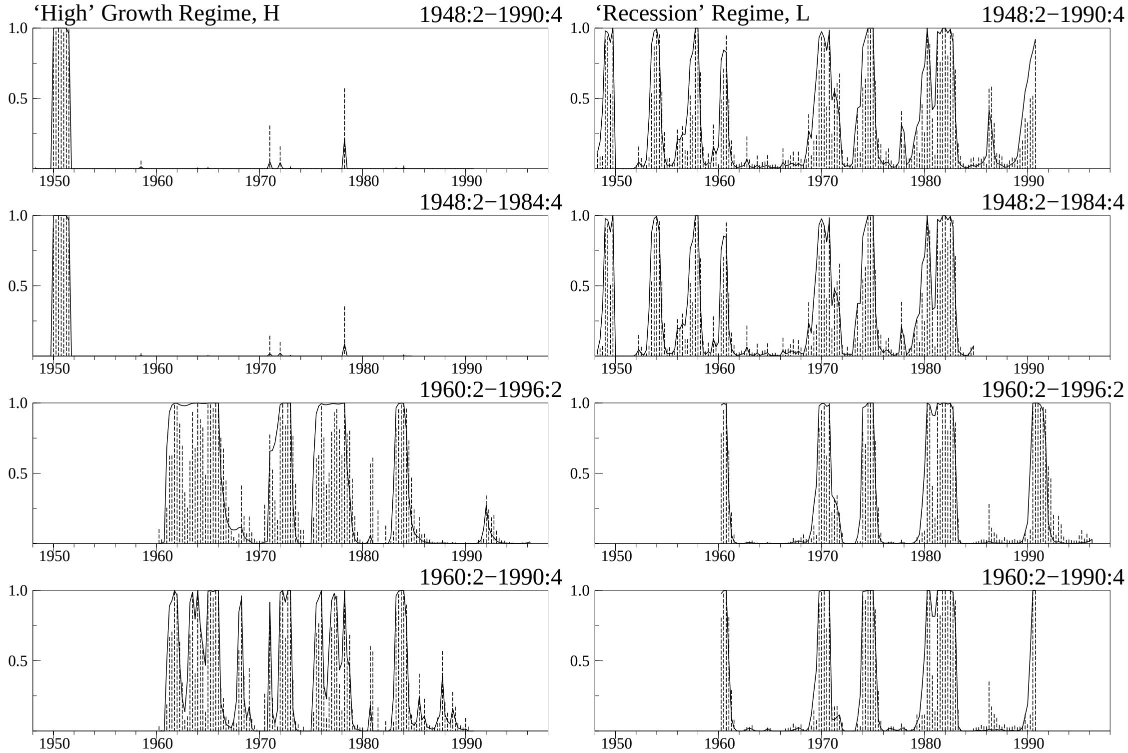

Figure 3 MSIH(3)-AR(4) model of US output growth: Smoothed and filtered probabilities of the high growth H and 'recession' L regime.

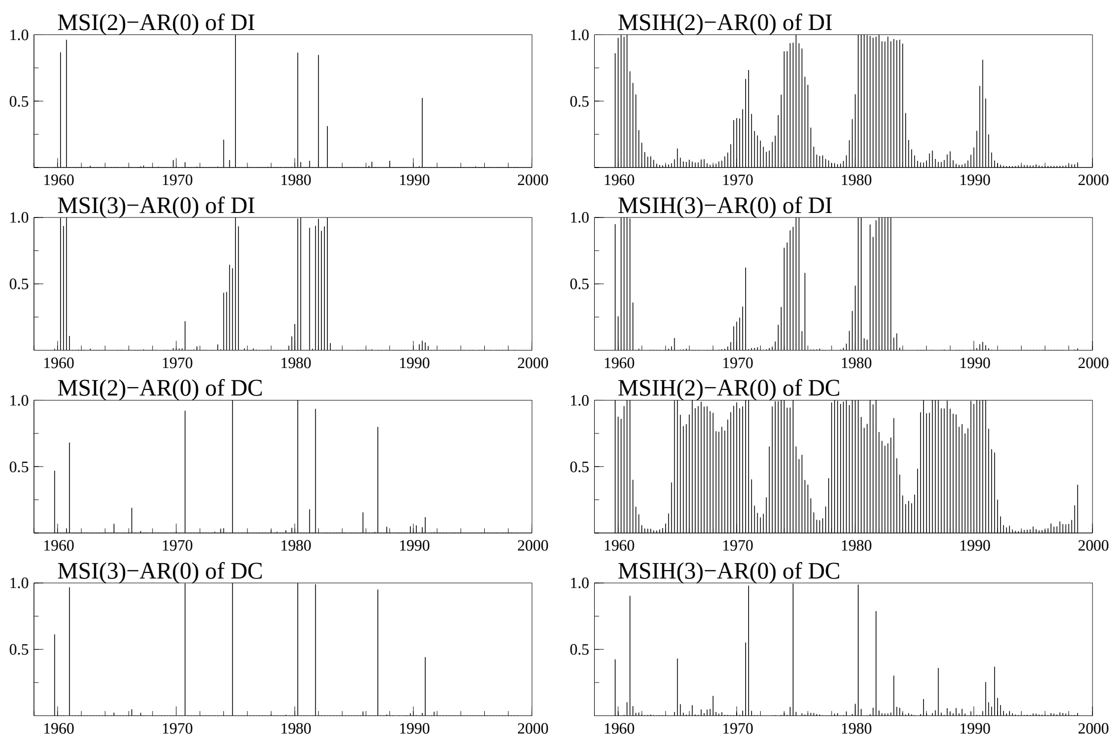

Figure 4 MS models of US investment and consumption growth: the estimated probabilities with which each observation falls in the recession regime for a variety of models.

Table 4 Tests for asymmetries using the MSM(2)-AR(4)model of US output growth, 1952:2 – 1984:4.

Test	$\phi(\lambda)$	Test statistic value	p-value
NonSharpness test:		2.7714	[0.0906] *
NonDeepness test:	-0.2971	1.7354	[0.1877]
Skewness (x):	-0.4900	5.2025	[0.0226] **
Skewness (Δx):	-0.0015	0.0000	[0.9945]

The NP and SD(S) test statistics are χ^2 with one degree of freedom under the null of symmetry. * indicates significance at 10% level and ** significance at the 5% level. A positive (negative) value of $\phi(\lambda)$ flags positive (negative) skewness.

Note: Non-steepness is a property of the 2-regime model: see Corollary 1.

Table 5 MSIH(3)-AR(4) models of US output growth.

Sample	48:2-90:4	60:2-96:2
Mean μ_1	-0.081	-0.050
Mean μ_2	1.413	0.838
Mean μ_3	3.430	1.406
α_1	-0.102	0.016
α_2	0.109	0.022
α_3	-0.172	-0.100
α_4	-0.191	-0.098
σ_1^2	0.816	0.796
σ_3^2	0.018	0.406
p_{12}	0.187	0.021
p_{13}	0.021	0.128
p_{21}	0.093	0.075
p_{23}	0.000	0.000
p_{31}	0.000	0.000
p_{32}	0.162	0.091
p_1	0.298	0.231
p_2	0.663	0.447
p_3	0.038	0.322
Duration 1	4.814	6.745
Duration 2	10.701	13.089
Duration 3	6.145	10.946
Observations	171	145

μ_i, σ_2^i and p_i denote the intercept, disturbance variance, and ergodic probability of regime i. The α_j are the autoregressive parameters, which are constant across regimes, and the p_{ij} are the transition probabilities.

Table 6 Tests for asymmetries using the MSIH(3)-AR(4) model of US output growth, 1948:2 – 1990:4

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Test	$\phi(\lambda)$	Test statistic value	p-value
NonSharpness test:		31828.7577	[0.0000] **
$p_{12} = p_{32}$:		0.0225	[0.8808]
$p_{13} = p_{31}$:		0.0417	[0.8382]
$p_{21} = p_{23}$:		31701.5198	[0.0000] **
NonDeepness test:	0.1271	0.1820	[0.6696]
NonSteepness test:	0.1966	17.7305	[0.0000] **
Skewness (x):	-0.1587	0.7055	[0.4010]
Skewness (Δx):	0.3451	3.3736	[0.0662] *

The NP and SDS test statistics are χ^2 with one degree of freedom under the null of symmetry. * indicates significance at 10% level and ** significance at the 5% level. A positive (negative) value of $\phi(\lambda)$ flags positive (negative) skewness.

Note: As p_{31} and p_{23} are close to zero, the matrix of second derivatives used for the calculation of parameter covariance is singular and the generalized inverse has been used, which explains the magnitude of the test statistics for non-sharpness.

Table 7 Tests for asymmetries using the MSIH(3)-AR(4) model of US output growth, 1960:2 – 1996:2

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Test	$\phi(\lambda)$	Test statistic value	p-value
NonSharpness test:		0.7539	[0.8605]
$p_{12} = p_{32}$:		0.7271	[0.3938]
$p_{13} = p_{31}$:		0.0225	[0.8809]
$p_{21} = p_{23}$:		0.0045	[0.9462]
NonDeepness test:	-0.0839	0.4681	[0.4939]
NonSteepness test:	0.0649	4.2702	[0.0388] *
Skewness (x):	-0.6412	9.8659	[0.0017] **
Skewness (Δx):	0.1874	0.8426	[0.3587]

The NP and SDS test statistics are χ^2 with one degree of freedom under the null of symmetry. * indicates significance at 10% level and ** significance at the 5% level. A positive (negative) value of $\phi(\lambda)$ flags positive (negative) skewness.

Table 8 Tests for asymmetries using MS-AR models of US investment and consumption growth.

	DI	DC
MSM(2)		
NonSharpness	0.0187 [0.8914]	7.6712 [0.0056]**
CK NonDeepness	3.6811 [0.0550]	4.2533 [0.0392]*
CK NonSteepness	———	———
MSMH(2)		
NonSharpness	2.1796 [0.1398]	0.2571 [0.6121]
CK NonDeepness	0.1213 [0.7276]	0.3191 [0.5722]
CK NonSteepness	———	———
MSM(3)		
NonSharpness	1.5690 [0.6664]	3.8275 [0.2807]
$p_{12} = p_{32}$	0.3339 [0.5633]	0.0309 [0.8605]
$p_{13} = p_{31}$	1.1252 [0.2888]	3.0928 [0.0786]
$p_{21} = p_{23}$	0.1455 [0.7029]	0.7363 [0.3908]
CK NonDeepness	0.7933 [0.3731]	6.7898 [0.0092]**
CK NonSteepness	7.6104 [0.0058]**	9.1463 [0.0025]**
MSMH(3)		
NonSharpness	1.0429 [0.7909]	1.7330 [0.6296]
$p_{12} = p_{32}$	1.0057 [0.3159]	0.1231 [0.7257]
$p_{13} = p_{31}$	0.0066 [0.9353]	1.6853 [0.1942]
$p_{21} = p_{23}$	0.0077 [0.9302]	0.0049 [0.9440]
CK NonDeepness	1.3505 [0.2452]	0.1335 [0.7149]
CK NonSteepness	4.7713 [0.0289]*	0.4420 [0.5062]
Non-parametric tests		
NP NonDeepness	14.3245 [0.0002]**	19.1923 [0.0000]**
NP NonSteepness	0.1256 [0.7230]	7.1955 [0.0073]**

The data are taken from the FRED database (see http://www.stls.frb.org/fred/data/gdp.html) and cover the period 1960:4 to 1999:2. DI is the first difference of the log of investment (FRED database mnemonic GPDIC92), and DC is the first difference of the logarithm of consumers' expenditure (mnemonic PCEDGC92).