Comparative Statics of the Weak Axiom

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Abstract: This paper examines the comparative statics of Walrasian economies with excess demand functions which obey the weak axiom. We show that in these economies there is a precise sense in which goods that are in excess supply (demand) after some perturbation will experience a fall (rise) in its price. We apply this to an exchange economy with additive utility functions, which can be interpreted as a financial economy with von Neumann-Morgenstern utility functions. We show that when the subjective probabilities which agents attribute to a particular state falls, so will the price of consumption in that state. Another interesting issue is the impact of changes to the endowment on the equilibrium price. We develop conditions under which, for an exchange economy, this *equilibrium map* - from endowment to equilibrium price - will obey the weak axiom and another stronger, monotonicity property.

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1. Introduction

The indeterminacy theorems of Debreu, Sonnenschein and Mantel say that without suitable restrictions on endowments, preferences or technology, there is no structure to the excess demand function of an exchange or production economy (see the survey in Shafer and Sonnenschein (1982)). It follows from these results that, in general, an economy's equilibrium prices are neither unique nor stable with respect to Walras' tatonnement. To guarantee these properties, additional structural conditions beyond continuity and Walras' Law need to be imposed on an economy's excess demand function. The two most extensively studied conditions which do guarantee uniqueness and stability are gross substitubility and the weak axiom. These are aggregate conditions, so one is lead quite naturally to the study of its microfoundations. For example, Mas-Colell (1991) and Grandmont (1992) have identified primitive assumptions which lead to gross substitubility; Hildenbrand (1983), Jerison (1999) and Quah (1997, 1999, 2000) have done the same for the weak axiom. (For the weak axiom, see Quah (1999) for a more exhaustive list of references.) On the whole these theoretical models suggest that aggregate properties like gross substitubility or the weak axiom could arise under plausible scenarioes, though a more definite judgment would require more empirical work.

This paper examines the comparative statics of economies with excess demand functions that obey a property we call *the weak axiom at equilibrium*, a property which is weaker than both gross substitubility and the weak axiom. (see Mas-Colell et al (1995)). We assume that this property holds in a set of prices containing an equilibrium price, and study the restrictions this imposes on equilibrium price changes as endowments and preferences are varied. We show that with this property, there is a sense in which goods that are in excess demand after some perturbation of the economy's parameters, will experience an increase in their prices relative to those goods in excess supply. A straightforward corollary of this result is that when the endowment of a good falls in an exchange economy where all agents have normal demands for all goods, the price of that good will rise relative to the price of some bundle of all other goods. In another application, we consider an exchange economy with additive utility functions, which can be interpreted as a financial economy with von Neumann-Morgenstern utility functions. We show that when the subjective probabilities which agents attribute to a particular state falls, so will the price of consumption in that state.

Another interesting issue in comparative statics is the following. Suppose an exchange economy's endowments are changed (with preferences held fixed) causing the economy's equilibrium price to change as well; when could we expect the economy's mean endowment and its corresponding equilibrium price to vary in a way that obeys the weak axiom? This is a different question from asking whether an economy's excess demand function satisfies the weak axiom, and its answer will in general depend, not just on the change in the mean endowment, but also on precisely how that change is distributed among agents in the economy. It is useful for this purpose to restrict ourselves to endowment changes which satisfy an *aggregate normality* condition, i.e, if an endowment change raises mean income at the original equilibrium price, then it should also raise the demand for *all* goods at that price. If all agents have normal demands for all goods, then aggregate normality will hold if the endowment perturbation increases the income of every agent (and not just mean income); if this does not hold, and the endowment perturbation, even though it raises average income also causes the income of some agents to fall, then it is quite clear that different income elasticities of demand across agents could lead to a violation of aggregate normality. So aggregate normality is a joint restriction on demand behaviour and the endowment perturbation.

We show that for endowment changes which satisfy the aggregate normality condition, the equilibrium price will vary with mean endowment in a way that obeys the weak axiom if and only if the economies' excess demand functions satisfy the weak axiom at their equilibrium prices. Indeed for this class of endowment changes one can go further: we show that the equilibrium price change following a change in endowment must obey a monotonicity property that was first studied by Nachbar (1999). Nachbar (1999) established a local version of this result; our result generalizes this to a non-local, non-differentiable context.

2. Comparative Statics

Let $Z : R^l_{++} \to R^l$ be the excess demand function of an exchange or production economy with goods in the set $L = \{1, 2, ..., l\}$. We say that Z is *standard* if it satisfies Walras' Law, i.e., $p \cdot Z(p) = 0$ for all p in R^l_{++}, and is homogeneous of degree zero. A price $p' \gg 0$ is an equilibrium price if $Z(p') = 0$. Let P be a set in R^l_{++}; then Z satisfies the *weak axiom at equilbrium (WAE) in P* if P contains an equilibrium price p' and $p' \cdot Z(p) > 0$ whenever p is in P and not collinear to p'. It is known that if WAE holds in an open neighborhood of p' then p' will be locally stable with respect to Walras' tatonnement. Clearly the definition also requires the equilibrium to be unique (up to scalar multiples) in P, but unless $P = R^l_{++}$ global uniqueness is not implied. A sufficient condition for Z to satisfy WAE in an open and convex neighborhood of p' is for Z to be C^1 and for $v^T \partial_p Z(p')v < 0$ when v is not collinear with p'; we shall refer to this property as the *differentiable WAE*. We will now explore the comparative statics implications of WAE.

PROPOSITION 2.1: *Suppose that $\tilde{Z} : R^l_{++} \to R^l$ is a standard excess demand function of an economy with an equilibrium price at $\tilde{p}$ and also that $\tilde{Z}$ satisfies $\tilde{p} \cdot \tilde{Z}(p) > 0$ for some $p \gg 0$ (in other words, $\tilde{Z}$ satisfies WAE in the set $\{p, \tilde{p}\}$).*

(i) Defining $L^+ = \{i \in L : \tilde{Z}^i(p) > 0\}$ and $L^- = \{i \in L : \tilde{Z}^i(p) < 0\}$, we have

$$\frac{\sum_{L^+} \tilde{p}^i \tilde{Z}^i(p)}{\sum_{L^-} \tilde{p}^i [-\tilde{Z}^i(p)]} > \frac{\sum_{L^+} p^i \tilde{Z}^i(p)}{\sum_{L^-} p^i [-\tilde{Z}^i(p)]}. \tag{1}$$

(ii) There is a good j in L^+ and a good k in L^- such that $\tilde{p}^j/\tilde{p}^k > p^j/p^k$.

Proof: Clearly, $\tilde{Z}(p) \neq 0$ so L^+ and L^- are non-empty. By the intermediate value theorem, there is $\lambda > 0$, such that $\sum_{L^+}(\tilde{p}^i - \lambda p^i)\tilde{Z}^i(p) = 0$. By WAE and Walras' Law, $(\tilde{p} - \lambda p) \cdot \tilde{Z}(p) > 0$. Expanding the left of this inequality, and following from our choice of λ, $\sum_{L^-} \tilde{p}^i \tilde{Z}^i(p) > \lambda \sum_{L^-} p^i \tilde{Z}^i(p)$. Therefore, we have

$$\frac{\sum_{L^+} \tilde{p}^i \tilde{Z}^i(p)}{\sum_{L^+} p^i \tilde{Z}^i(p)} = \lambda > \frac{\sum_{L^-} \tilde{p}^i [-\tilde{Z}^i(p)]}{\sum_{L^-} p^i [-\tilde{Z}^i(p)]}. \tag{2}$$

Re-arranging this gives us (1).

The second part of the proposition follows immediately from (2) once we notice that there must be a good j in L^+ such that

$$\frac{\tilde{p}^j}{p^j} \geq \frac{\sum_{L^+} \tilde{p}^i \tilde{Z}^i(p)}{\sum_{L^+} p^i \tilde{Z}^i(p)}$$

and a good k in L^- such that

$$\frac{\sum_{L^-} \tilde{p}^i[-\tilde{Z}^i(p)]}{\sum_{L^-} p^i[-\tilde{Z}^i(p)]} \geq \frac{\tilde{p}^k}{p^k}.$$

QED

This result could be interpreted in the following way. Suppose that the economy is initially at equilibrium at the price p. It experiences a perturbation which changes its excess demand function to $\tilde{Z}$. The set L^+ are those goods which are in excess demand after the perturbation, while L^- are those goods in excess supply, at the original price of p. The new equilibrium price is $\tilde{p}$; the proposition says that there is a bundle of goods in L^+ (weighted by the size of the excess demands) and a bundle of goods in L^- (weighted by the size of the excess supplies) such that the price of the former relative to the latter increases after the perturbation. This implies, in particular, that there is at least one good in L^+ whose price relative to a good in L^- increases after the perturbation.

Note that the theorem requires that WAE be satisfied by $\tilde{Z}$, the excess demand of the perturbed economy and not Z, the excess demand of the original economy. This requirement could be guaranteed for small perturbations in the following fashion. Suppose that Z satisfies differentiable WAE at its equilibrium price p. Then p is also *regular*, i.e., the rank of $\partial_p Z(p)$ is $l-1$. (see Mas-Colell et al (1995)). Furthermore, suppose that excess demand is a smooth function of price and a parameter q in R^s, which represents possible perturbations to the economy. Assume that at $q = 0$, the excess demand is Z. Then one could establish, with standard arguments that for an economy with a perturbation q which is sufficiently close to 0, its excess demand $\tilde{Z}$ will be sufficiently close to Z and its equilibrium price $\tilde{p}$ sufficiently close to p for the condition of Proposition 1, $\tilde{p} \cdot \tilde{Z}(p) > 0$, to hold.

Using Proposition 1 it is possible to form a direct link between the change in equilibrium price and the change in the model's parameters, provided we can say something about which goods will be in excess supply or demand as a result of the parameter change. We will illustrate this idea with two examples from exchange economies.

We consider the class of *standard exchange economies*, whose properties we now describe. The agents in these economies form a finite set A. To each agent a in A is associated an endowment ω_a in R^l_+ and a demand function $f_a : R^l_{++} \times R_{++} \to R^l_{++}$, defined as a function of price and income, (p, w). At price p, the agent a's income is $p \cdot \omega_a$, so his demand is $f(p, p \cdot \omega_a)$. We denote by ω the vector $(\omega_a)_{a \in A}$ in $R^{l|A|}_+$. and by f the function from p to $(f_a(p))_{a \in A}$. The economy with endowment ω and demand f is denoted by $\mathcal{E}(\omega, f)$. We assume that f_a satisfies the budget identity and is homogeneous of degree zero. Following from this, the excess demand function $Z : R^l_{++} \to R^l$, defined by $Z(p) = F(p) - \mu(\omega)$ must also be standard, where $F(p) = [\sum_{a \in A} f_a(p, p \cdot \omega_a)]/|A|$ is the mean market demand and $\mu(\omega) = [\sum_{a \in A} \omega_a]/|A|$ is the mean endowment.

Example 1. This is a simple and well-known example which could also be be found in Mas-Colell et al (1995). Consider an economy whose endowment has been perturbed from ω to $\tilde{\omega}$. For all agents a, $\tilde{\omega}^1_a \le \omega^1_a$, with a strict inequality for at least one agent, and for all other goods i, $\tilde{\omega}^i_a = \omega^i_a$. In other words, the endowment of good 1 has fallen for some agents, while the other endowment of all other goods stay the same. The demand function remains at f and we assume that it is such that all goods are normal for all agents. At the original equilibrium price of p, the change in endowment causes the income of some agents fall, while no agent has an increase in income. So at the price p, the demand for all goods must fall, and consequently, for any good $k > 1$, $\tilde{Z}^k(p) < 0$. By Walras' Law, $\tilde{Z}^1(p) > 0$. In the notation of Proposition 2.1, $L^+ = \{1\}$ and $L^- = \{2, 3, ..., l\}$; the same proposition tell us that at the new equilibrium price of $\tilde{p}$, the price of good 1 relative to the price of some bundle of all other goods must have risen.

Example 2. In this example, we assume that endowments are fixed and consider the effect of a change in preferences on the equilibrium price. Two standard exchange economies, $\mathcal{E}(\omega, g)$ and $\mathcal{E}(\omega, \tilde{g})$ have different demand functions but the same endowments. We assume that the demand function g_a is generated by the smooth utility function $U_a(x) = \sum_{i=1}^l \pi_a^i u_a^i(x^i)$, while $\tilde{g}_a$ is generated by $\widetilde{U}_a(x) = \sum_{i=1}^l \tilde{\pi}_a^i u_a^i(x^i)$, where $\sum_{i=1}^l \pi_a^i = \sum_{i=1}^l \tilde{\pi}_a^i = 1$. We may interpret this as a finance model with complete markets inhabited by agents with von Neumann-Morgenstern utility functions; U_a and $\widetilde{U}_a$ are the utility functions of the agent a under the subjective probabilites π_a and $\tilde{\pi}_a$ (both understood as vectors in R_{++}^l) respectively. Can we say anything about the connection between the equilibrium price of state i consumption and the subjective probabilities agents attach to state i? It turns out that we can, using Propostion 2.1 and the following lemma.

LEMMA 2.2: *Suppose that $\tilde{\pi}_a \neq \pi_a$. Then*

$$L_a^* = \text{argmax}_{i \in L} \frac{\tilde{\pi}_a^i}{\pi_a^i} \text{ and } L_a^{**} = \text{argmin}_{i \in L} \frac{\tilde{\pi}_a^i}{\pi_a^i}$$

are both non-empty and for all $(p, w) \gg 0$, $\tilde{g}_a^i(p, w) > g_a^i(p, w)$ if i is in L_a^*, and $\tilde{g}_a^i(p, w) < g_a^i(p, w)$ if i is in L_a^{**}.

Proof: The non-emptiness of L_a^* and L_a^{**} are obvious, following from the fact that $\pi_a \neq \tilde{\pi}_a$ and $\sum_{i \in L} \pi_a^i = \sum_{i \in L} \tilde{\pi}_a^i = 1$. Suppose that $\tilde{\pi}_a^i / \pi_a^i = K$ if i is in L_a^*. Note that K must be bigger than 1. We define $\hat{\pi}_a = \tilde{\pi}_a / K$; clearly $\pi_a > \hat{\pi}_a$, with $\pi_a^i = \hat{\pi}_a^i$ if and only if i is in L_a^*. The utility function $\widehat{U}_a(x) = \sum_{i=1}^{l} \hat{\pi}_a^i u_a^i(x^i)$, generates the same demand as $\widetilde{U}_a$. We could think of U_a being changed to $\widehat{U}_a$ step by step, where at each step, the weight attached to each i is lowered from π_a^i to $\hat{\pi}_a^i$, for all i in $L \setminus L_a^*$. It is known that lowering the weight attached to some good, keeping the other weights the same has the effect of raising the demand of every other good (see, for example, Grandmont (1998)); therefore, at each step, the demand for good i in L_a^* strictly increases. It follows that $\tilde{g}_a^i(p, w) > g_a^i(p, w)$, if i is in L_a^*. The proof for the other part is the same. QED

To apply this result consider the following scenario. For some subset of agents A' in A, π_a undergoes a change to $\tilde{\pi}_a$. This change in subjective beliefs is binary, i.e., the probability of a state either goes up by a common factor k_a (in which case the state is in L_a^*) or it goes down by a common factor k_a' (in which case the state is in L_a^{**}); so $L_a^* \cup L_a^{**} = L$. Assume that $L_a^* = L^*$ is common across all agents in A', and similarly, that $L_a^{**} = L^{**}$ is common across all agents in A'. Then we know from the lemma that at the prevailing equlibrium price of p, the goods in L^* (respectively L^{**}) must experience an increase (decrease) in demand. It follows that (using the notation of Proposition 2.1) $L^* \subset L^+$ and $L^{**} \subset L^-$. Together with the fact that $L^* \cup L^{**} = L$, we obtain $L^+ = L^*$ and $L^- = L^{**}$. Finally we assume that $\tilde{Z}$ the excess demand of $\mathcal{E}(\omega, \tilde{g})$ and $\tilde{p}$, its equilibrium price, satisfy $\tilde{p} \cdot \tilde{Z}(p) > 0$, where p is the equilibrium price of $\mathcal{E}(\omega, g)$. By Proposition 2.1 there is a bundle of goods in L^* whose price will rise relative to some bundle of goods in L^{**}. Even more specifically, suppose that for the agents in A', the subjective probability of state 1 goes up, and the probability of all other states fall by the same proportion. Then the price of state 1 consumption, relative to the price of some consumption bundle in all other states, will rise.

Proposition 2.1 and the two examples discussed show that assuming WAE leads to comparative statics that correspond broadly with our intuition: when a good is in excess supply, its price falls; when the endowment of a good falls, its price will increase; and when the probability of an event increases, so will its state price. The next section takes up a different theme.

3. The Equilibrium Map

We keep the agents' demand function fixed at f, and consider the collection of economies $\{\mathcal{E}(\omega, f) : \omega \in R_+^{l|A|}\}$. Modifying our earlier notation, we write $F(p,\omega)$ to refer to the mean demand of the economy $\mathcal{E}(\omega, f)$ at price p and $Z(p,\omega)$ as its excess demand. This notation highlights the dependence of demand and excess demand on the endowment ω; it also means, of course, that we shall be thinking of F and Z as functions of (p,ω). We say that F satisfies *weak aggregate normality between* (p,ω) *and* $(p,\tilde{\omega})$ if the following holds: when $p \cdot \mu(\omega) > p \cdot \mu(\tilde{\omega})$, $F(p,\omega) > F(p,\tilde{\omega})$, when $p \cdot \mu(\omega) < p \cdot \mu(\tilde{\omega})$, $F(p,\omega) < F(p,\tilde{\omega})$, and when $p \cdot \mu(\omega) = p \cdot \mu(\tilde{\omega})$, $F(p,\omega) = F(p,\tilde{\omega})$. This concept will be central to our exposition in this section; the name we give to it follows Nachbar (1999) and is entirely appropriate since the condition essentially says that if the endowment ω at price p has a higher mean value than $\tilde{\omega}$ at the same price, then aggregate demand must be weakly higher for all goods and strictly higher for at least one good.

Let Ω be a subset of $R_+^{l|A|}$. To each economy $\mathcal{E}(\omega, f)$, with ω in Ω, we associate an equilibrium price $P(\omega)$ which is normalized to satisfy $P(\omega) \cdot \mu(\omega) = 1$. We shall refer to P as the *equilibrium map* or *function.* This map satisfies the *weak axiom at* $\bar{\omega}$ if for any ω in

Ω, with $\mu(\bar{\omega}) \neq \mu(\omega)$, either $P(\bar{\omega}) \cdot \mu(\omega) > P(\bar{\omega}) \cdot \mu(\bar{\omega}) = 1$ or $P(\omega) \cdot \mu(\bar{\omega}) > P(\omega) \cdot \mu(\omega) = 1$. The next result identifies conditions which guarantee that P obeys the weak axiom at $\bar{\omega}$.

PROPOSITION 3.1: *Let $P : \Omega \to R^l_{++}$, $\Omega \subset R^{l|A|}_{+}$ be an equilibrium map. Then P satisfies the weak axiom at $\bar{\omega}$ if*

(i) $P(\omega) \cdot Z(P(\bar{\omega}), \omega) > 0$ for all ω in Ω with $P(\omega) \neq P(\bar{\omega})$ (equivalently, $Z(\cdot, \omega)$ satisfies WAE in $\{P(\omega), P(\bar{\omega})\}$), and

(ii) F satisfies weak aggregate normality between $(P(\bar{\omega}), \bar{\omega})$ and $(P(\bar{\omega}), \omega)$ for all ω in Ω.

Proof: Suppose that $\mu(\omega) \neq \mu(\bar{\omega})$ and $\mu(\omega) \cdot P(\bar{\omega}) \leq 1 = \mu(\bar{\omega}) \cdot P(\bar{\omega})$. By aggregate normality, $F(P(\bar{\omega}), \omega) \leq F(P(\bar{\omega}), \bar{\omega}) = \mu(\bar{\omega})$. Assuming that $P(\omega) \neq P(\bar{\omega})$, we have $P(\omega) \cdot Z(P(\bar{\omega}), \omega) > 0$ from which we obtain, as required,

$$P(\omega) \cdot \mu(\bar{\omega}) \geq P(\omega) \cdot F(P(\bar{\omega}), \omega) > P(\omega) \cdot \mu(\omega).$$

Suppose now that $P(\omega) = P(\bar{\omega})$. Then if $\mu(\omega) \cdot P(\bar{\omega}) < 1$, we also have $\mu(\omega) \cdot P(\omega) < 1$, which by the definition of P, cannot be true. So $\mu(\omega) \cdot P(\bar{\omega}) = 1$. By aggregate normality,

$$\mu(\omega) = F(P(\omega), \omega) = F(P(\bar{\omega}), \omega) = F(P(\bar{\omega}), \bar{\omega}) = \mu(\bar{\omega}),$$

contrary to our initial assumption that $\mu(\omega) \neq \mu(\bar{\omega})$. QED

Proposition 3.1 does not require the perturbation considered to be small, so ω in Ω can be far away from $\bar{\omega}$, nor does it require differentiability in any form. If these assumptions are made, then standard arguments will allow us to say a bit more. Let $\bar{p}$ be an equilibrium price of $\mathcal{E}(\bar{\omega}, f)$ satisfying $\bar{p} \cdot \mu(\bar{\omega}) = 1$. Assume that Z is a C^1 function and that $Z(\cdot, \bar{\omega})$ is regular at $\bar{p}$. By the implicit function theorem, there is an open neighborhood of $\bar{\omega}$, M, and an open neighborhood of $\bar{p}$, N, such that for all ω in M, the economy $\mathcal{E}(\omega, f)$

has a unique (normalized) equilibrium price in N, which we denote by $P(\omega)$, and the map $P : M \to N$ is C^1. Furthermore, if $Z(\cdot, \bar{\omega})$ satisfies differentiable WAE at $\bar{p}$ (so in particular, $Z(\cdot, \bar{\omega})$ is regular), M and N could be chosen such that the equilibrium map P also satisfies $P(\omega) \cdot Z(P(\bar{\omega}), \omega) > 0$ for all ω in N, provided $P(\bar{\omega}) \neq P(\omega)$. In other words, P satisfies condition (i) in Proposition 3.1. If we now choose $\Omega \subset N$ so that condition (ii) in the proposition also holds, then (abusing notation slightly), $P : \Omega \to R^l_{++}$ will satisfy the weak axiom at $\bar{\omega}$.

Condition (ii) - aggregate normality - is essentially a joint restriction on agents' Engel curves and the distribution of the endowment perturbation across agents. It is quite clear that unless agents have parallel Engel curves, aggregate normality must mean that Ω cannot include *all* perturbations from $\bar{\omega}$; in other words, Ω is a proper subset of, and cannot be equal to, N. A simple but extreme situation where the condition is satisfied is to assume that all goods are normal to all agents and that

$$\Omega = \{\omega \in M : \omega_a = \bar{\omega}_a + k_a \theta \text{ where } k_a > 0 \text{ and } \theta \in R^l \text{ satisfies } \|\theta\| = 1\}.$$

The important thing to note here is that perturbation is collinear across agents, in the direction of θ, so all agents either simultaneously experience an increase or decrease in income at the original equilbrium price $\bar{p}$, depending respectively on whether $\bar{p} \cdot \theta$ is positive or negative. Clearly, condition (ii) is now guaranteed and Proposition 3.1 can be applied to guarantee that P obeys the weak axiom at $\bar{p}$.

The next result is a partial converse of Proposition 3.1. Loosely speaking, we show that if P obeys the weak axiom, and aggregate normality holds, then the excess demand function must satisfy WAE for a particular set of prices.

PROPOSITION 3.2: *Let* $\bar{p}$ *be an equilibrium price of* $\mathcal{E}(\bar{\omega}, f)$ *with* $\bar{p} \cdot \mu(\bar{\omega}) = 1$ *and let* $\Omega \subset \{\omega \in R_+^{l|A|} : \bar{p} \cdot \mu(\omega) = 1\}$*. Suppose that*

(i) an equilibrium map $P : \Omega \to R_{++}^l$ *with* $P(\bar{\omega}) = \bar{p}$ *satisfies the weak axiom at* $\bar{\omega}$*, and*

(ii) F *satisfies weak aggregate normality between* $(P(\omega), \bar{\omega})$ *and* $(P(\omega), \omega)$ *for all* ω *in* Ω*.*

Then $Z(\cdot, \bar{\omega})$ *satisfies WAE in* $P(\Omega)$*.*

Proof: By (i), $P(\omega) \cdot \mu(\bar{\omega}) > 1 = P(\omega) \cdot \mu(\omega)$, which implies, by (ii) that $F(P(\omega), \bar{\omega}) > F(P(\omega), \omega) = \mu(\omega)$. It follows that

$$P(\bar{\omega}) \cdot F(P(\omega), \bar{\omega}) > P(\bar{\omega}) \cdot \mu(\omega) = P(\bar{\omega}) \cdot \mu(\bar{\omega})$$

and so $P(\bar{\omega}) \cdot Z(P(\omega), \bar{\omega}) > 0$. QED

To understand Proposition 3.2 a little better, imagine that the economy $\mathcal{E}(\bar{\omega}, f)$, with equilibrium price $\bar{p}$, is perturbed with the endowment of agent a changed from ω_a to $\omega_a + \theta$, where θ is in $\Theta = \{\theta \in R^l : \bar{p} \cdot \theta = 0\}$. Clearly, at the price $\bar{p}$, the income of all agents are left unchanged by the perturbation and mean income is also preserved at $\bar{p} \cdot \mu(\bar{\omega})$, while mean endowment becomes $\mu(\bar{\omega}) + \theta$. Since θ is common across all agents, the aggregate normality required in condition (ii) of Proposition 3.2 will hold if all agents have normal demands for all goods. Departing from our convention so far, we assume that $\bar{p}$ is in $T = \{p \in R_{++}^l : p^l = 1\}$. So long as $\bar{p}$ is a regular equilibrium price, we can find a diffeomorphic map $P : \Theta' \to T'$, where Θ' is open in Θ and T' is open in T such that $P(\theta)$ is an equilibrium price of $\mathcal{E}(\bar{\omega} + \theta, f)$ and $P(0) = \bar{p}$. (Note that we have abused the notation P.) If condition (i) in Proposition 3.2 holds, i.e., if $P(\theta) \cdot \theta < 0$ for all θ in Θ', then the proposition says that $\bar{p} \cdot Z(p, \bar{\omega}) > 0$ for all p in T'. In other words, if there is an open neighborhood of collinear endowment perturbations (as represented by θ) in which P

obeys the weak axiom, then there must also be an open neighborhood of $\bar{p}$ in which $Z(\cdot, \bar{\omega})$ satisfies WAE.

Besides the weak axiom, there are other properties which the equilibrium map P could usefully satisfy. We say that P is *monotonic* at $\bar{\omega}$ if $(\mu(\bar{\omega})-\mu(\omega)\cdot(P(\bar{\omega})-P(\omega)) < 0$ whenever $\mu(\omega) \neq \mu(\bar{\omega})$. It is easy to check that this property is stronger than the weak axiom; it was first investigated by Malinvaud (1972) and a recent study could be found in Quah (1999). We shall not discuss it any further in this paper, but it is worth mentioning that the conditions developed in Quah (1999) to guarantee the monotonicity of P are neither stronger nor weaker than the assumptions of Proposition 3.1: in particular, aggregate normality is not required, so this condition is not necessary for P to satisfy the weak axiom.

Another property of P, closely related to the weak axiom, has recently been studied by Nachbar (1999). We say that the equilibrium map $P : \Omega \to R^l_{++}$ is *N-monotonic at* $\bar{\omega}$ if for all ω in Ω, there is a vector $a \gg 0$, and real number $\lambda > 0$ (both of which depend on ω) such that $a \cdot P(\bar{\omega}) = a \cdot \lambda P(\omega)$ and $(\mu(\bar{\omega}) - \mu(\omega)) \cdot (P(\bar{\omega}) - \lambda P(\omega)) \leq 0$, with equality only if (i) $\mu(\omega) = \mu(\bar{\omega})$ or (ii) $\mu(\omega) \cdot P(\bar{\omega}) \neq 1$ and $P(\bar{\omega})$ and $P(\omega)$ are collinear. Again, it is straightforward to check that this property is stronger than the weak axiom. The property says that there is a way of normalizing prices such that the change in price and the change in mean endowment move in opposite directions. The normalization gives an easy way of interpreting a price change. For example, suppose that the price vector changes from p to q, with $a \cdot p = a \cdot q$, and $p^i > q^i$ for i in $K \subset L$. Then $\sum_{i \in K} p^i a^i > \sum_{i \in K} q^i a^i$ and

$\sum_{i\in L\setminus K} p^i a^i < \sum_{i\in L\setminus K} q^i a^i$, so

$$\frac{\sum_{i\in K} p^i a^i}{\sum_{i\in L\setminus K} p^i a^i} > \frac{\sum_{i\in K} q^i a^i}{\sum_{i\in L\setminus K} q^i a^i}.$$

In other words, the price of the bundle $\{a^i\}_{i\in K}$ relative to the bundle $\{a^i\}_{i\in L\setminus K}$ has fallen.

Our eventual goal is to find conditions which guarantee the N-monotonicity of P, but a short digression is helpful at this stage. All the three properties on P we have stated have natural analogues for any map $\Pi : R^l_{++} \to R^l_{++}$, satisfying the *budget identity* $x \cdot \Pi(x) = 1$. In this case, Π could be interpreted as the supporting price or inverse demand of the commodity bundle x. We say that Π satisfies the *weak axiom at* $\bar{x}$ if for all $x \neq \bar{x}$, either $x \cdot \Pi(\bar{x}) > 1$ or $\bar{x} \cdot \Pi(x) > 1$; we say that Π is *monotonic at* $\bar{x}$ if $(\bar{x} - x) \cdot (\Pi(\bar{x}) - \Pi(x)) < 0$ for all $x \neq \bar{x}$; and, finally, we say that Π is *N-monotonic at* $\bar{x}$ if for any $x \neq \bar{x}$, there is $a \gg 0$ and $\lambda > 0$ such that $a \cdot \Pi(\bar{x}) = a \cdot \lambda\Pi(x)$ and $(\bar{x} - x) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) \leq 0$, with equality only if (i) $\bar{x} = x$ and (ii) $x \cdot \Pi(\bar{x}) \neq 1$ and $\Pi(x)$ and $\Pi(\bar{x})$ are collinear.

We will not in this paper investigate the monotonicity of Π, a topic which has already been extensively studied: necessary and sufficient conditions could be found in Milleron (1974), Mitjuschin and Polterovich (1978), Kannai (1989) and Quah (2000). We shall focus instead on the N-monotonicity of Π; we show that N-monotonicity is a simple consequence of a weaker form of the weak axiom and a normality condition. More specifically, Π satisfies the *limited weak axiom at* $\bar{x}$ if for all $x \neq \bar{x}$, with $x \cdot \Pi(\bar{x}) = 1$, we have $\bar{x} \cdot \Pi(x) > 1$. Assuming that Π^{-1} exists, we say that Π^{-1} is *normal* (or *satisfies normality*) *at* p if $\Pi^{-1}(\gamma p) \gg \Pi^{-1}(p)$ whenever $\gamma < 1$, and $\Pi^{-1}(\gamma p) \ll \Pi^{-1}(p)$ whenever $\gamma > 1$.

PROPOSITION 3.3: *Suppose that* $\Pi : R^l_{++} \to R^l_{++}$ *is an invertible function satisfying the budget identity and the limited weak axiom. Then the following are equivalent:*

(i) There is a vector $a \gg 0$ such that for all x with $x \cdot \Pi(\bar{x}) = k \neq 1$, we obtain $(\bar{x} - x) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) \leq 0$, where λ is chosen to satisfy $a \cdot \Pi(\bar{x}) = a \cdot \lambda\Pi(x)$.

(ii) Π^{-1} is a normal function at $\Pi(\bar{x})$.

Furthermore, if (ii) is true, the vector a in (i) is unique up to scalar multiples and $(\bar{x} - x) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) = 0$ if and only if $\Pi(\bar{x})$ and $\Pi(x)$ are collinear.

Proof: We first show that (ii) implies (i). Assume that $k < 1$ (the case of $k > 1$ can be dealt with analogously); then $m = 1/k > 1$ satisfies $m\Pi(\bar{x}) \cdot x = 1$. Since Π is invertible, there is a unique y such that $\Pi(y) = m\Pi(\bar{x})$. Note that $y \cdot \Pi(\bar{x}) = y \cdot \Pi(y)/m = k$ By (ii), $y \ll \bar{x}$, so $a = \bar{x} - y \gg 0$. For any x satisfying $x \cdot \Pi(x) = k$, we can choose λ to satisfy $a \cdot \Pi(\bar{x}) = a \cdot \lambda\Pi(x)$. Provided $x \neq y$, the limited weak axiom guarantees that $y \cdot \Pi(x) > 1$, so

$$\begin{aligned}(\bar{x} - x) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) &= (\bar{x} - y - (x - y)) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) \\ &= -(x - y) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) \\ &= \lambda(x - y) \cdot \Pi(x)) < 0.\end{aligned}$$

So we have shown that (ii) implies (i) and also that given our choice of the vector a, $(\bar{x} - x) \cdot (\Pi(\bar{x}) - \lambda\Pi(x)) = 0$ if and only if $\Pi(\bar{x})$ and $\Pi(x)$ are collinear.

Suppose now that (i) is true; we want to show that for $\gamma > 1$, $\bar{x} \gg \Pi^{-1}(\gamma\Pi(\bar{x}))$. (Once again, the case of $\gamma < 1$ can be dealt with analogously.) Consider x satisfying $x \cdot \Pi(\bar{x}) = 1/\gamma > 1$; for such x there is an associated vector $a \gg 0$ satisfying the conditions laid out in (i). Choose x' such that $\bar{x} - x'$ is parallel to a and $x' \cdot \Pi(\bar{x}) = 1/\gamma$. We claim that $x' = \Pi^{-1}(\gamma\Pi(\bar{x}))$. If it is not, we write $x'' = \Pi^{-1}(\gamma\Pi(\bar{x}))$, and consider $x''' = tx' + (1 - t)x''$.

For a suitably chosen t strictly between 0 and 1, x''' is in the positive orthant, so $\Pi(x''')$ exists, and since it is distinct from x'', the limited weak axiom guarantees that $x''\cdot\Pi(x''') > 1$. Since $x'''\cdot\Pi(x''') = 1$ by definition, we obtain $x'\cdot\Pi(x''') < 1$. Note also that $x'''\cdot\Pi(\bar{x}) = 1/\gamma$. Choose λ so that $a\cdot\Pi(\bar{x}) = a\cdot\lambda\Pi(x''')$. Then

$$\begin{aligned}(\bar{x}-x''')\cdot(\Pi(\bar{x})-\lambda\Pi(x''')) &= (\bar{x}-x'-(x'''-x'))\cdot(\Pi(\bar{x})-\lambda\Pi(x''')) \\ &= -(x'''-x')\cdot(\Pi(\bar{x})-\lambda\Pi(x''')) \\ &= \lambda(x'''-x')\cdot\Pi(x''') \\ &= \lambda(1-x'\cdot\Pi(x''')) > 0,\end{aligned}$$

which is a violation of (i). So we conclude that $x' = \Pi^{-1}(\gamma\Pi(\bar{x}))$. Since $a \gg 0$ by assumption, and by our construction a is collinear to $\bar{x}-x'$, we have $x' \ll \bar{x}$. This establishes (ii) and the uniqueness (up to scalar multiples) of a. QED

Note that the case of $k = 1$ in condition (i) of Propositon 3.3 is trivially true if Π satisfies the limited weak axiom. If $x\cdot\Pi(\bar{x}) = 1$, we have $(\bar{x}-x)\cdot(\Pi(\bar{x})-\lambda\Pi(x)) = \lambda(1-\bar{x}\cdot\Pi(x)) < 0$ for *any* positive λ, which means that *any* vector $a \gg 0$ could be chosen. This observation and Proposition 3.3 are summarized in the following corollary.

COROLLARY 3.4: *Let* $\Pi : R^l_{++} \to R^l_{++}$ *be a function with the following properties: (i)* $x\cdot\Pi(x) = 1$ *for all* x, *(ii) it obeys the limited weak axiom, and (iii)* Π^{-1} *exists and is normal. Then* Π *satisfies N-monotonicity.*

Notice that the conditions of Corollary 3.4 are analogous to those of Proposition 3.1, with the limited weak axiom similar to condition (i), and normality similar to condition (ii). It turns out, not surprisingly, that a slight strengthening of the conditions in Proposition 3.1

are sufficient to guarantee the N-monotonicity of P. In particular, (ii) needs to be replaced by a slightly stronger condition. We say that F satisfies *aggregate normality between* (p, ω) *and* $(p, \tilde{\omega})$ if the following holds: (i) when $p \cdot \mu(\omega) > p \cdot \mu(\tilde{\omega})$, $F(p, \omega) \gg F(p, \tilde{\omega})$, (ii) when $p \cdot \mu(\omega) < p \cdot \mu(\tilde{\omega})$, $F(p, \omega) \ll F(p, \tilde{\omega})$, (iii) when $p \cdot \mu(\omega) = p \cdot \mu(\tilde{\omega})$, $F(p, \omega) = F(p, \tilde{\omega})$.

PROPOSITION 3.5: *Let* $P : \Omega \to R^l_{++}$, $\Omega \subset R^{l|A|}_+$ *be an equilibrium map. Then* P *satisfies N-monotonicity at* $\bar{\omega}$ *if*

(i) $P(\omega) \cdot Z(P(\bar{\omega}), \omega) > 0$ *for all* ω *in* Ω *with* $P(\omega) \neq P(\bar{\omega})$ *(equivalently,* $Z(\cdot, \omega)$ *satisfies WAE in* $\{P(\omega), P(\bar{\omega})\}$*), and*

(ii) F *satisfies aggregate normality between* $(P(\bar{\omega}), \bar{\omega})$ *and* $(P(\bar{\omega}), \omega)$ *for all* ω *in* Ω.

Proof: If ω satisfies $\mu(\omega) \cdot P(\bar{\omega}) = 1$, we have

$$\begin{aligned}(\mu(\bar{\omega}) - \mu(\omega) \cdot (P(\bar{\omega}) - \lambda P(\omega)) &= -\lambda(\mu(\bar{\omega}) - \mu(\omega)) \cdot P(\omega) \\ &= -\lambda(F(P(\bar{\omega}), \omega) - \mu(\omega)) \cdot P(\omega) < 0\end{aligned}$$

for any positive number λ, which means that the vector a can be chosen to be any vector with strictly positive entries. (Note that the second equality follows from (ii) and the final inequality from (i).)

We now consider the case when ω satisfies $\mu(\omega) \cdot P(\bar{\omega}) < 1 = \mu(\bar{\omega}) \cdot P(\bar{\omega})$. (The case of $\mu(\omega) \cdot P(\bar{\omega}) > 1$ can be handled by an analogous argument.) By (ii), $F(P(\bar{\omega}), \omega) \ll F(P(\bar{\omega}), \bar{\omega}) = \mu(\bar{\omega})$, so $a = \mu(\bar{\omega}) - F(P(\bar{\omega}), \omega) \gg 0$. Choosing λ so that $a \cdot P(\bar{\omega}) = \lambda a \cdot P(\omega)$, we obtain

$$\begin{aligned}(\mu(\bar{\omega}) - \mu(\omega)) \cdot (P(\bar{\omega}) - \lambda P(\omega)) &= [\mu(\bar{\omega}) - F(P(\bar{\omega}), \omega) - (\mu(\omega) - F(P(\bar{\omega}), \omega))] \cdot (P(\bar{\omega}) - \lambda P(\omega)) \\ &= -(\mu(\omega) - F(P(\bar{\omega}), \omega)) \cdot (P(\bar{\omega}) - \lambda P(\omega))\end{aligned}$$

$$
\begin{aligned}
&= \lambda(\mu(\omega) - F(P(\bar{\omega}), \omega)) \cdot P(\omega) \\
&= -\lambda P(\omega) \cdot Z(P(\bar{\omega}), \omega)
\end{aligned}
$$

which, by (i), must be negative. QED

The first version of this result could be found in Nachbar (1999). This statement is different from Theorem 1 in Nachbar (1999) in several ways: the statement is non-local, differentiability is no longer assumed, and perturbations which preserve mean income, i.e., $P(\bar{\omega}) \cdot \mu(\bar{\omega}) = P(\bar{\omega}) \cdot \mu(\omega)$ are not excluded.

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