Higher order variation and stochastic volatility models

Ole E. Barndorff-Nielsen
The Centre for Mathematical Physics and Stochastics (MaPhySto),
University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark.
oebn@mi.aau.dk

Neil Shephard
Nuffield College, Oxford OX1 1NF, UK
neil.shephard@nuf.ox.ac.uk

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Abstract

Limit distribution results on quadratic and higher order variation quantities are derived for certain types of continuous local martingales, in particular for a class of OU-based stochastic volatility models.

Some key words: Mixed asymptotic normality; Realised volatility; Quadratic variation.

1. Introduction

In a recent study by Barndorff-Nielsen and Shephard (2001d) of the properties of realised volatility, that is the sum of squares of intra-day returns on speculative assets, it became necessary in addition to quadratic variation of the stochastic processes to consider also aspects of higher order variation. The requisite mathematical results on higher order variation seem of some independent interest and are therefore discussed separately here.

The modelling framework in the paper referred to is that of stochastic volatility models of the form

$$y^*(t) = \mu t + \beta\tau^*(t) + \int_0^t \tau^{1/2}(s)\mathrm{d}w(s), \tag{1.1}$$

where μ and β are parameters, $w(t)$ denotes Brownian and $\tau(t)$, the stochastic volatility, is a stationary and positive stochastic process, assumed independent of τ; finally,

$$\tau^*(t) = \int_0^t \tau(s)\mathrm{d}s.$$

(For some general information on processes y^* of this type, see for example Barndorff-Nielsen and Shephard (2001a-c) and Ghysels, Harvey and Renault (1996)).

Because of the independence between w and τ we may, and shall in the context of the present paper, simply consider τ in (1.1) as a deterministic positive cadlag or caglad function on $[0, \infty)$. For the general setting, with τ random, the same conclusions will hold with probability

1 provided τ has, almost surely, the properties just mentioned. This is the case, in particular, when (as in Barndorff-Nielsen and Shephard (2001a)) τ is a superposition of non-Gaussian OU processes or if τ is a continuous solution of a stochastic differential equation driven by a Brownian motion independent of $.w$.

Our results continue to hold for a more general type of semimartingale

$$y^*(t) = a(t) + \int_0^t \tau^{1/2}(s)\mathrm{d}w(s) \tag{1.2}$$

where a and τ are assumed to be jointly independent of w and a is a cadlag or caglad process satisfying a certain additional requirement.

Section 2 lists the results of the paper and proofs are provided in Section 3.

Extension to several dimensions will be discussed in a separate paper, which will also contain empirical work additional to that presented in Barndorff-Nielsen and Shephard (2001d).

2. Results

We first introduce some notation for higher order variation quantities of an arbitrary semimartingale x. Fix t, let δ be positive real and write $M = \lfloor t/\delta \rfloor$, where $\lfloor t \rfloor$ for any positive number t denotes the largest integer less and or equal to t, and let

$$x_\delta(t) = x(\lfloor t/\delta \rfloor \, \delta).$$

Further, for r positive real we define

$$[x_\delta]^{[r]}(t) = \sum_{j=1}^{M} |x\,(j\delta) - x((j-1)\delta|^r. \tag{2.1}$$

Then, in particular, for $M \to \infty$,

$$[x_\delta]^{[2]}(t) \xrightarrow{p} [x](t),$$

where $[x]$ is the quadratic variation process of the semimartingale x. Note also that

$$[x_\delta]^{[2]} = [x_\delta].$$

When $r > 2$ we speak of (2.1) and similar quantities as *higher order variations.*

Now, returning to processes of the form (1.2) we impose throughout the condition

(C) $\tau > 0$ and a are cadlag or caglad functions on $[0, \infty)$ and a has the property

$$\overline{\lim_{\delta \downarrow 0}} \max_{1 \le j \le M} \delta^{-1} |a(j\delta) - a((j-1)\delta)| < \infty \tag{2.2}$$

This condition is satisfied in particular if a is of the form

$$a(t) = \mu t + \beta \tau^*(t),$$

as in the stochastic volatility model (1.1), or in the more general setting when

$$a(t) = \int_0^t g(\tau(s))\mathrm{d}s.$$

where g is a smooth function.

Note also that condition **(C)** implies that τ and a are bounded Riemann integrable functions.

Define

$$\tau^{r*}(t) = \int_0^t \tau^r(s)\mathrm{d}s.$$

Theorem 2.1 For $\delta \downarrow 0$ and r positive real

$$\delta^{-r+1}[\tau_\delta^*]^{[r]}(t) \to \tau^{r*}(t).$$

□

Henceforth q denotes a positive integer and $c_q = \{1 \cdot 3 \cdot \cdots \cdot (2q-1)\}^{-1}$.

Theorem 2.2 Let y^* be a stochastic process of the form (1.2). Then, for $\delta \downarrow 0$

$$\delta^{-q+1} c_q [y_\delta^*]^{[2q]}(t) \xrightarrow{p} \tau^{q*}(t).$$

□

Theorem 2.3 Let y^* be a stochastic process of the form (1.2). Then, for $\delta \downarrow 0$,

$$\frac{[y_\delta^*](t) - \tau^*(t)}{\sqrt{\frac{2}{3}[y_\delta^*]^{[4]}(t)}} \xrightarrow{\mathcal{L}} N(0,1). \tag{2.3}$$

□

This considerably sharpens the well known important result that for models (1.2) $[y^*] = \tau^*$. Note also that Theorems 2.2 and 2.3 together imply that when τ is a stationary process whose sample paths satisfy condition **(C)** then $\delta^{-1/2}\{[y_\delta^*](t) - \tau^*(t)\}$ follows asymptotically a normal variance mixture. Otherwise put,

$$\frac{\delta^{-1/2}\{[y_\delta^*](t) - \tau^*(t)\}}{\sqrt{2\tau^{2*}(t)}} \xrightarrow{\mathcal{L}} N(0,1)$$

as $\delta \downarrow 0$.

3. Proofs

Let

$$y_0^*(t) = \int_0^t \tau^{1/2}(s) \mathrm{d}w(s).$$

It is helpful to introduce the following notation

$$u_{0j} = y_0^*(j\delta) - y_0^*((j-1)\delta)$$

and

$$\tau_j = \tau^*(j\delta) - \tau^*((j-1)\delta),$$

$$\varepsilon_j = a(j\delta) - a((j-1)\delta).$$

We may now rewrite $[\tau_\delta^*]^{[r]}(t)$ and $[y_\delta^*]^{[2q]}(t)$ as

$$[\tau_\delta^*]^{[r]}(t) = \sum_{j=1}^{M} \tau_j^r \tag{3.1}$$

and

$$[y_\delta^*]^{[2q]}(t) = [y_{0\delta}^*]^{[2q]}(t) + \sum_{k=1}^{2q} \binom{2q}{k} \sum_{j=1}^{M} {\varepsilon_j}^k {u_{0j}}^{2q-k}. \tag{3.2}$$

Proof of Theorem 2.1 For every $j = 1, ..., M$ there exists a constant θ_j such that

$$\inf_{(j-1)\delta \le s \le j\delta} \tau(s) \le \theta_j \le \inf_{(j-1)\delta \le s \le j\delta} \tau(s)$$

and

$$\tau_j = \theta_j \delta.$$

Hence, by (3.1),

$$\delta^{-r+1}[\tau_\delta^*]^{[r]}(t) = \delta^{-r+1} \sum_{j=1}^{M} \tau_j^r = \sum_{j=1}^{M} {\theta_j}^r \delta$$
$$\to \int_0^t \tau^r(s) \mathrm{d}s = \tau^{r*}(t)$$

□

To handle the expression (3.2) we establish some lemmas (that also are of some independent interest). Let

$$D_0 = c_q [y_{0\delta}^*]^{[2q]}(t) - [\tau_\delta^*]^{[q]}(t)$$

and recall Taylor's formula with remainder term:

$$f(x) = f(0) + f'(0)x + x^2 \int_0^1 (1-u) f''(ux) \mathrm{d}u. \tag{3.3}$$

Lemma 3.1 The cumulant function of D_0 is of the form

$$\log \mathrm{E}\{\exp(i\zeta D_0)\} = -\frac{1}{2}\zeta^2\delta^{2q}R$$

where

$$R = 2\sum_{j=1}^{M} \theta_j{}^{2q} \int_0^1 (1-u)k_q''(\delta^q\theta_j{}^q\zeta u)\mathrm{d}u$$

and k_q denotes the log Laplace transform of $c_q\xi^{2q}$ for ξ a standard normal random variable. $\square$

Proof Note first that u_{0j} is distributed as $N(0,\tau_j)$. Hence

$$c_q[y^*_{0\delta}]^{[2q]}(t) = c_q\sum_{j=1}^{M} u_{0j}^{2q} \sim \sum_{j=1}^{M} c_q\xi_j^{2q}\tau_j^q$$

where the ξ_j are independent copies of the standard normal variate ξ and $\sim$ means 'distributed as'. Since $\mathrm{E}\{c_q\xi^{2q}\} = 1$ we find, using (3.3), that

$$\begin{aligned}\log \mathrm{E}\{\exp(i\zeta D_0)\} &= -\zeta^2\sum_{j=1}^{M}\tau_j^{2q}\int_0^1(1-u)k_q''(\tau_j^q\zeta u)\mathrm{d}u\\ &= -\zeta^2\delta^{2q}\sum_{j=1}^{M}\theta_j{}^{2q}\int_0^1(1-u)k_q''(\delta^q\theta_j{}^q\zeta u)\mathrm{d}u\end{aligned}$$

$\square$

Lemma 3.2 For $\delta \downarrow 0$

$$\delta^{-q+1/2}\left\{c_q[y^*_{0\delta}]^{[2q]}(t) - [\tau^*_\delta]^{[q]}(t)\right\} \xrightarrow{\mathcal{L}} N(0, k_q''(0)\tau^{2q*}(t)).$$

$\square$

Proof From Lemma 3.1 we find

$$\log \mathrm{E}\{\exp(i\zeta\delta^{-q+1/2}D_0)\} = -\frac{1}{2}\zeta^2\delta R,$$

with

$$R = 2\sum_{j=1}^{M}\theta_j{}^{2q}\int_0^1(1-u)k_q''(\delta^q\theta_j{}^q\zeta u)\mathrm{d}u.$$

By the boundedness of τ on $[0,1]$ we have

$$\lim_{\delta\downarrow 0}\delta^q\max_j\theta_j{}^q = 0$$

and hence, for $\delta \downarrow 0$,

$$\delta R \to k_q''(0)\tau^{2q*}(t).$$

Therefore

$$\log \mathrm{E}\{\exp(i\zeta\delta^{-q+1/2}D_0)\} = -\frac{1}{2}\zeta^2 k_q''(0)\tau^{2q*}(t) + o(1) \tag{3.4}$$

and the lemma follows.

□

Lemma 3.3 For $\delta \downarrow 0$,

$$\log \mathrm{E}\{\exp(i\zeta\delta^{-1/2}([y_{0\delta}^*](t) - \tau^*(t))\} = -\frac{1}{2}\zeta^2 2\tau^{2*}(t) + O(\delta^{1/2}).$$

□

Proof This follows from (3.4) on setting $q = 1$ and noting that $k_1''(0) = \mathrm{Var}\{\xi^2\} = 2$, $[y_{0\delta}^*]^{[2]}(t) = [y_{0\delta}^*](t)$, and

$$[\tau_\delta^*]^{[1]}(t) = \tau^*(\lfloor t/\delta \rfloor\, \delta) = \tau^*(t) + O(\delta).$$

□

Lemma 3.4 For $\delta \downarrow 0$,

$$\delta^{-q}\{[y_\delta^*]^{[2q]}(t) - [y_{0\delta}^*]^{[2q]}(t)\} = O_p(1).$$

□

Proof By formula (3.2) the left hand side, say L, in the above formula may be written

$$L = L_1 + L_2$$

where

$$L_1 = 2q\delta^{-q}\sum_{j=1}^{M}\varepsilon_j u_{0j}^{2q-1}$$

$$L_2 = \delta^{-q}\sum_{k=2}^{2q}\binom{2q}{k}\sum_{j=1}^{M}{\varepsilon_j}^k {u_{0j}}^{2q-k}.$$

Recall that $u_{0j} \sim N(0, \tau_j)$ and let $\mu_r = \mathrm{E}\{|\xi|^r\}$ for $r > 0$ and $\xi \sim N(0,1)$. Then, with $m_q = \max_{0\le l\le 2q-2}\mu_l$ we have

$$\mathrm{E}\{|L_2|\} \le m_q\tbinom{2q}{q}\sum_{k=2}^{2q}\delta^{k/2}\sum_{j=1}^{M}(\delta^{-1}|\varepsilon_j|)^k(\delta^{-1}\tau_j)^{q-k/2}.$$

On account of condition **(C)** there exists a constant $c > 1$ such that, for all δ,

$$\max_{1\le j\le M}\delta^{-1}|\varepsilon_j| \le c \tag{3.5}$$

and hence, for $\delta < 1$,

$$\begin{aligned}
\mathrm{E}\{|L_2|\} &\le m_q \tbinom{2q}{q} c^{2q} \delta^{-1} \sum_{k=2}^{2q} \delta^{k/2} \sum_{j=1}^{M} \delta^{-(q-k/2)+1} \tau_j{}^{q-k/2} \\
&\le m_q \tbinom{2q}{q} c^{2q} K,
\end{aligned}$$

where

$$\begin{aligned}
K &= \sum_{k=2}^{2q} \delta^{-(q-k/2)+1} [\tau_\delta^*]^{[q-k/2]}(t) \\
&\to \sum_{k=2}^{2q} \tau^{(q-k/2)*}(t)
\end{aligned}$$

for $\delta \downarrow 0$ (and where $\tau^{(0)*}(t) = t$). Consequently $L_2 = O_p(1)$.

The mean of L_1 is 0 and for its variance we find, using again (3.5),

$$\begin{aligned}
\mathrm{Var}\{L_1\} &= 4q^2 \mu_{4q-2} \delta^{-2q} \sum_{j=1}^{M} \varepsilon_j^2 \tau_j^{2q-1} \\
&\le 4q^2 \mu_{4q-2} c^2 \sum_{j=1}^{M} \theta_j{}^{2q-1} \delta
\end{aligned}$$

and since

$$\sum_{j=1}^{M} \theta_j{}^{2q-1} \delta \to \tau^{(2q-1)*}(t),$$

also $L_1 = O_p(1)$, and the proof is complete. □

Proof of Theorem 2.2 We have

$$\begin{aligned}
\delta^{-q+1} c_q [y_\delta^*]^{[2q]}(t) - \tau^{q*}(t) &= \delta^{-q+1} c_q \{[y_\delta^*]^{[2q]}(t) - [y_{0\delta}^*]^{[2q]}(t)\} \\
&\quad + \delta^{-q+1} \left\{ c_q [y_{0\delta}^*]^{[2q]}(t) - [\tau_\delta^*]^{[q]}(t) \right\} \\
&\quad + \delta^{-q+1} \{[\tau_\delta^*]^{[q]}(t) - \tau^{q*}(t)\}
\end{aligned}$$

and the result now follows as a consequence of Lemmas 3.4 and 3.2 and Theorem 2.1.

□

Proof of Theorem 2.3 By Lemma 3.3

$$\frac{\delta^{-1/2}\{[y_{0\delta}^*](t) - \tau^*(t)\}}{\sqrt{2\tau^{2*}(t)}} \xrightarrow{\mathcal{L}} -\frac{1}{2}\zeta^2$$

and since, by Lemma 3.4,

$$\delta^{-1/2}\{[y_\delta^*](t) - [y_{0\delta}^*](t)\} = O_p(\delta^{1/2})$$

we have that $\delta^{-1/2}\{[y_\delta^*](t) - \tau^*(t)\}$ has the same limit law as $\delta^{-1/2}\{[y_{0\delta}^*](t) - \tau^*(t)\}$, i.e.

$$\frac{[y_\delta^*](t) - \tau^*(t)}{\sqrt{\delta 2\tau^{2*}(t)}} \xrightarrow{\mathcal{L}} -\frac{1}{2}\zeta^2$$

Finally, Theorem 2.2 with $q = 2$ shows that $\delta^{-1}\frac{1}{3}[y_\delta^*]^{[2q]}(t)$ is a consistent estimator of $\tau^{2*}(t)$ and this implies (2.3).

□

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