Power Variation and Time Change

Ole E. Barndorff-Nielsen
The Centre for Mathematical Physics and Stochastics (MaPhySto), University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark
oebn@imf.au.dk

Neil Shephard
Nuffield College, University of Oxford, Oxford OX1 1NF, UK
neil.shephard@nuf.ox.ac.uk

December 16, 2002

Abstract

This paper provides limit distribution results for power variation, that is sums of powers of absolute increments, for certain types of time-changed Brownian motion and α-stable processes. Special cases of these processes are stochastic volatility models used extensively in financial econometrics.

Keywords: Power variation; r-variation; Realised variance; Semimartingales; Stochastic volatility; Time-change.

Contents

1. Introduction

In Barndorff-Nielsen and Shephard (2003) limit distribution results were derived for quantities of the form

$$\sum_{j=1}^{n} |X(j\delta) - X((j-1)\delta)|^r, \tag{1.1}$$

where X denotes a certain type of semimartingale, r is a positive number and $n\delta = t$ for some time $t > 0$. The theory is based on $\delta \downarrow 0$. More specifically, X was assumed to be a special semimartingale with canonical decomposition of the form

$$X = A + H \bullet W, \tag{1.2}$$

where $\bullet$ indicates stochastic integration, W is Brownian motion, A and H are assumed to be jointly independent of W and to satisfy some rather mild regularity conditions, specified in Section 3 below. We refer to (1.1) and similar quantities as *power variations*. From the applied point of view the results in question provide, in particular, a versatile basis for drawing inference on the process H, which expresses the *volatility* of X. This is discussed in Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2003) and Barndorff-Nielsen and Shephard (2002b). See also Shiryaev (1999, p. 349-350) who mentions interest in the limits of sums of absolute returns.

The present paper extends the results mentioned in several directions (assuming for simplicity that $A = 0$). We consider more general time changes than those implicit in (1.2) (via the Dambis-Dubins-Schwarz theorem), as well as nonequidistant partitions of $[0, t]$. Furthermore, settings where instead of Brownian motion W in (1.2) we have a stable process will be discussed.

The structure of the paper is as follows. In Section 2 we review various elementary inequalities involving sums, and remind our selves of the definitions of power variation and time-change. Section 3 reviews the existing literature on power variation and stochastic volatility models. The latter constitute, in fact, an important subclass of the semimartingales. In Section 4 we establish some consequences of the general central limit theory which will be useful further on. Section 5 is the core of our paper. It is where we derive new limit law results for power variation in the case of Brownian motion which is time-changed. This Section also gives our new treatment of power variation for unequally spaced time intervals. In Section 6 we extend part of the above work to cover the situation where the Brownian motion is generalised to be a symmetric α-stable process. Finally, Section 7 provides a discussion of related work.

2. Variations

Throughout this paper, r denotes a positive number.

2.1. Some elementary inequalities

For later use we here list a few elementary inequalities.

(i) If $a_1, ..., a_n$ are nonnegative then (see Hardy, Littlewood, and Polya (1959, p. 30))

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \tag{2.1}$$

is decreasing in r (unless all but one of $a_1, ..., a_n$ is 0).

(ii) *Minkovsky's inequality* If $a_1, ..., a_n$ and $b_1, ..., b_n$ are nonnegative then for $r > 1$

$$\left(\sum_{j=1}^{n}(a_j+b_j)^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^r\right)^{1/r} + \left(\sum_{j=1}^{n} b_j^r\right)^{1/r} \tag{2.2}$$

while for $r < 1$

$$\left(\sum_{j=1}^{n}(a_j+b_j)^r\right)^{1/r} \geq \left(\sum_{j=1}^{n} a_j^r\right)^{1/r} + \left(\sum_{j=1}^{n} b_j^r\right)^{1/r}. \tag{2.3}$$

(iii) If $r \geq 1$ then, for arbitrary real $a_1, ..., a_n$ and $b_1, ..., b_n$,

$$\left|\left(\sum_{j=1}^{n}|a_j+b_j|^r\right)^{1/r} - \left(\sum_{j=1}^{n}|b_j|^r\right)^{1/r}\right| \leq \left(\sum_{j=1}^{n}|a_j|^r\right)^{1/r}. \tag{2.4}$$

To prove the latter inequality, suppose first that $r = 1$. For a and b real we have

$$|a+b|-|b| = \begin{cases} a & \text{for} \quad a \geq 0, b \geq 0 \\ a & \text{for} \quad a < 0, b \geq 0, a+b \geq 0 \\ -a-2b & \text{for} \quad a < 0, b \geq 0, a+b < 0 \\ a+2b & \text{for} \quad a > 0, b \leq 0, a+b \geq 0 \\ -a & \text{for} \quad a > 0, b \leq 0, a+b < 0 \\ -a & \text{for} \quad a \leq 0, b \leq 0 \end{cases}$$

implying $||a+b|-|b|| \leq |a|$.

Next, for $r > 1$ we find by (2.2)

$$\begin{aligned} \left(\sum_{j=1}^{n}|a_j+b_j|^r\right)^{1/r} &\leq \left(\sum_{j=1}^{n}(|a_j|+|b_j|)^r\right)^{1/r} \\ &\leq \left(\sum_{j=1}^{n}|a_j|^r\right)^{1/r} + \left(\sum_{j=1}^{n}|b_j|^r\right)^{1/r} \end{aligned}$$

so that

$$\left(\sum_{j=1}^{n}|a_j+b_j|^r\right)^{1/r} - \left(\sum_{j=1}^{n}|b_j|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n}|a_j|^r\right)^{1/r}. \tag{2.5}$$

On the other hand,

$$\begin{aligned} \left(\sum_{j=1}^{n}|b_j|^r\right)^{1/r} &= \left(\sum_{j=1}^{n}|a_j+b_j-a_j|^r\right)^{1/r} \\ &\leq \left(\sum_{j=1}^{n}(|a_j+b_j|+|a_j|)^r\right)^{1/r} \\ &\leq \left(\sum_{j=1}^{n}|a_j+b_j|^r\right)^{1/r} + \left(\sum_{j=1}^{n}|a_j|^r\right)^{1/r}, \end{aligned}$$

implying that

$$\left(\sum_{j=1}^{n}|b_j|^r\right)^{1/r}-\left(\sum_{j=1}^{n}|a_j+b_j|^r\right)^{1/r}\leq\left(\sum_{j=1}^{n}|a_j|^r\right)^{1/r} \tag{2.6}$$

and (2.5) and (2.6) together give (2.4).

2.2. Power variation

Let Δ denote a subdivision $0=t_0<t_1<\cdots<t_n=t$ of $[0,t]$ and let $\delta_j=t_j-t_{j-1}$ and $|\Delta|=\max\delta_j$. When considering a sequence of such subdivisions Δ we say that the sequence is *balanced* if $\max\delta_j/\min\delta_j$ is bounded above and ε-*balanced*, $\varepsilon\in(0,1)$, if $\max\delta_j/(\min\delta_j)^{\varepsilon}\to 0$ as $|\Delta|\to 0$. Note that here and in the following we usually have in mind a single, generally unspecified, sequence of subdivisions Δ with $|\Delta|\to 0$; however, for notational simplicity, we do not indicate this by attaching a sequence index to Δ. Clearly, if Δ is balanced then it is a fortiori ε-balanced for every $\varepsilon\in(0,1)$.

We consider arbitrary real functions f and g on the interval $[0,t]$ and introduce the notation

$$[f_\Delta]^{[r]}=\sum|f(t_j)-f(t_{j-1})|^r \tag{2.7}$$

where the sum is over $j=1,...,n$. We call $[f_\Delta]^{[r]}$ the *r-th order power variation* of f relative to Δ, or *r-tic variation* for short. In the special case where the subdivision Δ is equidistant, whence $\delta_j=\delta$ for all j, we will write f_δ instead of f_Δ, etc. Thus when δ occurs as an index the subdivision is understood to be equidistant. Furthermore, we write $[f]^{[r]}$ for the r-th order *sup-variation* or *sup-r-variation*[1] of f, that is

$$[f]^{[r]}=\sup_{\Delta\in\mathcal{D}}\sum|f(t_j)-f(t_{j-1})|^r \tag{2.8}$$

where $\mathcal{D}$ denotes the class of all possible subdivisions of $[0,t]$. When we wish to indicate the dependence on t we shall write $[f_\Delta]^{[r]}(t)$ instead of $[f_\Delta]^{[r]}$, etc.

For $r\geq 1$ we have, by the inequalities (2.2) and (2.4), that

$$\left([(f+g)_\Delta]^{[r]}\right)^{1/r}\leq\left([f_\Delta]^{[r]}\right)^{1/r}+\left([g_\Delta]^{[r]}\right)^{1/r}$$

and

$$\left|\left([(f+g)_\Delta]^{[r]}\right)^{1/r}-\left([f_\Delta]^{[r]}\right)^{1/r}\right|\leq\left([g_\Delta]^{[r]}\right)^{1/r}.$$

It is furthermore convenient to generalise the above setup to allow for weighted power variations, as follows. For B a function of locally bounded variation we let

$$[f_\Delta]^{[B,r]}=\sum|f(t_j)-f(t_{j-1})|^r\,(B(t_j)-B(t_{j-1})).$$

In particular,

$$[f_\delta]^{[B,r]}=\sum|f(j\delta)-f((j-1)\delta)|^r\,(B(j\delta)-B((j-1)\delta)).$$

Finally, when $f\geq 0$, we use the notation

$$f^*(t)=\int_0^t f(s)\mathrm{d}s \tag{2.9}$$

and, more generally,

$$f^{r*}(t)=\int_0^t f^r(s)\mathrm{d}s. \tag{2.10}$$

[1] We adopt this term rather than the more usual *r-variation*, for clarity in the context of the present paper. We will refer to some of the literature on r-variation in Section 7 of this paper.

2.3. Time-change

We define a *time-change* to be a non-decreasing function $T : [0, \infty) \to [0, \infty)$ with $T(0) = 0$ and $T(t) \to \infty$ as $t \to \infty$.

For an arbitrary function f (as above) and time-change T we have

$$[(f \circ T)_\Delta]^{[r]} = [f_{T(\Delta)}]^{[r]} \circ T$$

(where $\circ$ means composition of mappings) or, more specifically,

$$[(f \circ T)_\Delta]^{[r]}(t) = [f_{T(\Delta)}]^{[r]}(T(t))$$

where $T(\Delta)$ is the subdivision $0 = T(t_1) < \cdots < T(t_n) = T(t)$.

Henceforth, unless otherwise mentioned, we assume that T is continuous and strictly increasing. Then T is uniformly continuous on any compact interval and $|\Delta| \to 0$ will imply $|T(\Delta)| \to 0$. Hence, in wide generality it will hold that

$$[(f \circ T)]^{[r]} = [f]^{[r]} \circ T. \tag{2.11}$$

3. Review of power variation for SV models

Let X denote an arbitrary semimartingale, with decomposition $X = A + M$ into a process of bounded variation A and a local martingale M. Then X is said to be a stochastic volatility semimartingale model, or an $\mathcal{SVSM}$ model, provided M is of the form

$$M = H \bullet W$$

where W is a Brownian motion and H is a nonnegative process representing the stochastic volatility.

For reference below we introduce the following three conditions on H and A.

(R) The processes A and H are pathwise locally Riemann integrable (hence, in particular, locally bounded).

(V) The volatility process H is (pathwise) locally bounded away from 0 and has, moreover, the property that for some $\gamma > 0$

$$\lim_{\delta \downarrow 0} \delta^{1/2} \sum_{j=1}^{m} |H^\gamma(\eta_j) - H^\gamma(\xi_j)| = 0 \tag{3.1}$$

for any sequences $\xi_j = \xi_j(\delta)$ and $\eta_j = \eta_j(\delta)$ satisfying

$$0 \le \xi_1 \le \eta_1 \le \delta \le \xi_2 \le \eta_2 \le 2\delta \le \cdots \le \xi_n \le \eta_n \le t.$$

(M) The mean process A satisfies (pathwise)

$$\overline{\lim_{\delta \downarrow 0}} \max_{1 \le j \le n} \delta^{-1} |A(j\delta) - A((j-1)\delta)| < \infty.$$

Note When condition **(R)** holds the equality (3.1) is satisfied for all positive γ if and only if it holds for one such γ. $\square$

In Barndorff-Nielsen and Shephard (2003) the following result was proved (recall the notation (2.10)).

Theorem 3.1 Suppose the semimartingale $X = A + H \bullet W$ satisfies conditions **(R)**, **(V)** and **(M)**, and assume that the pair of processes (A, H) is independent of the Brownian motion W.

Then, for any $t > 0$ and $\delta \downarrow 0$, we have

$$\delta^{1-r/2}[X_\delta]^{[r]}(t) \xrightarrow{p} \mu_r H^{r*}(t)$$

where $\mu_r = \mathrm{E}\{|u|^r\}$ and $u \sim N(0,1)$.

Furthermore, if either $r \geq \frac{1}{2}$ or $A = 0$ then, additionally

$$\frac{\delta^{1-r/2}[X_\delta]^{[r]}(t) - \mu_r H^{r*}(t)}{\delta^{1-r/2}\sqrt{\mu_{2r}^{-1} v_r [X_\delta]^{[2r]}(t)}} \overset{law}{\to} N(0,1) \tag{3.2}$$

where $v_r = \mathrm{Var}\{|u|^r\}$ is the variance of $|u|^r$. □

Thus, in particular, we have that

$$\frac{[X_\delta]^{[2]}(t) - H^{2*}(t)}{\sqrt{\frac{2}{3}[X_\delta]^{[4]}(t)}} \overset{law}{\to} N(0,1) \tag{3.3}$$

and

$$\frac{\delta^{1/2}[X_\delta]^{[1]}(t) - \sqrt{2/\pi}H^*(t)}{\sqrt{(1-2/\pi)\delta[X_\delta]^{[2]}(t)}} \overset{law}{\to} N(0,1). \tag{3.4}$$

In Section 5 we will extend these results (assuming for simplicity that $A = 0$), by using a line of argument different from that applied in Barndorff-Nielsen and Shephard (2003). As a preliminary to this, the next Section lists some Central Limit Theory results.

Remark Relation (3.2) may be rewritten as

$$\frac{\delta^{1-r/2}\mu_r^{-1}[X_\delta]^{[r]}(t) - H^{r*}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2}\nu_r\mu_{2r}^{-1}\delta^{1-r}[X_\delta]^{[2r]}(t)}} \overset{law}{\to} N(0,1).$$

Here,

$$\mu_{2r}^{-1}\delta^{1-r}[X_\delta]^{[2r]}(t) \xrightarrow{p} H^{2r*}(t)$$

so that

$$\frac{\delta^{1-r/2}\mu_r^{-1}[X_\delta]^{[r]}(t) - H^{r*}(t)}{\delta^{1/2}\sqrt{\mu_r^{-2}\nu_r H^{2r*}(t)}} \overset{law}{\to} N(0,1).$$

In other words, $\delta^{1-r/2}\mu_r^{-1}[X_\delta]^{[r]}(t) - H^{r*}(t)$ follows asymptotically a mixed normal distribution. □

4. Some Central Limit Theory results

We shall need the following special cases of the general central limit theory.

Let $y_{n1}, ..., y_{nk_n}$ ($n = 1, 2, ...,$ with $k_n \to \infty$ as $n \to \infty$) be a triangular array of independent random variables and let $y_n = y_{n1} + \cdots + y_{nk_n}$.

4.1. Asymptotic normality

Theorem 4.1 (Gnedenko and Kolmogorov (1954, p. 102-103)) Suppose that $\mathrm{E}\{y_{nj}\} = 0$ for all n and j and that $\mathrm{Var}\{y_n\} = 1$ for all n. Then $y_n \stackrel{law}{\to} N(0,1)$ if and only if for arbitrary $\gamma > 0$

$$\sum_{j=1}^{k_n} \mathrm{E}\{y_{nj}^2 \mathbf{1}_{(\gamma,\infty)}(|y_{nj}|)\} \to 0. \tag{4.1}$$

Corollary 4.1 Suppose that y_{nj} is of the form $y_{nj} = c_{nj}x_{nj}$ where the c_{nj} are real constants and the x_{nj} are independent copies of a random variable x that has mean 0 and variance 1. If $c_{n1}^2 + \cdots + c_{nk_n}^2 = 1$ and $c_n = \max_j c_{nj} \to 0$ as $n \to \infty$ then y_n converges in law to the standard normal distribution $N(0,1)$. $\square$

Proof In the present case

$$\begin{aligned}\sum_{j=1}^{k_n} \mathrm{E}\{y_{nj}^2 \mathbf{1}_{(\gamma,\infty)}(|y_{nj}|)\} &= \sum_{j=1}^{k_n} c_{nj}^2 \mathrm{E}\{x^2 \mathbf{1}_{(\gamma,\infty)}(|c_{nj}x|)\} \\ &\leq \mathrm{E}\{x^2 \mathbf{1}_{(c_n^{-1}\gamma,\infty)}(|x|)\} \to 0\end{aligned}$$

and hence Theorem 4.1 applies. $\square$

4.2. Probability limit results

Theorem 4.2 *Degenerate Convergence Criterion* (Loève (1977, p. 329)) We have that $y_n \stackrel{p}{\to} 0$ and the uniform asymptotic neglibility condition is satisfied if and only if for every $\varepsilon > 0$ and for some $\gamma > 0$

$$\sum_{j=1}^{k_n} P\{|y_{nj}| \geq \varepsilon\} \to 0 \tag{4.2}$$

$$\sum_{j=1}^{k_n} \mathrm{E}\{y_{nj} \mathbf{1}_{(-\gamma,\gamma)}(y_{nj})\} \to 0 \tag{4.3}$$

and

$$\sum_{j=1}^{k_n} \left(\mathrm{E}\{y_{nj}^2 \mathbf{1}_{(-\gamma,\gamma))}(y_{nj})\} - \mathrm{E}\{y_{nj}\mathbf{1}_{(-\gamma,\gamma)}(y_{nj})\}^2 \right) \to 0 \tag{4.4}$$

for $n \to \infty$. $\square$

Now, let x_{nj}, $n = 1, 2, ...$, $j = 1, 2, ..., k_n$ be independent copies of a random variable x having distribution function F and mean 0, suppose that c_{ni} are arbitrary positive reals and let

$$y_{nj} = c_{nj}x_{nj}$$

and $y_n = y_{n1} + \cdots + y_{nk_n}$.

Corollary 4.2 Suppose that x has mean 0, let $c_n = \max_j c_{nj}$ and assume that, as $n \to \infty$,

$$c_n \to 0 \tag{4.5}$$

$$k_n P\{|x| \geq c_n^{-1}\varepsilon\} \to 0 \tag{4.6}$$

$$\sup_n \sum_{j=1}^{k_n} c_{nj} < \infty \tag{4.7}$$

and, for some $\gamma > 0$,

$$c_n \int_{-c_n^{-1}\gamma}^{c_n^{-1}\gamma} \xi^2 \mathrm{d}F(\xi) \to 0. \tag{4.8}$$

Then $y_n \xrightarrow{p} 0$. $\square$

Proof In the present setting the conditions of Theorem 4.2 take the form

$$\sum_{j=1}^{k_n} P\{|x| \geq c_{nj}^{-1}\varepsilon\} \to 0 \tag{4.9}$$

$$\sum_{j=1}^{k_n} c_{nj} \int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi \mathrm{d}F(\xi) \to 0 \tag{4.10}$$

and

$$\sum_{j=1}^{k_n} c_{nj}^2 \left(\int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi^2 \mathrm{d}F(\xi) - \left(\int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi \mathrm{d}F(\xi) \right)^2 \right) \to 0. \tag{4.11}$$

The first of these conditions is implied by (4.5) and (4.6). Next, since $\mathrm{E}\{x\} = 0$ and $c_n \to 0$,

$$\int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi \mathrm{d}F(\xi) \to 0$$

uniformly in j. Combined with (4.7) the latter entails (4.10) and also

$$\sum_{j=1}^{k_n} c_{nj}^2 \left(\int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi \mathrm{d}F(\xi) \right)^2 \to 0.$$

Finally (4.8) gives

$$\sum_{j=1}^{k_n} c_{nj}^2 \int_{-c_{nj}^{-1}\gamma}^{c_{nj}^{-1}\gamma} \xi^2 \mathrm{d}F(\xi) \leq c_n \int_{-c_n^{-1}\gamma}^{c_n^{-1}\gamma} \xi^2 \mathrm{d}F(\xi) \sum_{j=1}^{k_n} c_{nj} \to 0.$$

$\square$

Corollary 4.3 Suppose that x has mean 0 and finite variance and assume that

$$c_n \to 0 \tag{4.12}$$

$$k_n P\{|x| \geq c_n^{-1}\varepsilon\} \to 0 \tag{4.13}$$

$$\sup_n \sum_{j=1}^{k_n} c_{nj} < \infty. \tag{4.14}$$

Then $y_n \xrightarrow{p} 0$. $\square$

Proof Condition (4.8) follows from the assumed finiteness of $\mathrm{Var}\{x\}$. $\square$

5. Power variation and time changed Brownian motion

Our focus in this Section is on time changed Brownian motion, that is we are considering local martingales of the form

$$X = B \circ T \tag{5.1}$$

and we aim to generalise the results of Theorem 3.1 to this setting, moreover allowing the subdivisions Δ to be non-equidistant.

SV models, as discussed in Section 3 but with $A = 0$, i.e.

$$X = H \bullet W \tag{5.2}$$

fall within this group. In fact, supposing that

$$H^{2*}(t) = \int_0^t H^2(s)\mathrm{d}s \to \infty$$

for $t \to \infty$ we have, by the Dambis-Dubins Schwarz theorem[2], that the process $X = H \bullet W$ can be reexpressed a.s. as $B \circ T$ where $T = H^{2*}$ and the Brownian motion B is defined from X by $B = X \circ \overleftarrow{T}$ where denotes the inverse of the time change of T. (Of course, T and $\overleftarrow{T}$ are themselves determined by X since $T = H^{2*} = [X]$, the quadratic variation of X.) It will be notationally convenient to write Q for H^2, and then $Q^* = T$.

As before, we assume that the time-change T is continuous and strictly increasing, and Δ stands for a subdivision $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$ (with t and n suppressed in some of the notation). Further, in line with Theorem 3.1, we assume that T is independent of B, and therefore we may argue conditionally on T. Otherwise put, we may consider T to be deterministic.

5.1. Limit laws

Letting

$$T_{\Delta j} = T(t_j) - T(t_{j-1})$$

we have

$$[X_\Delta]^{[r]} - \mu_r [T_\Delta]^{[r/2]} \overset{law}{=} \sum_{j=1}^{M} T_{\Delta j}^{r/2}(|u_j|^r - \mu_r)$$

where $u_1, ..., u_n$ are independent copies of a standard normal variate u (and, as before, $\mu_r = \mathrm{E}\{|u|^r\}$). Consequently,

$$\frac{[X_\Delta]^{[r]} - \mu_r [T_\Delta]^{[r/2]}}{\sqrt{v_r [T_\Delta]^{[r]}}} \overset{law}{=} y_\Delta$$

where $y_\Delta = y_{\Delta 1} + \cdots + y_{\Delta n}$ and $y_{\Delta j} = c_{\Delta j} x_{\Delta j}$ with

$$c_{\Delta j} = \frac{T_{\Delta j}^{r/2}}{\sqrt{[T_\Delta]^{[r]}}}$$

and $x_{\Delta j} \overset{law}{=} (|u|^r - \mu_r)/\sqrt{v_r}$. By Corollary 4.1 we obtain

Theorem 5.1 Suppose that

[2]The extension of this theorem to the case where instead of the Brownian motions W and B one considers stable processes is discussed in the recent paper by Kallsen and Shiryaev (2002), the results of which are summarised in Section 6 below.

$$\frac{\max_j T_{\Delta j}^{r/2}}{\sqrt{[T_\Delta]^{[r]}}} \to 0 \tag{5.3}$$

as $|\Delta| \to 0$. Then

$$\frac{[X_\Delta]^{[r]} - \mu_r [T_\Delta]^{[r/2]}}{\sqrt{v_r [T_\Delta]^{[r]}}} \xrightarrow{law} N(0,1). \tag{5.4}$$

□

Note that the statements in Theorem 5.1 as well as in later propositions refer to a single, but arbitrary, sequence of subdivisions Δ with $|\Delta| \to 0$.

Example 5.1 Suppose $r = 1$. Then $[T_\Delta]^{[r]} = T(t)$ and, since T is uniformly continuous on $[0,t]$, condition (5.3) holds. More generally, since $\left([T_\Delta]^{[r]}\right)^{1/r}$ is decreasing in r (cf. point (i) in Subsection 2.1) we have for $r \leq 1$ that

$$\sqrt{[T_\Delta]^{[r]}} \geq T(t)^{r/2}$$

and hence (5.3) is, in fact, valid for all $0 < r \leq 1$. □

Note that, writing $\hat{T}_\Delta = \max_j T_{\Delta j}$ we have

$$\frac{\max_j T_{\Delta j}^{r/2}}{\sqrt{[T_\Delta]^{[r]}}} = \left\{\sum (T_{\Delta j}/\hat{T}_\Delta)^r\right\}^{-1/2}.$$

Example 5.2 Suppose $T(s) = s^\psi$ for some $\psi \in (0,1)$ and, for simplicity, take $t = 1$. Taking Δ to be the equidistant subdivision determined by $t_j = j/n$ we have $\hat{T}_\Delta = n^{-\psi}$ and

$$\sum (T_{\Delta j}/\hat{T}_\Delta)^r = \sum \{j^\psi - (j-1)^\psi\}^r$$

where for large j

$$(j^\psi - (j-1)^\psi)^r \sim \psi^r j^{-(1-\psi)r}.$$

Consequently, if $(1-\psi)r > 1$ condition (5.3) is not satisfied. In particular, this is the case if $r = 2$ and $\psi < \frac{1}{2}$. □

Example 5.3 In case $T = Q^*$, where

$$Q^*(s) = \int_0^s Q(u)\mathrm{d}u$$

for some positive Riemann integrable function Q on $[0,t]$, we have

$$\underline{Q} \leq \Delta_j^{-1} T_{\Delta j} \leq \overline{Q}$$

where $\underline{Q}$ and $\overline{Q}$ are, respectively, the infimum and the supremum of Q over $[0,t]$. Suppose further that Q is bounded away from 0, i.e. $\underline{Q} > 0$.

Then we have

$$\frac{\max_j T_{\Delta j}^{r/2}}{\sqrt{[T_\Delta]^{[r]}}} \leq \frac{1}{\sqrt{n}} \left(\frac{\max \delta_j}{\min \delta_j}\right)^{r/2} (\overline{Q}/\underline{Q})/^{r/2} \to 0$$

and it follows that condition (5.3) is satisfied and Theorem 5.1 applies if $\max \delta_j / \min \delta_j$ is bounded above, as is the case in particular if the subdivision Δ is equidistant. $\square$

Now suppose that $[T_\Delta]^{[r/2]}$ converges as $|\Delta| \to 0$, with limit $[T]^{[r/2]}$, irrespectively of which sequence of subdivisions is considered. It is then of interest to consider conditions under which

$$\frac{[X_\Delta]^{[r]} - \mu_r [T]^{[r/2]}}{\sqrt{v_r [T_\Delta]^{[r]}}} \overset{law}{\to} N(0,1). \tag{5.5}$$

Clearly this will be the case provided

$$\frac{[T_\Delta]^{[r/2]} - [T]^{[r/2]}}{\sqrt{[T_\Delta]^{[r]}}} \to 0$$

as $|\Delta| \to 0$. In particular, for $r = 2$ we have simply $[T_\Delta]^{[r/2]} = [T]^{[r/2]}$ and therefore the following Corollary to Theorem 5.1.

Corollary 5.1 If

$$\frac{\max_j T_{\Delta j}}{\sqrt{[T_\Delta]^{[2]}}} \to 0$$

as $|\Delta| \to 0$ then

$$\frac{[X_\Delta]^{[2]}(t) - T(t)}{\sqrt{2[T_\Delta]^{[2]}}} \overset{law}{\to} N(0,1).$$

$\square$

Example 5.4 Let the setting be as in Example 5.3 and assume that Δ is equidistant. Then for any $\gamma > 0$, by the Riemann integrability of Q,

$$\delta^{1-\gamma}[T_\Delta]^{[\gamma]} = \delta^{1-\gamma}[T_\delta]^{[\gamma]} = \delta^{1-\gamma} \sum \left(\int_{(j-1)\delta}^{j\delta} Q(s)\mathrm{d}s \right)^\gamma \to \int_0^t Q^\gamma(s)\mathrm{d}s = Q^{\gamma *}(t). \tag{5.6}$$

In the present case the left hand side of (5.5) can be rewritten

$$\frac{[X_\delta]^{[r]} - \mu_r [T_\delta]^{[r/2]}}{\sqrt{v_r [T_\delta]^{[r]}}} = \frac{\delta^{1-r/2}[X_\delta]^{[r]} - \mu_r \delta^{1-r/2}[T_\delta]^{[r/2]}}{\delta^{1/2}\sqrt{v_r \delta^{1-r}[T_\delta]^{[r]}}}$$

and therefore, if

$$\frac{\delta^{1-r/2}[T_\delta]^{[r/2]} - Q^{r/2*}}{\delta^{1/2}} \to 0 \tag{5.7}$$

as $\delta \to 0$, then

$$\frac{\delta^{1-r/2}[X_\delta]^{[r]} - \mu_r Q^{r/2*}}{\delta^{1/2}\sqrt{v_r Q^{r*}}} \overset{law}{\to} N(0,1). \tag{5.8}$$

Now, in view of (5.6),

$$\frac{\delta^{1-r/2}[T_\Delta]^{[r/2]} - Q^{r/2*}}{\delta^{1/2}} = \delta^{1/2} \sum (\phi_j^{r/2} - \psi_j^{r/2})$$

where ϕ_j and ψ_j satisfy

$$\int_{(j-1)\delta}^{j\delta} Q(s)\mathrm{d}s = \delta\phi_j \quad \text{and} \quad \int_{(j-1)\delta}^{j\delta} Q^{r/2}(s)\mathrm{d}s = \delta\psi_j^{r/2}$$

with both ϕ_j and ψ_j bounded above and below by the upper and lower limit of Q on the interval $[(j-1)\delta, j\delta]$. Thus (5.7) is in fact satisfied.

Note further that since (5.8) holds for arbitrary $r > 0$ we have that $\mu_{2r}^{-1}\delta^{1-r}[X_\delta]^{[2r]}$ provides a consistent estimator of Q^{r*}. Therefore, from (5.8) we obtain

$$\frac{\delta^{1-r/2}[X_\delta]^{[r]} - \mu_r Q^{r/2*}}{\delta^{1/2}\sqrt{v_r \mu_{2r}^{-1}\delta^{1-r}[X_\delta]^{[2r]}}} \stackrel{law}{\rightarrow} N(0,1).$$

We have thus rederived the result (3.2) (under the assumption that $A = 0$). In Subsection 5.3 we will go on to extend this result by looking at sums of the form $\sum \delta_j^{1-r/2}|X(t_j) - X(t_{j-1})|^r$. $\square$

5.2. Probability limits

In the setting of Example 5.4, where $X = H \bullet W$, we found that

$$\delta^{1-r/2}\left([X_\delta]^{[r]} - \mu_r [T_\delta]^{[r/2]}\right) \stackrel{p}{\rightarrow} 0$$

and here $T = Q^*$. We now consider the possibility of having a similar probability limit result for the more general case where $X = B \circ T$, cf. the introduction to the present Section.

We have

$$\delta^{1-r/2}\left([X_\delta]^{[r]} - \mu_r [T_\delta]^{[r/2]}\right) \stackrel{law}{=} y_n$$

where $y_n = y_{n1} + \cdots + y_{nn}$ with $y_{nj} = c_{nj}x_{nj}$,

$$c_{nj} = \delta^{1-r/2}T_{\delta j}^{r/2},$$

and where $\delta = n^{-1}$, $T_{\delta j} = T(j\delta) - T((j-1)\delta)$ and $x_{nj} \stackrel{law}{=} x$ with x given by $x = |u|^r - \mu_r$.

By Corollary 4.3, in order that $y_n \stackrel{p}{\rightarrow} 0$ it suffices, letting $\hat{T}_\delta = \max T_{\delta j}$, to have

$$\delta^{1-r/2}\hat{T}_\delta^{r/2} \rightarrow 0 \qquad (5.9)$$

and for every $\varepsilon > 0$

$$nP\{|x| \geq \delta^{-1+r/2}\hat{T}_\delta^{-r/2}\varepsilon\} \rightarrow 0 \qquad (5.10)$$

and

$$\sup_n \delta \sum_{j=1}^{n} \delta^{-r/2}T_{\delta j}^{r/2} < \infty. \qquad (5.11)$$

We are mostly interested in the range $0 < r < 2$. For that (5.9) is automatically satisfied. As regards (5.10), the tail behaviour of $|x|$ is equivalent to the tail behaviour of $|u|^r$ and

$$P(|u|^r > \xi) = 2(1 - \Phi(\xi^{1/r})) \sim \sqrt{\frac{2}{\pi}}\xi^{-1/r}e^{-\frac{1}{2}\xi^{2/r}}.$$

Thus

$$\begin{aligned} MP\{|x| &\geq \delta^{-1+r/2}T_M^{-r/2}\varepsilon\} \leq \delta^{-1}P(|x| > \delta^{-(1-r/2)}T(t)^{-r/2}\varepsilon) \\ &\sim \sqrt{\frac{2}{\pi}}T(t)^{1/2}\varepsilon^{-1/r}\delta^{-\frac{3}{2}+\frac{1}{r}}\exp(-\frac{1}{2}\varepsilon^{2/r}T(t)^{-1}\delta^{-(\frac{2}{r}-1)}) \\ &\rightarrow 0. \end{aligned}$$

Hence we have

Theorem 5.2 Let $0 < r < 2$ and $\bar{T}_{\delta j} = \delta^{-1} T_{\delta j}$. Suppose that

$$\sup_n \delta \sum_{j=1}^{n} \bar{T}_{\delta j}^{r/2} < \infty. \tag{5.12}$$

Then

$$\delta^{1-r/2} \left([X_\delta]^{[r]} - [T_\delta]^{[r/2]} \right) \xrightarrow{p} 0.$$

□

5.3. General subdivisions

We proceed to generalise Theorem 5.1 (supposing $A = 0$) to non-equidistant Δ. Thus, again, we assume X to be a process of the form $X = H \bullet W$, and $Q = H^2$. As before, convergence statements will refer to a sequence of subdivisions Δ with $|\Delta| \to 0$.

It is now convenient to introduce the notation

$$\overline{[X_\Delta]}^{[r]} = \sum \delta_j^{1-r/2} |X(t_j) - X(t_{j-1})|^r \tag{5.13}$$

and the condition

($\bar{\mathbf{V}}$) The volatility process H is (pathwise) bounded away from 0 and has, moreover, the property that for some $\gamma > 0$ (equivalently for all $\gamma > 0$)

$$\frac{\sum_{j=1}^{m} \delta_j |H^\gamma(\eta_j) - H^\gamma(\xi_j)|}{\sqrt{\min \delta_j}} \to 0 \tag{5.14}$$

for any sequences $\xi_j = \xi_j(\Delta)$ and $\eta_j = \eta_j(\Delta)$ satisfying

$$0 \le \xi_1 \le \eta_1 \le t_1 \le \xi_2 \le \eta_2 \le t_2 \le \cdots \le \xi_n \le \eta_n \le t.$$

In case Δ is equidistant condition **($\bar{\mathbf{V}}$)** reduces to condition **(V)**.

Now recall the definition of an ε-balanced sequence of subdivisions Δ, given in Subsection 2.2.

Theorem 5.3 Let X be a semimartingale of the form $X = H \bullet W$ and suppose that the volatility process H is independent of the Brownian motion W and satisfies conditions **(R)** and **($\bar{\mathbf{V}}$)**. Then, for any $t > 0$ and for any $\frac{1}{2}$-balanced sequence of subdivisions Δ we have

$$\overline{[X_\Delta]}^{[r]}(t) \xrightarrow{p} \mu_r H^{r*}(t) \tag{5.15}$$

as $|\Delta| \to 0$ and where $\mu_r = \mathrm{E}\{|u|^r\}$ and $u \sim N(0,1)$.

Furthermore, if the sequence of subdivisions Δ is $\frac{2}{3}$-balanced then

$$\frac{\overline{[X_\Delta]}^{[r]}(t) - \mu_r H^{r*}(t)}{\sqrt{\mu_{2r}^{-1} \nu_r \sum \delta_j^{2-r} |X(t_j) - X(t_{j-1})|^r(t)}} \xrightarrow{law} N(0,1) \tag{5.16}$$

where $\nu_r = \mathrm{Var}\{|u|^r\}$ is the variance of $|u|^r$. □

PROOF We have

$$\overline{[X_\Delta]}^{[r]} \stackrel{law}{=} \sum \delta_j^{1-r/2} |Q^*(t_j) - Q^*(t_{j-1})|^{r/2} |u_j|^r$$

where the u_j are i.i.d. standard normal. Hence, since

$$\overline{[Q^*_\Delta]}^{[r/2]} = \sum \delta_j^{1-r/2} |Q^*(t_j) - Q^*(t_{j-1})|^{r/2}$$

we find

$$\overline{[X_\Delta]}^{[r]} - \mu_r \overline{[Q^*_\Delta]}^{[r]} \stackrel{law}{=} \sum \delta_j^{1-r/2} |Q^*(t_j) - Q^*(t_{j-1})|^{r/2} (|u|^r - \mu_r)$$

and it follows from Corollary 4.1 that

$$\frac{\overline{[X_\Delta]}^{[r]}(t) - \mu_r \overline{[Q^*_\Delta]}^{[r/2]}(t)}{\sqrt{v_r \sum \delta_j^{2-r} |Q^*(t_j) - Q^{r*}(t_{j-1})|^r}} \stackrel{law}{\to} N(0,1) \tag{5.17}$$

provided

$$\frac{\max \left\{ \delta_j^{1-r/2} |Q^*(t_j) - Q^*(t_{j-1})|^{r/2} \right\}}{\sqrt{\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r}} \to 0. \tag{5.18}$$

To show that the latter is the case we note that

$$\begin{aligned} \frac{\max \left\{ \delta_j^{1-r/2} |Q^*(t_j) - Q^{r*}(t_{j-1})|^{r/2} \right\}}{\sqrt{\sum \delta_j^{2-r} |Q^*(t_j) - Q^{r*}(t_{j-1})|^r}} &\leq \frac{\max \delta_j}{\sqrt{\min \delta_j}} \frac{\max \left\{ \delta_j^{-r/2} |Q^*(t_j) - Q^{r*}(t_{j-1})|^{r/2} \right\}}{\sqrt{\sum \delta_j^{-r} |Q^*(t_j) - Q^{r*}(t_{j-1})|^r}} \\ &= \frac{\max \delta_j}{\sqrt{\min \delta_j}} \frac{\max \phi_j^{r/2}}{\sqrt{\sum \delta_j \phi_j^r}} \end{aligned} \tag{5.19}$$

where ϕ_j is given by

$$\delta_j^{-1} |Q^*(t_j) - Q^*(t_{j-1})| = \phi_j. \tag{5.20}$$

By condition **(R)**, $\phi_j \leq \sup_{0 \leq s \leq t} Q(s) < \infty$ and, for any $\gamma > 0$,

$$\sum \delta_j \phi_j^\gamma \to \int_0^t Q^\gamma(s) \mathrm{d}s = Q^{\gamma*}(t) \tag{5.21}$$

which, together with (5.19) and the assumption that the sequence of subdivisions Δ is $\frac{1}{2}$-balanced implies that (5.18) is fulfilled. Hence (5.17) has been shown to hold.

By (5.21) we also have

$$\overline{[Q^*_\Delta]}^{[\gamma]} \to Q^{\gamma*}(t) \tag{5.22}$$

for every $\gamma > 0$ and therefore, in view of (5.17), we will have

$$\frac{\overline{[X_\Delta]}^{[r]}(t) - \mu_r H^{r*}(t)}{\sqrt{v_r \sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r}} \stackrel{law}{\to} N(0,1) \tag{5.23}$$

provided

$$\frac{\overline{[Q^*_\Delta]}^{[r/2]}(t) - Q^{r/2*}(t)}{\sqrt{\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r}} \to 0. \tag{5.24}$$

The numerator in (5.24) may be rewritten as

$$\begin{aligned}\overline{[Q^*_\Delta]}^{[r/2]}(t) - Q^{r/2*}(t) &= \sum \delta_j \left(\delta_j^{-r/2} |Q^*(t_j) - Q^*(t_{j-1})|^{r/2} - \delta_j^{-1} \int_{t_{j-1}}^{t_j} Q^{r/2}(s)\mathrm{d}s \right) \\ &= \sum \delta_j \left(\phi_j^{r/2} - \psi_j^{r/2} \right)\end{aligned}$$

where ϕ_j was defined by (5.20) and

$$\psi_j = \left(\delta_j^{-1} \int_{t_{j-1}}^{t_j} Q^{r/2}(s)\mathrm{d}s \right)^{2/r}. \tag{5.25}$$

(For simplicity, we have suppressed the dependence of ψ_j on r in the notation.) For the denominator we have

$$\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r \geq \min \delta_j \sum \delta_k \phi_k^{r/2} \sim \min \delta_j Q^{r/2*}(t). \tag{5.26}$$

Thus

$$\frac{\overline{[Q^*_\Delta]}^{[r/2]}(t) - Q^{r/2*}(t)}{\sqrt{\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r}} \leq \frac{1}{\sqrt{\overline{[Q^*_\Delta]}^{[r]}(t)}} \frac{\sum \delta_j \left(\phi_j^{r/2} - \psi_j^{r/2} \right)}{\sqrt{\min \delta_j}}$$

and on account of (5.22) and condition $(\bar{\mathbf{V}})$ the right hand side tends to 0, verifying (5.24) and hence (5.23).

Since by (5.26) the denominator in (5.23) tends to 0 we have shown the first assertion in Theorem 5.3.

It remains to prove that, under the strengthened assumption that the sequence of subdivisions is $\frac{2}{3}$-balanced, we may substitute

$$\mu_{2r}^{-1} \sum \delta_j^{2-r} |X(t_j) - X(t_{j-1})|^r(t)$$

for

$$\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r \tag{5.27}$$

in (5.23). Noting that

$$\sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r = \sum \delta_j^2 \phi_j^r$$

and

$$\sum \delta_j^{2-r} |X(t_j) - X(t_{j-1})|^r - \mu_{2r} \sum \delta_j^{2-r} |Q^*(t_j) - Q^*(t_{j-1})|^r \overset{law}{=} \sum \delta_j^2 \phi_j^r (|u_j|^{2r} - \mu_{2r})$$

we must, in other words, prove that $\sum \delta_j^2 \phi_j^r (|u_j|^{2r} - \mu_{2r})$ is of smaller order of magnitude than $\sum \delta_j^2 \phi_j^r |u_j|^{2r}$, in probability as $|\Delta| \to 0$. For this it is enough to show that the standard variation of the former sum is of smaller order than the mean of the latter sum. The ratio of these two quantities is

$$\frac{\sqrt{\nu_{2r}}}{\mu_{2r}} \frac{\sqrt{\sum \delta_j^4 \phi_j^{2r}}}{\sum \delta_j^2 \phi_j^r}$$

where for the second ratio we have

$$\frac{\sqrt{\sum \delta_j^4 \phi_j^{2r}}}{\sum \delta_j^2 \phi_j^r} \leq \frac{(\max \delta_j)^{3/2}}{\min \delta_j} \frac{\sqrt{\sum \delta_j \phi_j^{2r}}}{\sum \delta_j \phi_j^r}$$

The second ratio on the right hand side of this inequality is bounded, by a previous argument, and the first ratio tends to 0 on account of the $\frac{2}{3}$-balancedness assumption. $\square$

Example 5.7 If the sequence of subdivisions Δ is balanced and if H is of local bounded variation then condition (5.14) is satisfied. The latter requirement is met in particular by the superpositions of OU processes used as models for H in Barndorff-Nielsen and Shephard (2001a), Barndorff-Nielsen and Shephard (2001b), cf. also Barndorff-Nielsen (2001) and Barndorff-Nielsen, Nicolato, and Shephard (2002). $\square$

5.4. Weighted variations

Suppose for the moment that the subdivision Δ is equidistant. Above we have discussed the asymptotic behaviour of sums $\sum |X(j\delta)-X((j-1)\delta)|^r$. It is sometimes of interest to allow for the single terms to occur with individual weights, thus considering sums of the form $\sum c_j|X(j\delta) - X((j-1)\delta)|^r$ where the b_j are chosen weights. In broad generality limit results like those established above will hold in this setting, as we shall now indicate.

Let B be an increasing function on $[0,\infty)$ with $B(0)=0$ and let

$$B_j = B(t_j) - B(t_{j-1}), \quad X_j = X(t_j) - X(t_{j-1}), \quad T_j = T(t_j) - T(t_{j-1}).$$

Then

$$\sum \delta_j^{-r/2}|X_j|^r B_j \overset{law}{=} \sum \delta_j^{-r/2} T_j^{r/2} B_j |u_j|^r$$

and it follows from Corollary 4.1 that

$$\frac{\sum \delta_j^{-r/2} T_j^{r/2} B_j \left(|u_j|^r - \mu_r\right)}{\sqrt{\nu_r \sum \delta_j^{-r} T_j^r B_j^2}} \overset{law}{\to} N(0,1)$$

provided

$$\frac{\max \delta_j^{-r/2} T_j^{r/2} B_j}{\sqrt{\sum \delta_j^{-r} T_j^r B_j^2}} \to 0 \tag{5.28}$$

as $|\Delta| \to 0$.

In case T is of the form Q^*, as above, we have

$$\sum \delta_j^{-r/2} T_j^{r/2} B_j \to \int_0^t Q^{r/2}(s)\mathrm{d}B(s)$$

and

$$\sum \delta_j^{-r} T_j^r B_j^2 \to 2\int_0^t Q^r(s)B(s)\mathrm{d}B(s).$$

Consequently, assuming (5.28) holds we will, subject to a further mild condition, similar to condition $(\bar{\mathbf{V}})$, have that

$$\frac{\sum \delta_j^{-r/2}|X_j|^r B_j - \mu_r \int_0^t Q^{r/2}(s)\mathrm{d}B(s)}{\sqrt{\nu_r 2\int_0^t Q^r(s)B(s)\mathrm{d}B(s)}} \overset{law}{\to} N(0,1).$$

Furthermore, the integral in the latter relation can be consistently estimated yielding as the final result

$$\frac{\sum \delta_j^{-r/2}|X_j|^r B_j - \mu_r \int_0^t Q^{r/2}(s)\mathrm{d}B(s)}{\sqrt{2\nu_r \mu_{2r}^{-1} \sum \delta_j^{-r}|X_j|^{2r} B_j^2}} \overset{law}{\to} N(0,1).$$

6. Power variation and time–changed stable processes

We now inquire into the question of the degree to which the results discussed above for time-changed Brownian motion can be extended to the class of α-stable processes. For simplicity we restrict attention to the case where X is of the form $X = H \bullet Z$ for some symmetric α-stable Lévy process, and we consider only equidistant subdivisions.

We first recall some known facts about symmetric α-stable processes. Let Z be the symmetric α-stable process with $0 < \alpha < 2$ and cumulant function

$$\mathrm{C}\{\zeta \ddagger Z(t)\} = \log \mathrm{E} e^{i\zeta Z(t)} = -t|\zeta|^{\alpha}. \tag{6.1}$$

This process is representable by subordination as

$$Z(t) \overset{law}{=} B(S(t)),$$

where S is the positive $\alpha/2$-stable subordinator with kumulant function

$$\bar{\mathrm{K}}\{\theta \ddagger S(t)\} = \log \mathrm{E} e^{-\theta S(t)} = -t(2\theta)^{\alpha/2}.$$

When $r < \alpha$, which is needed for the moments to exist, we will write

$$\mu_{\alpha,r} = \mathrm{E}\{|Z(1)|^r\} = \mu_r \mathrm{E}\left\{S(1)^{r/2}\right\}.$$

Furthermore, if H is a predictable process such that for all $t > 0$

$$\int_0^t |H|_s^{\alpha} \mathrm{d}s < \infty$$

and, for $t \to \infty$,

$$\int_0^t |H|_s^{\alpha} \mathrm{d}s \to \infty$$

then

$$H \bullet Z = \tilde{Z} \circ |H|^{\alpha *} \tag{6.2}$$

where $\tilde{Z}$ is a symmetric α-stable process and (in the previously established notation)

$$|H|_t^{\alpha *} = \int_0^t |H|_s^{\alpha} \mathrm{d}s.$$

Remark In case H is nonnegative the same conclusion holds for arbitrary, i.e. not necessarily symmetric, α-stable processes. For a proof and the history of these result, see Kallsen and Shiryaev (2002). These authors also show that, in essence, the results cannot be extended to more general Lévy processes. □

Henceforth, let $H(t)$ be a nonnegative and locally Riemann integrable function on $[0, \infty)$ and assume that

$$\int_0^t H^{\alpha}(s) \mathrm{d}Z(s) < \infty$$

for all $t > 0$. Then $X = H \bullet Z$ is a well-defined process. In line with the previous discussion, we assume that H and Z are independent, and we write $X_j = X(j\delta) - X((j-1)\delta)$ and $Z_j = Z(j\delta) - Z((j-1)\delta)$.

As an initial consideration we look at the asymptotic behaviour of unnormalised power variations and let $H \equiv 1$, i.e. we consider the simplest case, $X = Z$. Recall first that the sup-variation $[Z]^{[r]}(t)$ is finite or infinite according to whether $r > \alpha$ or $r \leq \alpha$ (cf. Fristedt and Taylor (1973), Mikosch and Norvaiša (2000)).

The law of $|X_j|$ is the same as the law of $\delta^{1/\alpha}|Z(1)|$ and thus

$$[X_\delta]^{[r]}(t) \stackrel{law}{=} \delta^{r/\alpha} \sum_{j=1}^{M} |Z_j|^r.$$

The random variables $|Z_j|$ belongs to the domain of normal attraction of a stable law with index α. Hence, on account of Feller (1971, pp. 580–581), we have the following limit properties, where for simplicity we are letting $r = 1$:

If $1 < \alpha < 2$ then, for a certain α-stable law S_α,

$$[X_\delta]^{[1]}(t) - \delta^{-1+1/\alpha}\mu_{\alpha,1} \stackrel{law}{\to} S_\alpha.$$

If $0 < \alpha < 1$ then, for a certain positive α-stable law $S_{+\alpha}$,

$$[X_\delta]^{[1]}(t) \stackrel{law}{\to} S_{+\alpha}.$$

If $\alpha = 1$ then, for a certain 1-stable law S_1,

$$[X_\delta]^{[1]}(t) - b_\delta \stackrel{law}{\to} S_1$$

where

$$b_\delta = \int_{-\infty}^{\infty} \sin(\delta x) \mathrm{d}P\{|Z(1)| \leq x\}.$$

In all three cases, $\delta^{1/2}[X_\delta]^{[1]}(t) \stackrel{p}{\to} 0$. Note that the above limit laws are more complicated than the mixed Gaussian limit laws obtained in Sections 3 and 5.

Next, for general H we have, by (6.1) and (6.2),

$$X_j \stackrel{law}{=} \left(\int_{(j-1)\delta}^{j\delta} H^\alpha(s)\mathrm{d}s \right)^{1/\alpha} Z(1)$$

so that

$$|X_j|^r \stackrel{law}{=} \left(\int_{(j-1)\delta}^{j\delta} H^\alpha(s)\mathrm{d}s \right)^{r/\alpha} |v_j|^r \tag{6.3}$$

where $v_1, ..., v_M$ are i.i.d. with the same distribution as $Z(1)$. Equivalently, by the subordination property, we have

$$|X_j|^r \stackrel{law}{=} \left(\int_{(j-1)\delta}^{j\delta} H^\alpha(s)\mathrm{d}s \right)^{r/\alpha} q_j^{r/2}|u_j|^r$$

where the $q_1, ..., q_M$ are i.i.d., with the same law as $S(1)$ and are independent of $u_1, ..., u_M$ which are i.i.d. standard normal.

In view of these representations of $|X_j|^r$ it would be rather simple to give a complete description of the various possible limiting behaviours of realised power variation as $\delta \to 0$. Here we shall only discuss some particular cases.

For $r=2$ we have that realised quadratic variation is

$$[X_\delta](t) \overset{law}{=} \left\{ \sum_{j=1}^{n} |u_j|^\alpha \int_{(j-1)\delta}^{j\delta} H^\alpha(s)\mathrm{d}s \right\}^{2/\alpha} S(1).$$

The term in braces satisfies, conditionally on H, as $\delta \downarrow 0$

$$\sum_{j=1}^{M} |u_j|^\alpha \int_{(j-1)\delta}^{j\delta} H^\alpha(s)\mathrm{d}s \xrightarrow{p} \mu_\alpha H^{\alpha *}(t).$$

This follows from Corollary 4.3. Consequently, for the quadratic variation we have

$$[X^*](t) \overset{law}{=} \{\mu_\alpha H^{\alpha*}(t)\}^{2/\alpha} S(1). \tag{6.4}$$

Much simpler and statistically more powerful results are available if we use realised power variation instead of realised quadratic variation.

Recall $\mathrm{E}\,|Z(1)|^\gamma$ exists if (and only if) $\gamma < \alpha$. Thus the moments of $|Z(1)|^r$ exist up to, but not including, order α/r. Hence, still given H, if $r < \alpha$ and $1 < \alpha < 2$ then

$$\delta^{1-r/\alpha}[X_\delta]^{[r]}(t) \xrightarrow{p} \mu_{\alpha,r} H^{r*}(t), \tag{6.5}$$

(where $\mu_{\alpha,r} = \mathrm{E}\{|Z(1)|^r\}$). This may be verified by means of Corollary 4.2. In fact, the assumptions made on H imply that it suffices to prove the statement in the case $H \equiv 1$. Then, in the notation of Corollary 4.2, $c_n = n^{-1}$ and the conditions (4.5)-(4.8) are easily checked using the well known tail behaviour of the α-stable laws. (The result (6.5) provides a simple generalisation of the use of quadratic variation for Brownian motion based stochastic volatility models, for then $r = 2$ and

$$[X_\delta]^{[2]}(t) \xrightarrow{p} H^{2*}(t)$$

exactly.)

In case $r < \alpha/2$ we have the stronger result that

$$\frac{\delta^{1-r/\alpha}\mu_{\alpha,r}^{-1}[X_\delta y_\delta^*]^{[r]}(t) - H^{r*}(t)}{\delta^{1/2}\sqrt{\mu_{\alpha,r}^{-2} v_{\alpha,r} H^{2r*}(t)}} \overset{law}{\to} N(0,1), \tag{6.6}$$

where $v_{\alpha,r} = \mathrm{Var}\{|z(1)|^r\}$. This result holds both conditionally and unconditionally. This is a consequence of Corollary 4.1.

Of course in practice the above limit theory is has an unknown denominator $H^{2r*}(t)$ and so could not be used even if we were to know α. However, in theory we could replace $H^{2r*}(t)$ by the consistent estimator

$$\delta^{1-2r/\alpha}\mu_{\alpha,2r}^{-1}[X_\delta]^{[2r]}(t).$$

7. Related work

There are in the literature a considerable number of important results on power variations of semimartingales generally, and Lévy processes in particular, that are related but not directly relevant to what we have discussed above. To complete the picture the following subsections contain a brief guide to those results.

7.1. Power variation and Lévy processes

A number of authors have investigated the relation between the Lévy measure ν of a Lévy process L and existence of sup-variations of the process.

The Blumenthal-Getoor index of a Lévy process is defined by

$$\beta = \inf\{r > 0 : \int_{[-1,1]} |x|^r \nu(\mathrm{d}x) < \infty\}.$$

If $\beta < r$ then pathwise (Lépingle (1976), Hudson and Mason (1976))

$$[L_\delta]^{[r]}(t) \to \sum_{0<s\leq t} |\Delta L(s)|^r < \infty$$

whereas in general $[L_\delta]^{[r]}(t) \to \infty$ when $r \leq \beta$.

Furthermore (see Sato (1999, Theorem 21.9)), with $r = 1$ we have $[L]^{[1]} < \infty$ or $= \infty$ according as $\beta \leq 1$ or $1 < \beta (< 2)$.

Some extensions to additive processes are considered in Woerner (2002).

7.2. Power variation and semimartingales

Let X be a semimartingale. Lépingle (1976) considered sup-variations of semimartingales generally and showed that $[X]^{[r]}(t) < \infty$ for every $r > 2$ while for $1 < r < 2$ we have

$$[X_\delta]^{[r]}(t) \to \sum_{0<s\leq t} |\Delta X(s)|^r$$

provided $\langle X \rangle_t = 0$ and

$$\sum_{0<s\leq t} |\Delta X(s)|^r < \infty.$$

7.3. Sub-ϕ variation and integration

We briefly recall the role of sup-variation in the theory of integration.

Young (1936) extended the Stieltjes integral to allow for integration in cases where the integrand and/or the integrator may be of unbounded variation. Dudley (1992) and Dudley and Norvaiša (1999) extended the concept further, and Mikosch and Norvaiša (2000) applies the theory to give path-by-path solutions to many basic stochastic integral equations. The main condition for the existence of such solutions is that $0 < r < 2$.

An annotated bibliography on power variation is available in Dudley, Norvaiša, and Jianghau Qian (1999). See also Dudley and Norvaiša (1998). We also refer to the related work of Lyons on *rough paths*, see Lyons (1994) and Bass, Hambly, and Lyons (2002) and references given there.

7.4. A general class of variation results

The limit behaviour as $M \to \infty$ of processes of the type

$$Y_M(t) = \sum_{j=1}^{[Mt]} f\left[\frac{j-1}{M}, \frac{1}{\sqrt{M}}\left\{X\left(\frac{j}{M}\right) - X\left(\frac{j-1}{M}\right)\right\}\right]$$

where f is function of two variables and X denotes a Brownian semimartingale (of a certain kind, see below) has been discussed in great depth in a thesis by Becker (1998). The diffusion case is especially important. Extensions to general continuous or purely discontinuous semimartingales and even combination of the two are presented. The thesis is partly based on an earlier report by Jacod (1992), see also Delattre and Jacod (1997) and Florens-Zmirou (1993). Both X and f may be multidimensional, and generalisations to cases where not only the increment of X over the j-th interval but the whole trajectory over that interval occurs in the second argument of f are also considered.

Of immediate interest in connection with the present paper are Becker's results when X is a Brownian semimartingale. More specifically, Becker considers the case where X is of the form

$$X(t) = \int_0^t C(s)\mathrm{d}s + \int_0^t H(s)\mathrm{d}W(s)$$

where W is Brownian motion and C and H are predictable and subject to restrictions on their variational behaviour. He shows, in particular, that $Y_M(t)$ after a suitable centering converges to a stochastic process which is representable as a certain type of stochastic integral where the integration is with respect to a 'martingale-measure tangential to X'. A key point of our work discussed above is that for the kind of functions f we consider, i.e. absolute powers, we are able to identify the limit behaviour as mixed Gaussian and, crucially for the statistical applicability, from this to establish standard normal limit statements using random rescaling by observable scale factors.

References

Barndorff-Nielsen, O. E. (2001). Superposition of Ornstein-Uhlenbeck type processes. *Theory of Probability and its Applications 45*, 175–194.

Barndorff-Nielsen, O. E., E. Nicolato, and N. Shephard (2002). Some recent developments in stochastic volatility modelling. *Quantitative Finance 2*, 11–23.

Barndorff-Nielsen, O. E. and N. Shephard (2001b). Modelling by Lévy processes for financial econometrics. In O. E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (Eds.), *Lévy Processes – Theory and Applications*, pp. 283–318. Boston: Birkhäuser.

Barndorff-Nielsen, O. E. and N. Shephard (2001a). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics (with discussion). *Journal of the Royal Statistical Society, Series B 63*, 167–241.

Barndorff-Nielsen, O. E. and N. Shephard (2002b). Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. Unpublished paper: Nuffield College, Oxford, Economics working paper 2002-W13.

Barndorff-Nielsen, O. E. and N. Shephard (2002a). Econometric analysis of realised volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B 64*, 253–280.

Barndorff-Nielsen, O. E. and N. Shephard (2003). Realised power variation and stochastic volatility. *Bernoulli 9*. Forthcoming.

Bass, R., B. Hambly, and T. Lyons (2002). Extending the Wong-Zakai theorem to reversible markov processes. *Journal of the European Mathematical Society 4*, 237–269.

Becker, E. (1998). Theorems limite pour des processus discretisés. Thése de Doctorat de l'Université Paris 6.

Delattre, S. and J. Jacod (1997). A central limit theorem for normalized functions of the increments of a diffusion process in the presence of round off errors. *Bernoulli 3*, 1–28.

Dudley, R. (1992). Fréchet differentiability, p-variation and uniform Donsker classes. *Annals of Probability 20*, 1968–1982.

Dudley, R. and R. Norvaiša (1998). An introduction to p-variation and Young integrals; with emphasis on sample functions of stochastic processes. MaPhySto Lecture Notes 1998-1, Aarhus University: MaPhySto.

Dudley, R. and R. Norvaiša (1999). Product integrals, Young integrals and p-variation. In R. Dudley and R. Norvaiša (Eds.), *Differentiability of Six Operators on Nonsmooth Functions and p-variation*. New York: Springer-Verlag. Lecture Notes in Mathematics 1703.

Dudley, R., R. Norvaiša, and Jianghau Qian (1999). Bibliographies on p-variations and ϕ-variation. In R. Dudley and R. Norvaiša (Eds.), *Differentiability of Six Operators on Nonsmooth Functions and p-variation*. New York: Springer-Verlag. Lecture Notes in Mathematics 1703.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* (2 ed.), Volume 2. New York: John Wiley.

Florens-Zmirou, D. (1993). On estimating the diffusion coefficient from discrete observations. *Journal of Applied Probability 30*, 790–804.

Fristedt, B. and S. Taylor (1973). Strong variation for the sample functions of a stable process. *Duke Mathematics Journal 40*, 259–278.

Gnedenko, B. W. and A. N. Kolmogorov (1954). *Limits Distributions for Sums of Independent Random Variables*. Cambridge, Mass: Additon-Wesley.

Hardy, G. H., J. E. Littlewood, and G. Polya (1959). *Inequalities*. Cambridge: Cambridge University Press.

Hudson, W. N. and J. D. Mason (1976). Variational sums for additive processes. *Proceeding of the Americian Mathematical Society 55*, 395–399.

Jacod, J. (1992). Limit of random measures associated with the increments of a Brownian semimartingale. Publ. Lab. Probabilite N. 120, Paris 6.

Kallsen, J. and A. N. Shiryaev (2002). Time change representation of stochastic integrals. *Theory of Probability and its Applications 46*, 522–528.

Lépingle, D. (1976). La variation d'ordre p des semi-martingales. *Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete 36*, 295–316.

Loève, M. (1977). *Probability Theory I* (4 ed.). New York: Springer.

Lyons, T. (1994). Differential equations driven by rough signals. I. An extension of an inequality by L.C.Young. *Mathematical Research Letters 1*, 451–464.

Mikosch, T. and R. Norvaiša (2000). Stochastic integral equations without probability. *Bernoulli 6*, 401–434.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Shiryaev, A. N. (1999). *Essentials of Stochastic Finance: Facts, Models and Theory*. Singapore: World Scientific.

Woerner, J. (2002). Power variation: an important tool for model selection and estimation in semimartingale models. Unpublished paper: OCIAM, Mathematical Institute, University of Oxford.

Young, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math 67*, 251–282.