Risk Aversion over Incomes and Risk Aversion over Commodities

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Abstract: This note determines the precise connection between an agent's attitude towards income risks and his attitude over risks in the underlying consumption space. Our results follow from a general mathematical theory connecting the curvature properties of an objective function with the ray-curvature properties of its dual.

Keywords: risk aversion, concavity, duality.

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1. Introduction

It is commonplace to assume that agents are averse to risk or uncertainty in their income. Yet the notion of aversion to income risks is really a derived concept, since what agents ultimately care about is the consumption of consumer goods, and dislike income risks only because of the risks it imposes on their consumption. So it is a little odd that the precise connection between risk attitudes over income and risk attitudes over consumption goods has never been formally explored. This note is an attempt to fill this lacuna.

Formally, the issue we have in mind is the following. An agent has a preference over bundles in the commodity space R^l_+ and also over lotteries of these bundles. Provided the von Neumann-Morgenstern axioms are satisfied, there exists a Bernoulli utility function $u : R^l_+ \to R$ such that the utility of any lottery will be given by the expected value of the Bernoulli utility.[2] In other words, a lottery which gives the bundle x' with probability t, and x'' with probability $1-t$, which we write as $t \bullet x' \oplus (1-t) \bullet x''$, will have a utility of $tu(x') + (1-t)u(x'')$. It is clear the the agent is risk averse in commodity space if and only if u is concave. How is this related to his attitude over income risks?

Consider the lottery $t \bullet y' \oplus (1-t) \bullet y''$, where the agent has incomes y' and y'' with probabilities t and $1-t$ respectively. Assume that the agent makes his consumer purchases only after the uncertainty is resolved, that there is no price risk (so price is held fixed at p) and that he maximizes his utility given his realized budget constraint. In this case, the utility of $t \bullet y \oplus (1-t) \bullet y''$ must be given by $tv(p, y') + (1-t)v(p, y'')$, where v is the indirect utility generated by u. The agent's risk attitudes over income is measured by the curvature of v as a function of income, with price held fixed at p. If the agent is risk averse in income then $v(p, \cdot)$ is concave. It is known that $v(p, \cdot)$ is concave at all p if and only if u is concave.[3] In other words, an agent is risk averse in income at all prices if and only if he is risk averse in commodity space. In this note, we sharpen this result by determining precisely the risk attitudes over commodity space that is consistent with a particular level of income risk aversion.

Suppose at the price p and income 1, the agent has a demand of x. Consider a lottery $t \bullet x' \oplus (1-t) \bullet x''$, where $p \cdot [tx' + (1-t)x''] = 1$. We know that a risk averse agent will surely prefer x to this lottery: his risk aversion means he prefers the sure bundle $tx' + (1-t)x''$ to the lottery, and this bundle in turn must not be preferred to x, since it is in the budget set. But suppose we now offer him the lottery $t \bullet [(p \cdot x')^{\theta-1} x'] \oplus (1-t) \bullet [(p \cdot x'')^{\theta-1} x'']$. By assumption $p \cdot (tx' + (1-t)x'') = 1$; the price p gives this lottery a mean valuation of $(p \cdot x')^{\theta} + (p \cdot x'')^{\theta}$, which must be greater than 1 if θ is greater than 1 or less than 0. So it is no longer the case that a risk averse agent *must* prefer the sure bundle x to the lottery; we will call an agent who always prefers x to such lotteries *θ-risk averse at p* (with income normalized at 1). We show that an agent is θ-risk averse at the prices λp for all $\lambda > 0$ if and only if, at the price p, the agent's coefficient of relative risk aversion over all income levels has an infimum of $1 - \theta^{-1}$, i.e., $\inf_{y>0} R(p, y) = 1 - \theta^{-1}$, where $R(p, y) = -y v_{yy}(p, y)/v_y(p, y)$. Section 2 of this note is devoted to this result and its variants.

When an agent is θ-risk averse, the Bernoulli utility function has a curvature property we call θ-concavity. Section 3 of this note is devoted to a result which is very useful in helping us visualize this property. If u is concave, we know that at any point $(x, u(x))$ on its graph, there is a tangent plane which sits on top of the *entire* graph, so that the family of such planes form a nest around the graph of u. Formally, there are functions g and h from R^l_{++} to R such that $u(x) = \inf_{r \in R^l_{++}} \{g(r) + h(r)(r \cdot x)\}$. (Note that the graph of the map from x to $g(r) + h(r)(p \cdot x)$ is a hyperplane.) When u is θ-concave, we obtain the following representation: $u(x) = \inf_{r \in R^l_{++}} \{g(r) + h(r)(r \cdot x)^\theta\}$. So the graph of u is now nested by a family of hypersurfaces of a particularly simple type, where each of them is the graph of a function from x to $g(r) + h(r)(r \cdot x)^\theta$.

While we have chosen in this note to focus on the connection between risk aversion over income and risk aversion over consumption, it should (or will) be clear to the reader that the arguments could be easily adjusted to deal other issues; for example, the relationship between the shape of the production function and the curvature of its cost curve. What we develop here is a general mathematical theory which relates the curvature properties of an objective function with the curvature, along rays emanating from the origin, of its dual, so its potential applications go beyond the one discussed in this note.

2. Characterizing Risk Aversion Over Incomes

A function $u : X \to R$ is called a *utility function* if it has the following properties: (i) X is convex subset of R^l_+ and if x is in X, so is λx for any strictly positive scalar λ; (ii) for any x in X, $u(\lambda x) > u(x)$ for any scalar $\lambda > 1$; (iii) for any (p, y) in $R^l_{++} \times R_+$, there is $\bar{x}$ that maximizes $u(x)$ in the set $B(p, y) = \{x \in X : p \cdot x \leq y\}$. We refer to $B(p, y)$ as the *budget set* at the *price-income situation* (p, y) and to $\bar{x}$ as a *demand* at (p, y). We denote the set of demands at (p, y) by $f(p, y)$. The *indirect utility function* generated by u refers to the function $v : R^l_{++} \times R_+ \to R$ defined by $v(p, y) = u(\bar{x})$ for any $\bar{x}$ in $f(p, y)$. In some instances, we will impose an additional condition on the utility function u called the *supporting price property.* By this, we mean that at every x in X, there is $(p, 1)$, with p in R^l_{++}, such that x is in $f(p, 1)$.

All our definitions are standard and conditions which guarantee that a function u has the properties we require are mild and well known (see Mas-Colell (1985)). It is most familiar to have X as R^l_+ or R^l_{++} but we also allow X to be smaller, for example, the intersection of R^l_+ with a proper subspace of R^l. In other words, we permit the market for consumption goods to be incomplete.

An agent's attitude towards income risks is measured by the *coefficient of relative risk aversion*, with its value at (p, y) given by $R(p, y) = -yv_{yy}(p, y)/v_y(p, y)$. Whenever we make a statement referring to this coefficient we are implicitly assuming that v is C^2. Conditions on u guaranteeing that v is C^2 are known (see Mas-Colell (1985)).

Instead of the coefficient of relative risk aversion, we will mostly rely in this paper on a measure of the curvature of v which does not require its differentiability. We say that an increasing function $F : R_+ \to R$ is *θ-concave* if the map from y to $F(y^\theta)$ is concave. There is a particular ordering of the real numbers which is appropriate to this concept. The total, transitive and irreflexive order $<^*$ is defined as follows: 0 is ranked lowest, any positive number has higher rank, and any positive number is ranked higher than smaller positive numbers; any negative number has a higher rank than a positive number, with large negative numbers ranked more highly than low negative numbers. For example, $0 <^* 2 <^* 2000 < -2 <^* -0.5 <^* -0.01$.... We denote by $\sup^*(X)$ and $\max^*(X)$ the supremum and maximum of the set X in R, with respect to the order $<^*$.[4] The reason for introducing this order comes from the next simple observation, whose proof we will omit.

Lemma 1: *An increasing function* $F : R_+ \rightarrow R$ *is* θ*-concave if and only if it is* α*-concave for all* $\alpha \leq^* \beta$*.*

The notion of θ-concavity can be regarded as the non-differentiable analogue to the coefficient of relative risk aversion, as the next lemma makes clear. Again we will skip the straightforward proof.

Lemma 2: $\mathrm{Inf}_{y>0} R(p, y) = c$ *if and only if* $\max^*\{\theta : \ v(p, \cdot)$ *is* θ*-concave*$\} = (1 - c)^{-1}$. (Note that this lemma is applicable to $c = 1$ under the following convention, which we adopt throughout this note: when $c = 1$, regard $(1 - c)^{-1} = \infty$, so $\inf_{y>0} R(p, y) = 1$ if and only if the map from y to $v(p, y^\theta)$ is concave for all $\theta \geq 0$ and for no other θ.)

We wish to relate the concavity of v as a function of income with an appropriate notion of concavity for the utility function u. We say that the utility function u is θ*-concave at* p if $u(x) \geq tu((p \cdot x')^{\theta-1}x') + (1 - t)u((p \cdot x'')^{\theta-1}x'')$, whenever x is in $f(p, 1)$, $0 \leq t \leq 1$, x' and x'' are in X, and $p \cdot (tx' + (1-t)x'') = 1$. We say that u is θ*-concave in the set* P (contained in R^l_{++}) if it is θ-concave at every p in P. Note that an agent's utility function is θ-concave at p if and only if he is θ-risk averse at p, in the sense defined in the introduction, i.e., the agent prefers the certain prospect x to the lottery $t \bullet (p \cdot x')^{\theta-1}x' \oplus (1 - t) \bullet (p \cdot x'')^{\theta-1}x''$.

The concept of θ-concavity is crucial to everything in this paper so we should discuss it in some detail. Note firstly, that u is *always* 0-concave. The bundles $(p \cdot x')^{-1}x'$ and $(p \cdot x'')^{-1}x''$ are both affordable at $(p, 1)$; since x is in $f(p, 1)$, $u(x)$ must not be less than than the utility of either bundle, nor any convex combination of their utilities. Secondly, we note that if u is concave in the standard sense then it is 1-concave at all prices. To see that, we consider t, x' and x'' satisfying $p \cdot (tx' + (1-t)x'') = 1$. Concavity guarantees that $tu(x') + (1-t)u(x'')$ is less than or equal to $u(tx' + (1-t)x'')$, which is in turn no greater than $u(x)$ since x is in $f(p, 1)$ and $tx' + (1-t)x''$ is in the budget. It is also clear that if u is 1-concave at all prices and has the supporting price property, then it is a concave function: let p be a supporting price at $tx' + (1-t)x''$; then 1-concavity at p guarantees that $u(tx' + (1-t)x'') \geq tu(x') + (1-t)u(x'')$. The next result shows that θ-concavity as applied to u can be ordered in the same way as θ-concavity in one dimension.

Proposition 1: *The utility function u is θ-concave at p if and only if it is α-concave at p for all $\alpha \leq^* \theta$.*

Proof: We first make the following observation: if u is θ-concave at p, with $\theta \geq 0$ (alternately, with $\theta \leq 0$), then $u(x) \geq tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'')$ whenever x is in $f(p, 1)$, $t \in [0, 1]$ and $p \cdot (tx' + (1-t)x'') \leq (\geq) 1$. To see this, we assume that $p \cdot (tx' + (1-t)x'') \leq 1$ and choose δ such that $p \cdot (t\delta x' + (1-t)\delta x'') = 1$. Clearly, $\delta \geq 1$. If u is θ-concave with $\theta \geq 0$ and since u is increasing in all arguments, we obtain

$$u(x) \geq tu(\delta^{\theta}(p \cdot x')^{\theta-1}x') + (1-t)u(\delta^{\theta}(p \cdot x'')^{\theta-1}x'') \geq tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'').$$

The proof of the other case is the same.

We we will first assume that $\theta \geq 0$ and show that u is α-concave for α in $[0, \theta]$. Choose z' and z'' so that $(p \cdot x')^{\alpha-1}x' = (p \cdot z')^{\theta-1}z'$ and $(p \cdot x'')^{\alpha-1}x'' = (p \cdot z'')^{\theta-1}z''$. Then

$$\begin{aligned} p \cdot [tz' + (1-t)z''] &= t(p \cdot x')^{\alpha/\theta} + (1-t)(p \cdot x'')^{\alpha/\theta} \\ &\leq [t(p \cdot x') + (1-t)(p \cdot x'')]^{\alpha/\theta} = 1, \end{aligned}$$

so by the observation we first made, $tu((p \cdot x')^{\alpha-1}x') + (1-t)u((p \cdot x'')^{\alpha-1}x'') = tu((p \cdot$

$z')^{\theta-1}z') + (1-t)u((p \cdot z'')^{\theta-1}z'')$ must be no greater than $u(x)$.

For $\theta < 0$, we wish to show that u is α-concave whenever $\alpha \leq \theta$ or $\alpha \geq 0$. To see this, we first define z' and z'' as above and then notice that $p \cdot [tz' + (1-t)z''] \geq 1$ provided $\alpha \geq 0$ or $\alpha \leq \theta$. The rest of the proof is then analogous to the case of $\theta \geq 0$. QED

The next theorem is our main result and gives us the relationship between the concavity of u and the concavity of its indirect utility function v, as a function of income.

Theorem 1: *The indirect utility function v is such that $v(p, \cdot)$ is θ-concave if and only if u is θ-concave in the set $[p] = \{\lambda p : \lambda > 0\}$.*

Proof: To prove necessity, we write $\bar{p} = \lambda p$ for some positive number λ, and suppose that x is in $f(\bar{p}, 1)$. If $\bar{p} \cdot [tx' + (1-t)x''] = 1$,

$$\begin{aligned} u(x) = v(\bar{p}, 1) &= v(\bar{p}, [t(\bar{p} \cdot x') + (1-t)(\bar{p} \cdot x'')]^{\theta}) \\ &\geq tv(\bar{p}, (\bar{p} \cdot x')^{\theta}) + (1-t)v(\bar{p}, (\bar{p} \cdot x'')^{\theta}) \\ &\geq tu((\bar{p} \cdot x')^{\theta-1}x') + (1-t)u((\bar{p} \cdot x'')^{\theta-1}x''), \end{aligned}$$

The first inequality follows from the θ-concavity of $v(\bar{p}, \cdot)$ (which in turn follows from that of $v(p, \cdot)$) and the second from the definition of v. So u is θ-concave at $\bar{p}$.

For sufficiency, we first assume that $ty' + (1-t)y'' = 1$. The agent has a demand at (p, y'^{θ}), which can always be expressed in the form $(p \cdot x')^{\theta-1}x'$. Similarly, we can write a demand at (p, y''^{θ}) as $(p \cdot x'')^{\theta-1}x''$. The budget identity guarantees that $p \cdot x' = y'$ and $p \cdot x'' = y''$. If x is in $f(p, 1)$, the θ-concavity of u at p guarantees that

$$\begin{aligned} tv(p, y'^{\theta}) + (1-t)v(p, y''^{\theta}) &= tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'') \\ &\leq u(x) = v(p, 1). \end{aligned}$$

In general, if $ty' + (1-t)y'' = \bar{y}$ then $t(y'/\bar{y}) + (1-t)(y''/\bar{y}) = 1$. Since u is θ-concave at $p/\bar{y}^\theta$, the preceding proof for this particular price guarantees that

$$tv\left(\frac{p}{\bar{y}^\theta}, \left(\frac{y'}{\bar{y}}\right)^\theta\right) + (1-t)v\left(\frac{p}{\bar{y}^\theta}, \left(\frac{y'}{\bar{y}}\right)^\theta\right) \leq v\left(\frac{p}{\bar{y}^\theta}, \left[t\left(\frac{y'}{\bar{y}}\right) + (1-t)\left(\frac{y''}{\bar{y}}\right)\right]^\theta\right);$$

equivalently, we have $tv(p, y'^\theta) + (1-t)v(p, y''^\theta) \leq v(p, (ty' + (1-t)y'')^\theta)$. QED

This theorem says in particular that $v(p, \cdot)$ is 1-concave, i.e., is a concave function, if and only if u is 1-concave in the set $[p]$. As we had pointed out earlier, the 1-concavity of u is equivalent to the concavity of u provided u has the supporting price property. So, assuming this property, Theorem 1 recovers the known result that v is a concave function of income at all prices if and only if u is concave. Of course the theorem says quite a bit more. The next corollary is an immediate consequence of Lemma 2 and Theorem 1: it gives the precise level of risk averse behavior over commodities which guarantees a particular infimum over the coefficient of relative risk aversion over income.

Corollary 1: $\mathrm{Inf}_{y>0} R(p, y) = c$ *if and only if* $\max^*\{\theta : \ u \textit{ is } \theta\textit{-concave in } [p]\} = (1 - c)^{-1}$.

The global results we have obtained so far can easily be re-stated as local results once the definitions are suitably modified. In particular, we say that u is *locally θ-concave* at p if there is $\epsilon > 0$ such that $u(x) \geq tu((p\cdot x')^{\theta-1}x') + (1-t)u((p\cdot x'')^{\theta-1}x'')$ whenever x is a demand at p, t is in $[0, 1]$, x' and x'' are in X, $p\cdot(tx' + (1-t)x'') = 1$, and $\max\{|p\cdot x' - 1|, |p\cdot x'' - 1|\} \leq \epsilon$. It is trivial to see that Proposition 1 remains true if θ-concavity is replaced by its local version. In other words, local θ-concavity has the same ranking property as θ-concavity. We say that $v(p, \cdot)$ is *locally θ-concave at $\bar{y}$* if there is a neighborhood around $\bar{y}$ such that the map from y to $v(p, y^\theta)$ is concave. With this definition, the following local analog to Theorem 1 is easy to check: $v(p,\cdot)$ is locally θ-concave at 1 if and only if u is locally θ-concave at p. The next result, which is the local analog of Corollary 1, should then be unsurprising.

Corollary 2: *$R(p,1) = c$ if and only if $\sup^*\{\theta : u$ is locally θ-concave at $p\} = (1-c)^{-1}$.*

Proof: Suppose $R(p,1) = c$; since v is C^2, for any $\epsilon > 0$, there is some neighborhood of income around 1 such that $R(p,y) > c-\epsilon$ in that neighborhood. It is easy to check that this guarantees that $v(p,\cdot)$ is locally $(1-c+\epsilon)^{-1}$-concave at 1, which implies that u is locally $(1-c+\epsilon)^{-1}$-concave at p. Letting ϵ go to zero, we obtain $\sup^*\{\theta : u$ is locally θ-concave at $p\} \geq (1-c)^{-1}$. But the inequality cannot be strict; if it were, there will be $\theta' > (1-c)^{-1}$ such that u is locally θ'-concave at p. This implies that $v(p,\cdot)$ is locally θ'-concave at income 1, which means that $R(p,1) > c$: a contradiction. QED

3. A Representation Theorem for θ-Concave Functions

Our main task in this section is to proof the following theorem.

Theorem 3: *Suppose $u : X \to R$ is a utility function with the supporting price property. If u is θ-concave at all prices, for $\theta \neq 0$, then there exists maps from g and h from R^l_{++} to R such that $u(x) = \inf_{r\in R^l_{++}}\{g(r) + h(r)(r\cdot x)^{1/\theta}\}$, where $h > 0$ if $\theta > 0$ and $h < 0$ if $\theta < 0$.*

Our proof will take an instructive roundabout approach, by first constructing a representation for the indirect utility function. This representation is analogous to that established for nondecreasing and convex-along-rays functions by Rubinov and Glover (1999).

Lemma 3: *Suppose $v(p,\cdot)$ is θ-concave at all prices p. If $\theta > 0$, there exists maps g and*

h from R^l_{++} to R, with $h > 0$, such that

$$v(p,1) = \inf_{r \in R^l_{++}} \left\{ g(r) + h(r) \left(\max_{1 \le i \le l} \frac{p_i^{-1/\theta}}{r_i} \right) \right\}.$$

For $\theta < 0$, the representation is the same, but with $h < 0$ and max *replaced by* min.

Proof: We denote $(p_1^{-\theta}, p_2^{-\theta}, ..., p_l^{-\theta})$ by $p^{-\theta}$. It is not hard to check that since v is homogeneous of degree zero, the θ-concavity of $v(p, \cdot)$ for all p will mean that the map $\tilde{v} : R^l_{++} \to R$, which maps p to $v(p^{-\theta}, 1)$ is concave along rays, i.e., $\tilde{v}(tp + (1-t)\lambda p) \geq t\tilde{v}(p) + (1-t)\tilde{v}(\lambda p)$, for any $\lambda > 0$ and t in $[0,1]$. If $\theta > 0$, $\tilde{v}$ is also an increasing function of p. At any point r in R^l_{++}, we let $h(r) > 0$ be a slope, along the ray, of $\tilde{v}$ at r. Note that since $\tilde{v}$ is increasing, for any r, $\tilde{v}(p) \leq \tilde{v}((\max_i p_i/r_i)r)$. The latter is in turn bounded above by $\tilde{v}(r) + h(r)((\max_i p_i/r_i) - 1)$. Therefore, we see that

$$\tilde{v}(p) \leq [\tilde{v}(r) - h(r)] + h(r) \left(\max_i \frac{p_i}{r_i} \right)$$

for all r. This gives us the result since the infimum of the right hand side of this inequality for r in R^l_{++} is achieved at $r = p$. The case of $\theta < 0$ has an analogous proof. QED

Proof of Theorem 3: Since u has the supporting price property, for any x in R^l_{++}, $u(x) = \min_{\{p \in R^l_{++} : p \cdot x = 1\}} v(p, 1)$. Theorem 1 tells us that since u is θ-concave in R^l_{++}, $v(p, \cdot)$ is θ-concave at all p. When $\theta > 0$ (the proof for $\theta < 0$ is analogous), Lemma 3 says that

$$\begin{aligned}
u(x) &= \inf_{\{p \in R^l_{++} : p \cdot x = 1\}} \inf_{r \in R^l_{++}} \left\{ g(r) + h(r) \left(\max_i \frac{p_i^{-1/\theta}}{r_i} \right) \right\} \\
&= \inf_{r \in R^l_{++}} \left\{ g(r) + h(r) \inf_{\{p \in R^l_{++} : p \cdot x = 1\}} \left(\max_i \frac{p_i^{-1/\theta}}{r_i} \right) \right\} \\
&= \inf_{r \in R^l_{++}} \left\{ g(r) + h(r) \left(r^{-\theta} \cdot x \right)^{1/\theta} \right\}.
\end{aligned}$$

. QED

Our final result uses Theorem 3 to give a sharper characterization of the θ-concavity of u in those cases where u is a concave function; in other words, when $1 \leq^* \theta$. This characterization is particularly nice since, unlike the definition of θ-concavity, it does not make any reference to the demand bundle at a price.

Corollary 4: *Suppose that the utility function $u : X \to R$ has the supporting price property. Then u is θ-concave at all prices for some $1 \leq^* \theta$ if and only if the following is true: for any $p \gg 0$, t in $[0,1]$, and x' and x'' in R^l_{++} satisfying $p \cdot (tx' + (1-t)x'') = 1$, we have $u(tx' + (1-t)x'') \geq tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'')$.*

Proof: Note that the 'if' part of this claim is clearly guaranteed by revealed preference. For the other direction, we first observe that when u has the representation given in Theorem 3, $tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'')$ is weakly less than

$$\inf_{r \in R^l_{++}} \{g(r) + h(r)\left(t((p \cdot x')^{\theta-1} r \cdot x')^{1/\theta} + (1-t)((p \cdot x'')^{\theta-1} r \cdot x'')^{1/\theta}\right)\}.$$

We can write

$$t((p \cdot x')^{\theta-1} r \cdot x')^{1/\theta} + (1-t)((p \cdot x'')^{\theta-1} r \cdot x'')^{1/\theta} = tp \cdot x' \left[\frac{r \cdot x'}{p \cdot x'}\right]^{1/\theta} + (1-t)p \cdot x'' \left[\frac{r \cdot x''}{p \cdot x''}\right]^{1/\theta}.$$

Since $tp \cdot x' + (1-t)p \cdot x'' = 1$, if $\theta \geq 1$ (< 0), the right hand side of this equation is weakly less than (more than)

$$\left(tp \cdot x' \left[\frac{r \cdot x'}{p \cdot x'}\right] + (1-t)p \cdot x'' \left[\frac{r \cdot x''}{p \cdot x''}\right]\right)^{1/\theta}.$$

As h is positive in the first instance and negative in the second, we have shown that $tu((p \cdot x')^{\theta-1}x') + (1-t)u((p \cdot x'')^{\theta-1}x'')$ is weakly less than

$$\inf_{r \in R^l_{++}} \{g(r) + h(r)\left(r \cdot (tx' + (1-t)x'')\right)^{1/\theta}\} = u(tx' + (1-t)x'').$$

This completes the proof since $1 \leq^* \theta$ if and only if $\theta \geq 1$ of $\theta < 0$. QED

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Footnotes:

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2. The description of u as the Bernoulli utility function is slightly non-standard, but it follows the usage in Mas-Colell et al (1995).

3. It is certainly known that when the utility function is concave, the indirect utility function is concave in income. For example, Mas-Colell et al (1995) have essentially the same result: they point out that when the production function is concave, the cost function is convex. The necessity of the condition is also terribly easy to prove and surely has to be known, though it is not easy to find a statement and proof of this result. One exists in Quah (2000).

4. Note that the difference between the maximum and the supremum is that in the case of the former, the supremum is achieved within the set X.