Two Criteria for Social Decisions*

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Abstract

This paper studies the ethical underpinnings of two social criteria which are prominent in the literature dealing with the problem of evaluating allocations of several consumption goods in a population with heteregenous preferences. The Pazner-Schmeidler criterion (Pazner-Schmeidler [22]) and the Walrasian criterion (Fleurbaey and Maniquet [7]) are prima facie quite different. But it is shown here that these criteria are related to close variants of the fairness condition that an allocation is better when every individual bundle in it dominates the average consumption in another allocation. In addition, the results suggest that the Pazner-Schmeidler criterion can be viewed as the best extension of the Walrasian criterion to non-convex economies.

JEL Classification: D63, D71.

Keywords: social welfare, social choice, fairness.

1 Introduction

The problem of defining criteria for social decisions has long been the topic of welfare economics and then of social choice, but an impressive array of difficulties and negative results have been obtained. Surplus criteria and related compensation tests have been criticized as unethical and inconsistent,[1] while Arrow's impossibility theorem of social choice (Arrow [1]) has been reproduced in all relevant contexts and came to be recognized as a major obstacle. The most trodden way out of this impossibility deadlock, which has been promoted in particular by Sen (e.g. [24]), is to rely on interpersonally comparable measures of individual well-being. An alternative approach consists in taking account of information about individual preferences at so-called "irrelevant alternatives".[2] This

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[1] For a recent survey on this issue, see Blackorby and Donaldson [2].

[2] Arrow's theorem involves an axiom of "independence of irrelevant alternatives" which requires the ranking of two alternatives to depend only on individuals' preferences over these two alternatives. This is

approach, which has recently produced a variety of interesting criteria in various contexts,[3] is adopted in this paper.

The Pazner-Schmeidler criterion (Pazner and Schmeidler [22], Pazner [21]) and the Walrasian criterion (Fleurbaey and Maniquet [7]) evaluate allocations of resources with the aim of equalizing the value of bundles consumed by individuals. This preference for equality is expressed by relying on the maximin principle, that is, by giving priority to the worst-off: Both criteria strictly prefer an allocation if the worst bundle in this allocation has a higher value than the worst bundle in another allocation. These two criteria differ only in how to measure the value of bundles. The Pazner-Schmeidler criterion measures the value of a bundle by *the percentage of total consumption* that his owner would accept in exchange for it. The Walrasian criterion is less simple, and measures the value of a bundle by the percentage of total consumption that his owner would accept in exchange for it, *with the possibility of making further trades* at reference market prices. These reference market prices are selected so as to maximize the smallest value of all bundles in the contemplated allocation. This means that, in general, reference market prices are different from one allocation to another. An important property of the Walrasian criterion, when individual preferences are convex, is that, among all feasible allocations in an exchange economy, it selects the egalitarian Walrasian equilibria (i.e. equilibria in which all agents have equal budgets) as the best allocations. Another important property of both criteria is that, by taking account of individual preferences, they satisfy the Pareto principle according to which an allocation must be deemed strictly better than another if all individuals strictly prefer the bundle they receive in it.[4]

The purpose of this paper is to compare these two criteria, through an analysis of their properties in economies with convex individual preferences, and also in economies with general (convex or non-convex) preferences. Such an analysis reveals the ethical underpinnings of the criteria, and should help in the choice of one criterion or the other. The purpose here is not simply to provide lists of properties satisfied by the criteria, and tightness of the analysis is obtained by looking for combinations of properties that logically imply the basic definition of the criteria. This reduces the choice between the criteria to a basic choice between mutually exclusive combinations of ethical principles. The main properties considered in this paper have to do with Pareto efficiency, preference for equality, informational parsimony. Detailed definitions of these notions will be provided after the formal framework has been introduced. As an outline of the main results, let

especially restrictive in an economic context, since it makes it impossible to prefer an allocation because it is, say, efficient or competitive and egalitarian in budgets, etc. This kind of evaluation typically requires information about preferences at alternatives other than the two considered. For further explanation and illustration, see below.

[3]See e.g. Pazner [21], Fleurbaey and Maniquet [7], [9] about the division of unproduced commodities, Fleurbaey and Maniquet [8] about production of a private good, Maniquet and Sprumont [20], [19] about production of a public good, Maniquet [18] about indivisibles, Fleurbaey [4] about health.

[4]Refinements of the criteria satisfy the stronger Pareto principle according to which strict preference for one individual only, and weak preference for the rest of the population, is enough to entail strict preference for the social criterion.

us first indicate that, in convex economies, the properties which separate the two criteria are not so divergent. In particular, it is shown that one and the same equity condition, formulated in two slightly different ways, leads either to the Walrasian criterion or to the Pazner-Schmeidler one. This equity condition is an extension to the case of multiple goods of the simple requirement that a distribution is better than another when its support is above the mean of the other. This may be called support-mean dominance. In a one-dimensional context, support-mean dominance implies generalized Lorenz dominance.[5] Here, the proposed adaptation of this condition to the multi-dimensional context says that an allocation x is at least as good as another, y, when all bundles in x dominate the average bundle of y. Stated in this way, however, this condition is incompatible with the Pareto principle, and restrictions are needed in its application. Depending on what restriction is applied, one obtains a (partial) characterization of either of the two criteria.

These results confirm those of earlier literature about the fact that these two criteria are prominent if not unchallengeable. They also show, by a joint characterization with similar structure, that the difference between the two criteria is not so strong as it may appear at first glance. The Walrasian criterion is just slightly more sensitive to efficiency of the allocation of resources, whereas the Pazner-Schmeidler criterion has a stronger preference for the equal-split allocation. Another, perhaps more striking, result is that, when considering the general case of convex or non-convex preferences, a weak requirement which is satisfied by both criteria in convex economies (namely, the intersection of the two axioms used for the characterization results in convex economies) uniquely singles out the Pazner-Schmeidler criterion in this wider domain. The analysis definitely leads to rejecting the Walrasian criterion in non-convex economies, even if it is well defined and still satisfies some good properties in that domain.

As briefly alluded to above, an essential feature of the approach adopted in this paper is that the only data about individual welfare are non-comparable preferences over consumption bundles, so that no interpersonal comparisons of utility are performed. The basis of interpersonal comparisons is the value of bundles, as measured with the help of individual indifference curves. For further explanations on this approach and its relation to the literature, see e.g. Fleurbaey and Maniquet [7], [9] and Fleurbaey, Suzumura and Tadenuma [11], [12]. In particular, this approach has been outlined long ago by Samuelson [23] and Pazner [21].[6] The latter introduced the Pazner-Schmeidler criterion as it is defined here. Fleurbaey and Maniquet [7] introduced the Walrasian criterion and Fleurbaey and Maniquet [9] have characterized the two criteria in convex economies, but on the basis of quite different properties for each of them. Fleurbaey [3] and Tadenuma [28] also characterized the Pazner-Schmeidler criterion but did not study the other one.[7] This

[5] The one-dimensional distribution $(x_1, ..., x_n)$ generalized-Lorenz dominates $(y_1, ..., y_n)$ when for all $k = 1, ..., n$, $\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}$, where $x_{(i)}$ denotes the i-th value in $(x_1, ..., x_n)$ by increasing order.

[6] Additional discussions of the approach can be found in Fleurbaey and Hammond [6] and Fleurbaey [5] in relation to interpersonal comparisons, and in Fleurbaey and Mongin [10] in relation to the Bergson-Samuelson brand of welfare economics.

[7] Tadenuma [27] studies the construction of social preferences on the basis of the no-envy criterion (defined below), which is indirectly related to the Walrasian criterion.

paper pursues the analysis, with different requirements which make it easier to compare the two criteria, and with an extension of the analysis to non-convex preferences.

The next section introduces the model and the main concepts. Then Section 3 presents basic ethical requirements that may be imposed on any reasonable social criterion. The support-mean equity condition is introduced in Section 4, and leads to a double characterization of the two social criteria mentioned above. Section 5 extends the analysis to the case of non-convex preferences, and Section 6 concludes. An appendix collects the proofs.

2 The Pazner-Schmeidler and Walras orderings

Like an important part of the literature (Kolm [16], Pazner and Schmeidler [22], Sprumont and Zhou [25], among many others), we focus here on the canonical consumption problem, i.e. the problem of distributing a fixed bundle $\Omega \in \mathbb{R}^\ell_{++}$ of ℓ goods ($\ell \geq 2$) to n individuals ($n \geq 2$). An *allocation* is a list of bundles, one for each agent: $x = (x_1, ..., x_n) \in \mathbb{R}^{n\ell}_+$. Every agent $i = 1, ..., n$ has a *preference* ordering[8] R_i over $\mathbb{R}^\ell_+$, and $x_i \ R_i \ y_i$ (resp., $x_i \ P_i \ y_i$, $x_i \ I_i \ y_i$) denotes weak preference (resp., strict preference, indifference). Preferences are assumed here to be monotonic ($x_i \geq y_i$ implies $x_i \ R_i \ y_i$ and $x_i \gg y_i$ implies $x_i \ P_i \ y_i$)[9] and continuous. Let $\mathcal{R}$ denote the set of such orderings, and $\mathcal{R}^c$ the subset of $\mathcal{R}$ containing convex preferences.

An allocation is *feasible* if $\sum_{i=1}^n x_i \leq \Omega$. The set of feasible allocations is denoted $F(\Omega)$. An allocation $x \in F(\Omega)$ is *efficient* if for no other allocation $y \in F(\Omega)$, $y_i \ R_i \ x_i$ for all i and $y_i \ P_i \ x_i$ for at least one i.

A *social ordering function* (SOF) is a mapping which, for every economy in a domain, determines a (social) ordering over the set of feasible allocations $F(\Omega)$, with standard notation $x \ R \ y$, $x \ P \ y$, $x \ I \ y$. Two domains of economies will be considered here. The domain $\mathcal{D}$ is the set of economies defined by a number $n \geq 1$, a profile $(R_1, ..., R_n) \in \mathcal{R}^n$ and a bundle $\Omega \in \mathbb{R}^\ell_{++}$, while the domain $\mathcal{D}^c$ is restricted to profiles in $(\mathcal{R}^c)^n$. The domain $\mathcal{D}^c$ will be referred to hereafter as the set of convex economies.

The Walrasian SOF, denoted R_W, is defined as follows:

$$x \ R_W \ y \Leftrightarrow \max_p \min_i u_i(x_i, p) \geq \max_p \min_i u_i(y_i, p),$$

where u_i is a money-metric utility function computed as the fraction of the value of Ω that the agent needs in order to reach the current satisfaction:

$$u_i(x_i, p) = \frac{1}{p\Omega} \min \left\{ pq \mid q \in \mathbb{R}^\ell_+, \ q \ R_i \ x_i \right\}.$$

An equivalent, more graphical, definition, goes by saying that this SOF relies on the minimal bundle proportional to Ω and contained in the convex hull of the union of the individual closed upper contour sets (see point W in Fig. 1 —the thick curves are the agents' indifference curves, and the thin line below them delineates the convex hull).

[8] An ordering is a reflexive, transitive and complete binary relation.

[9] Vector inequalities are denoted $\geq, >, \gg$.

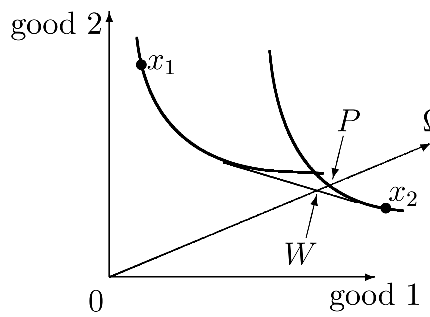

Figure 1: Evaluation of an allocation by R_W and R_{PS}

This SOF is closely related to the market mechanism, as its first best selection always coincides, in convex economies, with the subset of Walrasian equilibria with equal budgets. Pazner and Schmeidler [22], as an alternative to competitive solutions (and the related no-envy criterion —defined below) proposed to select the allocations which are Pareto-optimal, and such that every agent i is indifferent between his bundle and a particular bundle proportional to Ω, that is, such that for some real number λ, one has $x_i\ I_i\ \lambda\Omega$ for all i. As they mention in their paper, and Pazner [21] further clarified, this solution to the distribution problem can also be described by referring to the following SOF, denoted R_{PS} :

$$x\ R_{PS}\ y \Leftrightarrow \min_i v_i(x_i) \geq \min_i v_i(y_i),$$

where v_i is a representation of i's preferences defined by:

$$v_i(x_i) = \min\{v \mid v\Omega\ R_i\ x_i\}.$$

An equivalent definition, which shows a closer link to the Walrasian SOF, is as follows:

$$x\ R_{PS}\ y \Leftrightarrow \min_i \max_p u_i(x_i, p) \geq \min_i \max_p u_i(y_i, p).$$

Or, more graphically: This SOF relies on the minimal bundle proportional to Ω and contained in the union of the individual closed upper contour sets (see Fig. 1, point P). In this definition the only difference with the Walrasian SOF lies in the convex hull operation applied by the latter to the union of upper contour sets. This remark is at the root of the analysis of this paper.

3 Basic requirements

These two SOFs appear to be prominent in this model, according to the literature. And they are directly related to the two prominent allocation rules in this context, namely, the Walrasian equilibrium with equal budgets, and the egalitarian-equivalent allocation rule. The former has long since been identified as important in discussions about the existence of envy-free[10] and efficient allocations (e.g. Kolm [16]) and has later been axiomatically characterized by Gevers [14], Thomson [29], and many others. The latter has only recently been justified axiomatically by Sprumont and Zhou [25].

The two SOFs R_W and R_{PS} satisfy basic requirements, such as the following Pareto condition.

Weak Pareto: If x and y are such that for all i, $x_i \, P_i \, y_i$, then $x \, P \, y$.

They also satisfy an intuitive egalitarian requirement. It applies to pairs of agents with identical preferences, when one agent's bundle dominates the other's, and this inequality is reduced by a positive transfer. The axiom says that such a reduction of inequality is acceptable. The appeal of this axiom is rather obvious and it can be related to the Pigou-Dalton principle of transfer, which is central in the theory of inequality measurement. It can also be justified on grounds of reducing the intensity of envy, since the agent with the worse bundle envies the other one, while the other does not envy him.

Transfer Principle: If x and y are two allocations, and i and j are two agents with identical preferences, such that for some $\delta \gg 0$,

$$x_i = y_i - \delta \gg x_j = y_j + \delta,$$

whereas for all other agents k, $x_k = y_k$, then $x \, R \, y$.

Another kind of appealing condition is satisfied by the two SOFs. This condition, due to Hansson [15], is a weakening of Arrow's Independence of Irrelevant Alternatives (Arrow [1]). While Arrow's condition requires social preferences on a pair of allocations to depend only on individual preferences over this pair, Hansson's condition requires social preferences on a pair of allocations to depend only on individual closed upper contour sets at these allocations. In other words, only indifference curves at the bundles under consideration should matter, and the rest of the preference relations can be disregarded. This condition is very appealing because it guarantees that social preferences will not be sensitive to far-fetched details of the preferences. At the same time, it makes it possible to take account of relevant features which are excluded by Arrow's restrictive condition.[11]

Hansson Independence: Let x and y be two allocations, and R, R' be the social orderings for two profiles $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ respectively. If for all i, all $q \in \mathbb{R}^\ell_+$,

$$\begin{aligned} x_i \, I_i \, q \quad &\Leftrightarrow \quad x_i \, I'_i \, q \\ y_i \, I_i \, q \quad &\Leftrightarrow \quad y_i \, I'_i \, q, \end{aligned}$$

then

$$x \, R \, y \Leftrightarrow x \, R' \, y.$$

[10] Agent i is said to envy agent j if $x_j \, P_i \, x_i$. An allocation is envy-free if no agent envies any another, i.e. if for all i, j, $x_i \, R_i \, x_j$.

[11] For a more extensive discussion of Arrow's condition, see in particular Fleurbaey and Maniquet [9].

Finally, we will introduce two basic conditions of invariance, one relative to the scale of the allocation, the other relative to the size of the population. These conditions are quite uncontroversial and seem to be satisfied by all reasonable SOFs. The first one says that rescaling preferences and allocations alike does not change the ranking. It may be compared to the standard scale invariance condition for the measurement of relative inequality, usually formulated for one-dimensional distributions of income. The fact that preferences are rescaled too here (this cannot be made explicit in a one-dimensional framework) makes the condition even more acceptable, because this means that individual evaluations are attuned to the new scale.

Scale Independence: Let x and y be two allocations, and R the social ordering for the profile $(R_1, ..., R_n)$. Take $\lambda > 0$. Let $x' = \lambda x$, $y' = \lambda y$ and R' the social ordering for the profile $(R'_1, ..., R'_n)$ such that for all i and all bundles q, z, $qR_iz \Leftrightarrow \lambda q R'_i \lambda z$. Then

$$x \; R \; y \Leftrightarrow x' \; R' \; y'.$$

The second condition is formulated in terms of invariance to replication. Consider an economy with n agents, a profile $(R_1, ..., R_n)$, and a bundle Ω. A k-replicate of this economy has kn agents, a profile with R_i appearing k times for $i = 1, ..., n$, and a bundle $k\Omega$. An allocation $x = (x_1, ..., x_n)$ in the initial economy can be related to a replicated allocation $x^{(k)}$ in the k-replicate, where clones of agent i receive x_i.

Replication Independence: Consider an economy with a social ordering R, and let $R^{(k)}$ denote the social ordering for its k-replicate. Then, for any allocations x and y,

$$x \; R \; y \Leftrightarrow x^{(k)} \; R^{(k)} \; y^{(k)}.$$

All the axioms introduced in this section are satisfied by the two SOFs defined above (and many others). In the next sections, we turn to the differences between the two SOFs.

4 Analysis in convex economies

So far, the only axiom which expresses some concern for equity in the distribution of resources is Transfer Principle. This axiom displays a weak aversion to inequality and is quite innocuous as it applies only to agents with identical preferences. It certainly cannot capture all concerns for the distribution, and we will introduce here additional conditions having to do with simple comparisons of bundles. The leading idea here is that if all bundles in one allocation physically dominate the average bundle in another allocation, then there is a presumption in favor of the former. When dealing with one-dimensional distributions, the fact that the support of one distribution is greater than the mean of another distribution implies generalized Lorenz dominance and therefore guarantees that the former distribution is preferred by any reasonable social ranking. But in the multi-dimensional case, can we conclude that x is preferable to y? In other words, can we introduce the following axiom? Let $\overline{y}$ denote the mean $\frac{1}{n}\sum_i y_i$ (and similarly for any other allocation).

Support-Mean Dominance: For any pair of allocations x, y, if for all i, $x_i \gg \overline{y}$, then $x \; R \; y$.

Unfortunately, this axiom is incompatible with Weak Pareto, simply because it may happen that y dominates x for all individual preferences. This is another instance of the well-known difficulty of formulating simple equity conditions in terms of bundles without violating the Pareto principle.[12] In order to avoid a conflict with Weak Pareto, one must therefore weaken this axiom by restricting its scope. Here are various ways of doing so. Taking inspiration from Steinhaus [26] and Sprumont and Zhou [25], one may first restrict application of the axiom to the case when x is the equal-split allocation in which every agent receives $x_i = \Omega/n$. This is not enough to avoid a conflict with Pareto, and in addition one may require the other allocation y to be sufficiently unequal so that it contains a bundle, which can be denoted $\min y$, such that for all i, $y_i \geq \min y$. This yields the following axiom,[13] which is satisfied by R_{PS} but not by R_W :

Support-Mean Dominance I: For any pair of allocations x, y, if for all i,

$$x_i = \frac{\Omega}{n} \gg \overline{y} \gg \min y,$$

then $x \; R \; y$.

Another way to avoid the conflict is to require the equal-split allocation x to be efficient. This yields the following axiom, which is almost equivalent to a standard axiom of equity saying that when equal-split is efficient, it must be selected as one of the best allocations.

Efficient Equal-Split: For any pair of allocations x, y, if x is efficient and for all i,

$$x_i = \frac{\Omega}{n} \gg \overline{y},$$

then $x \; R \; y$.

This axiom is quite weak and is satisfied by both R_W and R_{PS}. But there is a quite natural way to strengthen it which separates the two solutions. Instead of requiring x to be the equal-split solution, one may require it simply to dominate an equal-split allocation for a smaller amount of resources, and the latter to dominate the mean of y.[14] This yields the following axiom, which is satisfied by R_W on the domain $\mathcal{D}^c$ but not by R_{PS} :

[12] See e.g. Fleurbaey and Trannoy [13] for an analysis of this difficulty.

[13] There are similar axioms in the literature. Fleurbaey and Maniquet [9] have a much stronger axiom saying that equal-split is at least as good as any allocation such that one agent prefers Ω/n to his own bundle. Fleurbaey [3] has an axiom saying that for allocations in which every bundle is proportional to Ω, equalizing the bundles does not yield a worse allocation. It is neither weaker nor stronger than this one.

[14] This way of strenghtening the axiom could be applied to Support-Mean Dominance I as well, without altering any of the results.

Support-Mean Dominance II: For any pair of allocations x, y, if x is efficient and for some $\lambda \leq 1$, for all i,

$$x_i \geq \lambda \frac{\Omega}{n} \gg \overline{y},$$

then $x \; R \; y$.

Notice that these axioms are compatible with any degree of aversion to inequality (including zero), since

$$x_i \gg \overline{y} \text{ for all } i \Rightarrow \overline{x} \gg \overline{y}.$$

The reasons why the two Support-Mean Dominance axioms are appealing are, however, slightly different. Support-Mean Dominance I expresses a stronger preference for equal-split whereas Support-Mean Dominance II reflects a greater sensitivity to efficiency.

The following results do not exactly characterize R_W and R_{PS}, but show that they are the coarsest orderings[15] satisfying these combinations of axioms.

Theorem 1 *Let R be a SOF which, on $\mathcal{D}^c$, satisfies Weak Pareto, Transfer Principle, Hansson Independence, Scale Independence and Support-Mean Dominance I. Then for all allocations x, y,*

$$x \; P_{PS} \; y \Rightarrow x \; P \; y.$$

Theorem 2 *Let R be a SOF which, on $\mathcal{D}^c$, satisfies Weak Pareto, Transfer Principle, Hansson Independence, Scale Independence, Replication Independence, and Support-Mean Dominance II. Then for all allocations x, y,*

$$x \; P_W \; y \Rightarrow x \; P \; y.$$

Theorem 1 does not involve Replication Independence and can be formulated for a subdomain relative to a fixed Ω (only changes of preferences are considered). In order to prove both theorems it is convenient to rely on the following lemma. It extracts an infinite inequality aversion from the first basic axioms.

Lemma 1 *If on $\mathcal{D}$ or $\mathcal{D}^c$, a SOF satisfies Weak Pareto, Transfer Principle, Hansson Independence and Scale Independence, then it satisfies the following property:*
If x and y are two allocations, and i and j are two agents with identical preferences denoted R_0, such that

$$y_i \; P_0 \; x_i \; P_0 \; x_j \; P_0 \; y_j,$$

whereas for all other agents k, $x_k \; P_k \; y_k$, then $x \; P \; y$.

The intuition for the proof of the two theorems can be explained as follows. Consider allocations x and y on Fig. 2. In this example, allocation x is better for both orderings.

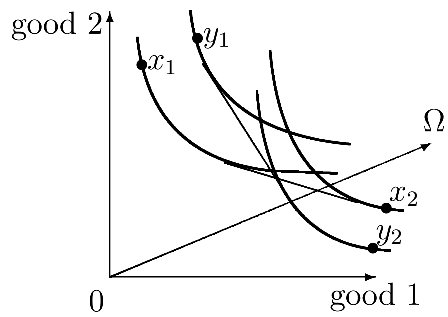

Figure 2: $x\ P_{PS}\ y$ and $x\ P_W\ y$

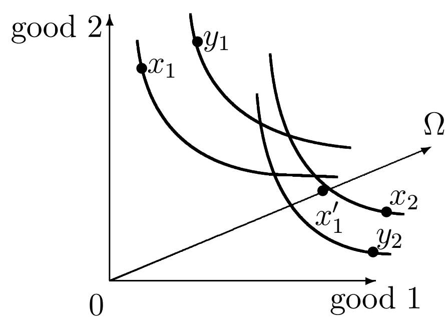

Figure 3: $x\ P\ x'$

Suppose that, contrary to the desired conclusion of Theorem 1, one has $y\ R\ x$. By Hansson Independence, other indifference curves could be anything. Moreover, by Hansson Independence and Lemma 1, it is always bad to separate further two agents who are on nested indifference curves, since they could have the same preferences.

Define a new allocation x' by giving the same bundle $x_1' = x_2'$ to both agents, just below $v_2(x_2)\Omega$, on the ray defined by Ω (Fig. 3). For a moment, assume that x' and all other constructed allocations are feasible. By Weak Pareto, one has $x\ P\ x'$, so that $y\ P\ x'$.

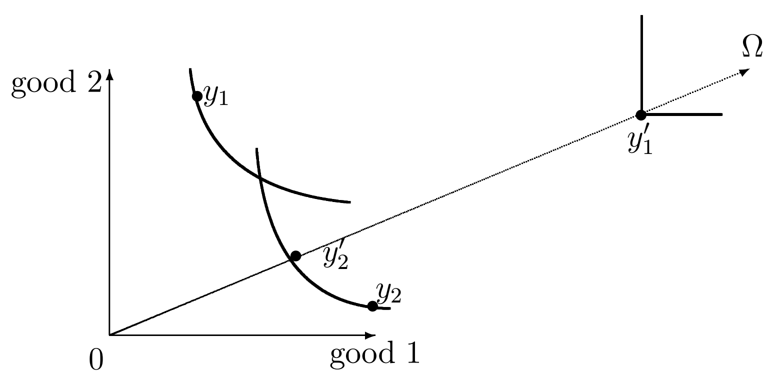

Figure 4: $y'\ P\ y$

[15]One may say that R' is (weakly) coarser than R when $x\ R\ y$ implies $x\ R'\ y$ or, equivalently, when $x\ P'\ y$ implies $x\ P\ y$.

Similarly, an allocation y' better than y by Weak Pareto can be constructed by raising agent 1 to y_1', and agent 2 to y_2' just above $v_2(y_2)\Omega$ (Fig. 4). By transitivity, $y'\ P\ x'$.

Then, because indifference curves are nested, one can invoke Hansson Independence and Lemma 1, and improve on y' by pulling down agent 1 from y_1' to a bundle just above x_1', and raising agent 2 to a bundle just above y_2'. This yields a new allocation y'' which should be better than y', and therefore than x'. But with an appropriate choice of bundles y_1'', y_2'' and x_1', one has $x_1' \gg \frac{1}{2}(y_1'' + y_2'') \gg y_2''$.

So far, we have assumed that all allocations involved in the reasoning were feasible. This cannot be true in general. But by a suitable scale reduction of allocations and preferences, all the allocations may be rendered feasible. The above reasoning then leads to allocations $\lambda y''$ and $\lambda x'$ which are homothetic reductions of y'' and x' and belong to $F(\Omega)$. By Scale Independence, $\lambda y''$ is preferred to $\lambda x'$. Now use a scale expansion, leading to two allocations $\lambda\lambda' y''$ and $\lambda\lambda' x'$ so as to have $\sum_{i=1}^n \lambda\lambda' x_i' = \Omega$. By Scale Independence, $\lambda\lambda' y''$ is preferred to $\lambda\lambda' x'$. But the latter is the equal-split allocation, so that Support-Mean Dominance II implies that it is at least as good as $\lambda\lambda' y''$, a contradiction. This contradiction proves that the assumption $y\ R\ x$ was wrong. Necessarily one must have $x\ P\ y$, which is the desired conclusion.

Here is an illustration of the proof of Theorem 2. Consider again allocations x and y on Fig. 2. Imagine that, contrary to the result of Theorem 2, one has $y\ R\ x$. Recall that by Hansson Independence, other indifference curves could be anything. Therefore, by a combined use of Weak Pareto, Lemma 1 and Hansson Independence, one can show that allocation x' as shown on Fig. 5, is worse than x.

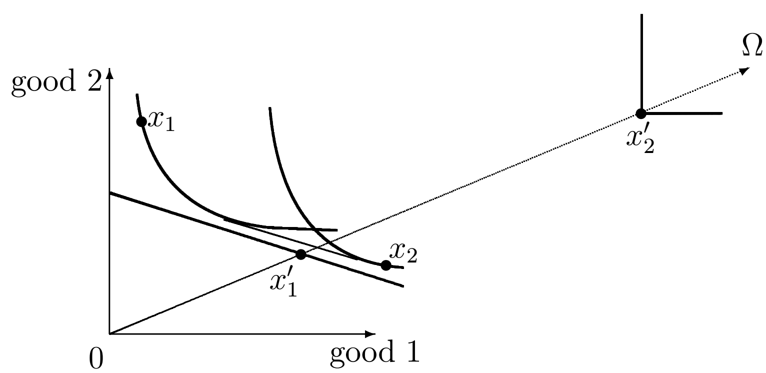

Figure 5: $x\ P\ x'$

By transitivity, $y\ P\ x'$. Now, by Replication Independence, every agent can be given an arbitrarily large (but equal) number $k-1$ of clones without altering the comparison, so that $y^{(k)}\ P^{(k)}\ x'^{(k)}$. Let y' be an allocation such that one agent of each sort is just better-off than in y, while all her $k-1$ clones are given x_2' (see Fig. 6). By Weak Pareto, $y'\ P^{(k)}\ y^{(k)}$. By transitivity, $y'\ P^{(k)}\ x'^{(k)}$.

Then, because indifference curves are nested, one can refer to Hansson Independence and notice that every clone who receives x_2' in y' could have the same preferences as any of the initial agents. Therefore, by Lemma 1, the clones at x_2' in y' can be brought back near

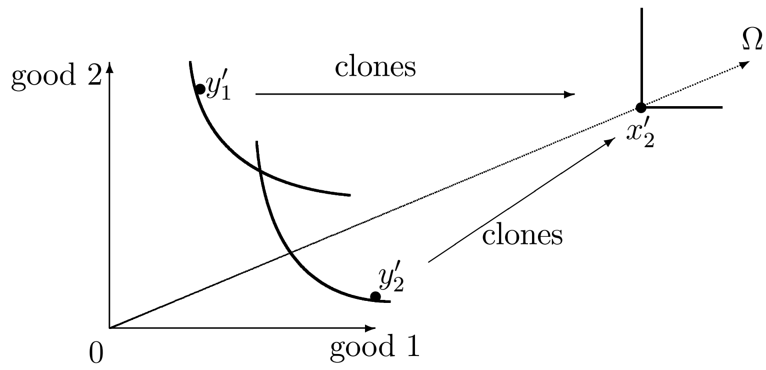

Figure 6: $y'\ P^{(k)}\ y$

the agents who are just above y, in arbitrary proportions. This yields a new allocation y'' which is just above the indifference curves at y, and if the bundles in y'' are well located, and the reallocation of clones among initial agents is well apportioned, one can obtain $\frac{1}{nk}\sum_i y''_i \ll x'_1$ (see Fig. 7). Then, by transitivity, one has y'' preferred to $x'^{(k)}$.

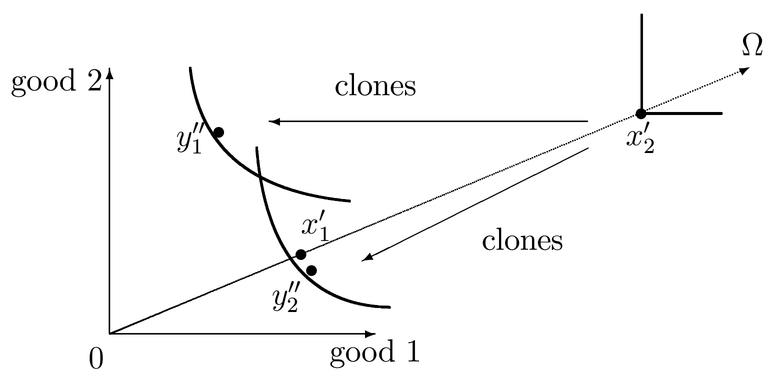

Figure 7: $y''\ P^{(k)}\ y'$

Again, Scale Independence makes it possible to deal with the feasibility of all allocations considered in the argument. As above, a λ-reduction and a λ'-expansion lead to two allocations $\lambda\lambda' y''$ and $\lambda\lambda' x'^{(k)}$ such that the former is preferred to the latter and $\sum_{i=1}^{n} \lambda\lambda' x'_i = \Omega$. But for the $\lambda\lambda'$-rescaled individual preferences, $\lambda\lambda' x'$ is efficient and so is $\lambda\lambda' x'^{(k)}$ in the replicated economy. Therefore Support-Mean Dominance II implies that $\lambda\lambda' x'^{(k)}$ is at least as good as $\lambda\lambda' y''$, a contradiction.

5 Extension to non-convex economies

On the larger domain $\mathcal{D}$, the Pazner-Schmeidler SOF still satisfies Support-Mean Dominance I, but the Walrasian SOF does no longer satisfy Support-Mean Dominance II. This is actually the symptom of an important ethical drawback of the latter SOF.

It is well known that the competitive mechanism may break down in non-convex economies. This difficulty has echoes in the theory of fairness, for instance in the possible non-existence of envy-free and efficient allocations (Varian [30]), or even of efficient allocations in which no individual bundle strictly dominates another (Maniquet [17]).

The Walrasian SOF R_W partly remedies this difficulty. It selects Walrasian allocations with equal budgets whenever they exist, even in non-convex economies, and is well defined in all economies, including the non-convex. But, unfortunately, this last fact does not guarantee that it always yields *appealing* social preferences in non-convex economies. The following example may show the problem.

Fig. 8 displays an economy with two agents, where the first-best optimal allocation for the Walrasian SOF R_W is quite unequal, and this seems rather unjustified when one looks at the agents' preferences. In particular, they have identical preferences on the Edgeworth box.

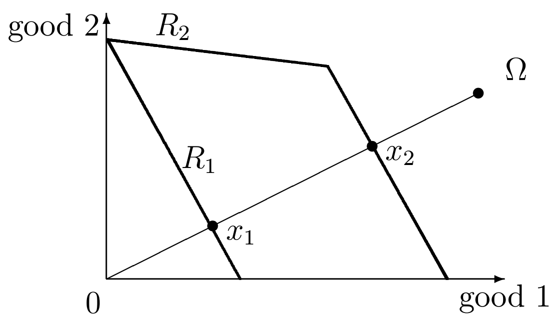

Figure 8: A first-best allocation for R_W

The problem displayed in this particular example can be partly alleviated by considering a constrained Walrasian solution, that focuses only on the part of upper contour sets which is included in the Edgeworth box. But this does not tackle similar problems due to strange shapes of indifference curves *within* the Edgeworth box. In contrast, the Pazner-Schmeidler SOF always avoids such gross inequalities and, at the minimum, always guarantees that, at the first-best optimal allocation, all agents are at least as well-off as at the equal split allocation.

Let us consider the axiom which combines the restrictions of application of Support-Mean Dominance I and II. This yields a very weak axiom, which is weaker than Efficient Equal-Split.

Minimal Preference for Equality: For any pair of allocations x, y, if x is efficient and for all i,

$$x_i = \frac{\Omega}{n} \gg \overline{y} \gg \min y,$$

then $x \; R \; y$.

It is satisfied by both R_W and R_{PS} in convex economies, and seems to be a minimal condition to require in non-convex economies as well. A natural question, now, is whether one can find an interesting extension of R_W to non-convex economies which, like R_{PS}, satisfies this minimal condition on the larger domain as well. Whether the answer is positive or negative is a matter of interpretation, but its substance is certainly quite precise, as shown in the following theorem.

Theorem 3 *Let R be a SOF which, on $\mathcal{D}$, satisfies Weak Pareto, Transfer Principle, Hansson Independence, Scale Independence, and Minimal Preference for Equality. Then for all allocations x, y,*

$$x \; P_{PS} \; y \Rightarrow x \; P \; y.$$

The main argument for this result is similar to the proof of Theorem 1. Notice that in the illustration of the proof of Theorem 1 in the previous section, allocation x' was egalitarian, but not efficient. In the domain of non-convex preferences, it is now possible to obtain an egalitarian and efficient allocation by giving a non-convex preference relation to agent 1, with an indifference curve just below the envelope of the union of upper contour sets at x, and by giving Leontief preferences to agent 2 (see Fig. 9).

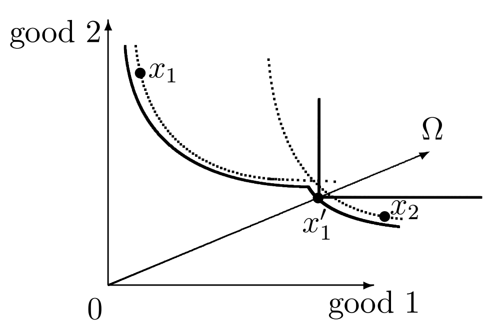

Figure 9: x' is egalitarian and efficient

Notice that this cannot be done in one step because the Leontief indifference curve of agent 2 would cut her indifference curve at x. But this can be done in three steps. First, one puts both agents down, so that agent 1's indifference curve already contains agent 2's one. Then by Lemma 1, they can be put further apart and agent 2 is raised to a bundle high enough so as to be on indiffference curves that (after use of Hansson Independence) do not cut the target Leontief curves at $x'_2 = x'_1$ (as well as at other relevant allocations). Finally, both are pulled down again to x', with the indifference curves as in Fig. 9. The rest of the argument is similar.

6 Conclusion

The results of this paper suggest that the differences between the Walrasian SOF and the Pazner-Schmeidler SOF are rather small. In convex economies, they diverge on the fact that the former favors efficient allocations more, while the latter favors equal-split in a more direct way. In non-convex economies, the Pazner-Schmeidler SOF appears to be the best one, even if one refers only to properties satisfied by the Walrasian SOF in convex economies. In a sense, then, for non-convex economies one can view R_{PS} as the best extension of R_W, a rather surprising idea.

The possibility to define appealing social preferences is important if one looks for social criteria applicable in cost-benefit analysis and the like. This was, after all, the main motivation in good old welfare economics. The two criteria characterized here can be extended and applied in various contexts, although significantly different models (with production, for instance, or public goods) deserve a full-blown analysis of their own.[16]

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Appendix: Proofs

Let coA denote the closed convex hull of set A (i.e., the closure of the ordinary convex hull). For any bundle $z \in \mathbb{R}_+^\ell$, and any set $A \subset \mathbb{R}_+^\ell$, let

$$A - z = \left\{q \in \mathbb{R}_+^\ell \mid q + z \in A\right\}.$$

The notation $z < A$ means that for all $q \in A$, $z < q$, and $A < B$ means that for all $q \in A$, $q' \in B$, $q < q'$.

Let $C(R_i, x_i)$, $c(R_i, x_i)$ and $I(R_i, x_i)$ denote the upper contour sets and indifference "curve"

$$\begin{aligned} C(R_i, x_i) &= \left\{q \in \mathbb{R}_+^\ell \mid q\ R_i\ x_i\right\} \\ c(R_i, x_i) &= \left\{q \in \mathbb{R}_+^\ell \mid q\ P_i\ x_i\right\} \\ I(R_i, x_i) &= \left\{q \in \mathbb{R}_+^\ell \mid q\ I_i\ x_i\right\}. \end{aligned}$$

Proof of Lemma 1: Let x and y be two allocations, and i and j two agents with identical preferences denoted R_0, such that

$$y_i\ P_0\ x_i\ P_0\ x_j\ P_0\ y_j,$$

whereas for all $k \neq i, j$, $x_k\ P_k\ y_k$.

We first focus on the domain $\mathcal{D}^c$.

First case: $C(R_0, y_i) \not> I(R_0, x_i) \cap \mathbb{R}^\ell_{++}$.[17] Choose $x_0 \in I(R_0, x_i) \cap \mathbb{R}^\ell_{++}$ such that $C(R_0, y_i) \not> x_0$. Let $x_i^1, y_i^1, x_j^1, y_j^1$ be bundles proportional to x_0 and such that

$$x_i^1 = x_0;\ y_i^1\ I_0\ y_i;\ x_j^1\ I_0\ x_j;\ y_j^1\ I_0\ y_j.$$

Since $C(R_0, y_i) \not> x_0$, there exists $\widehat{y}_0 \in I(R_0, y_i)$, such that for all $q < \widehat{y}_0$,

$$q \notin co\left[\{x_0\} \cup C(R_0, y_i)\right].$$

Necessarily $\widehat{y}_0 \neq y_i^1$ since $y_i^1 > x_0$ and obviously $x_0 \in co\left[\{x_0\} \cup C(R_0, y_i)\right]$.

If $\widehat{y}_0 \gg 0$, let $y_0 = \widehat{y}_0$. Otherwise, take a sequence $(\widehat{y}_{0t})_t \to \widehat{y}_0$ such that $\widehat{y}_{0t} \gg 0$ and $\widehat{y}_{0t}\ I_0\ y_i$ for all t. Let $\widehat{x}^2_{it}, \widehat{y}^2_{it}, \widehat{x}^2_{jt}, \widehat{y}^2_{jt}$ be bundles proportional to $\widehat{y}_{0t}$ and such that

$$\widehat{x}^2_{it}\ I_0\ x_i;\ \widehat{y}^2_{it} = \widehat{y}_{0t};\ \widehat{x}^2_{jt}\ I_0\ x_j;\ \widehat{y}^2_{jt}\ I_0\ y_j.$$

Let

$$\begin{aligned} \varepsilon_t^2 &= \frac{1}{3}\min\left\{\hat{x}^2_{jt} - \hat{y}^2_{jt}, \hat{y}^2_{it} - \hat{x}^2_{it}\right\}, \\ q_t &= \widehat{y}_{0t} - \frac{\varepsilon_t^2}{4}. \end{aligned}$$

For all t, $q_t < \widehat{y}_{0t}$ and, since $\lim \varepsilon_t^2 > 0$, $\lim q_t < \widehat{y}_0$. Therefore

$$\lim q_t \notin co\left[\{x_0\} \cup C(R_0, y_i)\right].$$

This convex hull is closed so that there is t^* such that

$$q_{t^*} \notin co\left[\{x_0\} \cup C(R_0, y_i)\right].$$

Then let $y_0 = \widehat{y}_{0t^*}$. Similarly let

$$x_i^2 = x^2_{it^*};\ y_i^2 = y_0;\ x_j^2 = x^2_{jt^*};\ y_j^2 = y^2_{jt^*}.$$

Let

$$\varepsilon^1 = \frac{1}{3}\min\left\{x_j^1 - y_j^1, y_i^1 - x_i^1\right\};\ \varepsilon^2 = \varepsilon^2_{t^*};$$

$$\begin{aligned} a_i &= y_i^2 + \frac{\varepsilon^2}{2};\ b_i = y_i^2 - \frac{\varepsilon^2}{2};\ c_i = x_i^1 + \frac{\varepsilon^1}{2};\ d_i = x_i^1 - \frac{\varepsilon^1}{2}; \\ a_j &= y_j^2 + \frac{\varepsilon^2}{4};\ b_j = a_j + \varepsilon^2;\ c_j = x_j^1 - \frac{5\varepsilon^1}{4};\ d_j = c_j + \varepsilon^1. \end{aligned}$$

Recall that $c_i \gg x_i^1 = x_0$, so that

$$co\left[\{c_i\} \cup C(R_0, y_i)\right] \subset co\left[\{x_0\} \cup C(R_0, y_i)\right],$$

[17] Continuity and monotonicity of preferences, together with $x_i\ P_0\ 0$, imply that $I(R_0, x_i) \cap \mathbb{R}^\ell_{++}$ is not empty.

and since $b_i \ll q_{t^*}$,

$$b_i \notin co\left[\{x_0\} \cup C(R_0, y_i)\right].$$

Therefore, for $\eta \gg 0$ small enough, one can have

$$b_i \notin co\left[\{c_i\} \cup (C(R_0, y_i) - \eta)\right] \subset c(R_0, x_i).$$

Besides, by construction (values of $\varepsilon^1, \varepsilon^2$), $b_i\ P_0\ x_i$ and $y_i\ P_0\ c_i$.

By Hansson Independence, the ranking of x and y depends only on $I(R_0, x_i)$, $I(R_0, y_i)$, $I(R_0, x_j)$, $I(R_0, y_j)$, so that the other indifference curves can be modified at will. Therefore one can let

$$C(R_0, c_i) = co\left[\{c_i\} \cup (C(R_0, y_i) - \eta)\right],$$

which implies that $c_i\ P_0\ b_i$. Similarly one can assume that

$$\frac{1}{2}\left(x_j^1 + y_j^1\right)\ I_0\ \frac{1}{2}\left(x_j^2 + y_j^2\right),$$

entailing that

$$c_j = \frac{1}{2}\left(x_j^1 + y_j^1\right) + \frac{\varepsilon^1}{4}\ P_0\ b_j = \frac{1}{2}\left(x_j^2 + y_j^2\right) - \frac{\varepsilon^2}{4}.$$

For $k \neq i, j$, let a_k, b_k, c_k, d_k be chosen so that

$$x_k\ P_k\ d_k = c_k\ P_k\ b_k = a_k\ P_k\ y_k.$$

Assume for the moment that the allocations a, b, c, d are feasible. By Weak Pareto, $a\ P\ y$, $c\ P\ b$ and $x\ P\ d$. By Transfer Principle, $b\ R\ a$ and $d\ R\ c$. By transitivity, $x\ P\ y$.

If any of the allocations a, b, c, d is not feasible, let $\lambda < 1$ be such that $\lambda a, \lambda b, \lambda c, \lambda d$ are feasible. Let R' be the social ordering for the profile $(R_1', ..., R_n')$ such that for all i and all bundles q, z, $qR_iz \Leftrightarrow \lambda q R_i' \lambda z$. By the above argument, one has $\lambda x\ P'\ \lambda y$, and by Scale Independence, $x\ P\ y$.

Second case: $C(R_0, y_i) > I(R_0, x_i) \cap \mathbb{R}^\ell_{++}$. One can find z_i such that $y_i\ P_0\ z_i\ P_0\ x_i$ and neither $C(R_0, y_i) > I(R_0, z_i) \cap \mathbb{R}^\ell_{++}$ nor $C(R_0, z_i) > I(R_0, x_i) \cap \mathbb{R}^\ell_{++}$. Take z_j such that $x_j\ P_0\ z_j\ P_0\ y_j$. By the above argument, one shows that $z\ P\ y$ and $x\ P\ z$ separately, so that by transitivity $x\ P\ y$.

On the domain $\mathcal{D}$, one can no longer be sure that for η small enough,

$$co\left[\{c_i\} \cup (C(R_0, y_i) - \eta)\right] \subset c(R_0, x_i).$$

One can instead let

$$C(R_0, c_i) = co\left[\{c_i\} \cup (C(R_0, y_i) - \eta)\right] \cap (C(R_0, x_0 + \eta)),$$

for η small enough so that $b_i \notin C(R_0, c_i)$ and $x_0 + \eta \ll c_i$. The rest of the proof is the same. ■

Lemma 2 *Let two allocations x, y and one agent i_0 be such that $x_{i_0}\ P_{i_0}\ y_{i_0}$ and $y_i\ P_i\ x_i$ for all $i \neq i_0$, while*

$$c(R_{i_0}, x_{i_0}) \supset \bigcup_{i \neq i_0} C(R_i, x_i).$$

For any SOF R satisfying Weak Pareto, Transfer Principle, Hansson Independence and Scale Independence, one then has $x\ P\ y$.

Proof: For simplicity of notation, let us assume that $i_0 = n$. Let $z^0, ..., z^{n-1}$ be a sequence of allocations such that $z^0 = y$, $z^{n-1} = x$,

$$x_n\ P_n\ z_{n-1}^{n-2}\ P_n ... P_n\ z_{n-1}^1\ P_n\ y_n$$

and for $i = 1, ..., n-2$,

$$z_i^{i-1}\ P_i ... P_i\ z_i^1\ P_i\ y_i\ P_i\ x_i\ P_i\ z_i^{n-2}\ P_i ... P_i\ z_i^i.$$

(For $i = 1$, this formula reads

$$y_1\ P_1\ x_1\ P_1\ z_1^{n-2}\ P_1 ... P_1\ z_1^1$$

and for $i = n-1$,

$$z_{n-1}^{n-2}\ P_{n-1} ... P_{n-1}\ z_{n-1}^1\ P_{n-1}\ y_{n-1}\ P_{n-1}\ x_{n-1}.)$$

In addition, let us require that

$$c(R_n, x_n) \supset \bigcup_{i \neq n} C(R_i, z_i^i).$$

Notice that necessarily $x_i\ P_i\ 0$ for all $i \neq n$, so that the allocations $z^1, ..., z^{n-2}$ exist. However, they may not all be feasible, but by Scale Independence one can work on a reduction of these allocations (details can be worked out as in the proof of Lemma 1). For $k = 1, ..., n-1$, let R^k be an individual preference relation such that

$$C(R^k, z_k^{k-1}) = C(R_k, z_k^{k-1}) \text{ and } C(R^k, z_k^k) = C(R_k, z_k^k),$$

$$\forall q,\ x_n\ R_n\ q \Rightarrow C(R^k, q) = C(R_n, q)$$

Such a relation exists thanks to the fact that

$$c(R_n, x_n) \supset C(R_k, z_k^k) \supset C(R_k, z_k^{k-1}).$$

For $k = 1, ..., n-1$, let $(R_1, ..., R_{k-1}, R^k, R_{k+1}, ..., R_{n-1}, R^k)$ be a profile with related social ordering $R^{(k)}$. In this profile, agents k and n have identical preferences R^k, and

$$z_k^{k-1}\ P^k\ z_k^k\ P^k\ z_n^k\ P^k\ z_n^{k-1},$$

while for all other agents i, $z_i^k\ P_i\ z_i^{k-1}$. Therefore, by Lemma 1, one has $z^k\ P^{(k)}\ z^{k-1}$.

Now, by construction one has

$$\begin{aligned} C(R^k, z_k^k) &= C(R_k, z_k^k) \text{ and } C(R^k, z_k^{k-1}) = C(R_k, z_k^{k-1}), \\ C(R^k, z_n^k) &= C(R_n, z_n^k) \text{ and } C(R^k, z_n^{k-1}) = C(R_n, z_n^{k-1}), \end{aligned}$$

so that by Hansson Independence, $z^k \; P^{(k)} \; z^{k-1}$ if and only if $z^k \; P \; z^{k-1}$.

By transitivity, one obtains $z^{n-1} \; P \; z^0$, that is, $x \; P \; y$. ■

Proof of Theorem 1:

Let x and y be two allocations such that $x \; P_{PS} \; y$. That is,

$$\min_i v_i(x_i) > \min_i v_i(y_i).$$

Suppose that, contrary to the desired result, one has $y \; R \; x$.

Let $\alpha, \beta, \gamma, \delta, \lambda, \lambda'$ and ω be such that

$$\begin{aligned} &\min_i v_i(x_i) > \alpha > \beta > \gamma > \delta > \min_i v_i(y_i), \\ &\beta > \tfrac{(n-1)\alpha+\gamma}{n}, \\ &\omega > \max\{\max_i v_i(x_i), \max_i v_i(y_i)\}, \\ &\lambda = \tfrac{1}{\max\{1, n\beta, (n-1)\omega+\delta, (n-1)\alpha+\gamma\}}, \\ &\lambda' = \tfrac{1}{\lambda n \beta}, \end{aligned}$$

and let i_0 be an agent such that $\min_i v_i(y_i) = v_{i_0}(y_{i_0})$.

Notice that $\lambda \leq 1$. Let $(R_1', ..., R_n')$ be the profile obtained by λ-reduction. That is, for every i, $qR_iz \Leftrightarrow \lambda q R_i' \lambda z$. Let R' denote the corresponding social ordering. By Scale Independence, $\lambda y \; R' \; \lambda x$.

Let x' be an allocation defined by $x_i' = \beta\Omega$ for all i. Since $\lambda n \beta \leq 1$, $\lambda x'$ belongs to $F(\Omega)$. By Weak Pareto, $\lambda x \; P' \; \lambda x'$. By transitivity, $\lambda y \; P' \; \lambda x'$.

Let $(R_1^1, ..., R_n^1)$ be a profile such that for all i,

$$\begin{aligned} C(R_i^1, \lambda\beta\Omega) &= C(R_i', \lambda\beta\Omega) \text{ and } C(R_i^1, \lambda y_i) = C(R_i', \lambda y_i), \\ C(R_i^1, \lambda\omega\Omega) &= \left\{q \in \mathbb{R}_+^\ell \mid q \geq \lambda\omega\Omega\right\}, \end{aligned}$$

and let R^1 denote the related social ordering. By Hansson Independence, $\lambda y \; R^1 \; \lambda x'$.

Let y' be an allocation defined by $y_{i_0}' = \delta\Omega$, and for all $i \neq i_0$,

$$y_i' = \omega\Omega.$$

Since $\lambda\left((n-1)\omega + \delta\right) \leq 1$, $\lambda y'$ belongs to $F(\Omega)$. By Weak Pareto, $\lambda y' \; P^1 \; \lambda y$. By transitivity, $\lambda y' \; P^1 \; \lambda x'$.

Let $(R_1^2, ..., R_n^2)$ be a profile such that for all i,

$$\begin{aligned} C(R_i^2, \lambda\beta\Omega) &= C(R_i^1, \lambda\beta\Omega) \text{ and } C(R_i^2, \lambda y_i') = C(R_i^1, \lambda y_i'), \\ C(R_i^2, \lambda\alpha\Omega) &= \left\{q \in \mathbb{R}_+^\ell \mid q \geq \lambda\alpha\Omega\right\}, \end{aligned}$$

and let R^2 denote the related social ordering. By Hansson Independence, $\lambda y' \; P^2 \; \lambda x'$.

Let y'' be an allocation defined by $y''_{i_0} = \gamma\Omega$, and for all $i \neq i_0$,

$$y''_i = \alpha\Omega.$$

Since $\lambda\left((n-1)\alpha + \gamma\right) \leq 1$, $\lambda y''$ belongs to $F(\Omega)$.

By construction,

$$c(R^2_{i_0}, \lambda y''_{i_0}) \supset \bigcup_{i \neq i_0} C(R^2_i, \lambda y''_i).$$

In addition, $\lambda y''_{i_0}\ P^2_{i_0}\ \lambda y'_{i_0}$ and $\lambda y'_i\ P^2_i\ \lambda y''_i$ for all $i \neq i_0$. By Lemma 2, one has $\lambda y''\ P^2\ \lambda y'$, so that by transitivity, $\lambda y''\ P^2\ \lambda x'$.

Notice that $\lambda' \geq 1$. Let $(R^{2\prime}_1, ..., R^{2\prime}_n)$ be the profile obtained from $(R^2_1, ..., R^2_n)$ by λ'-expansion. That is, for every i, $qR^2_i z \Leftrightarrow \lambda' q R^{2\prime}_i \lambda' z$. Let $R^{2\prime}$ denote the corresponding social ordering. By Scale Independence, $\lambda\lambda' y''\ P^{2\prime}\ \lambda\lambda' x'$. One has, for all i, $\lambda\lambda' x'_i = \Omega/n$. In addition, recall that $\beta > \left((n-1)\alpha + \gamma\right)/n$, so that for all i,

$$\lambda\lambda' x'_i = \frac{\Omega}{n} = \lambda\lambda'\beta\Omega \gg \lambda\lambda'\overline{y''} = \lambda\lambda'\frac{(n-1)\alpha + \gamma}{n}\Omega \gg \min \lambda\lambda' y'' = \lambda\lambda'\gamma\Omega.$$

Therefore, by Support-Mean Dominance I, one should have $\lambda\lambda' x'\ R^{2\prime}\ \lambda\lambda' y''$, a contradiction.

Finally, we check that no axiom is redundant. That is, if one axiom is dropped, then one can find a SOF R which satisfies the remaining axioms and such that $x\ P_{PS}\ y$ does not imply $x\ P\ y$.

Dropping Weak Pareto. Take the SOF $\widetilde{R}^1$ such that $x\ \widetilde{I}^1\ y$ for all x, y.

Dropping Transfer Principle. Let $I(x)$ the set of allocations which are Pareto-indifferent to x, and D the set of allocations in which one bundle is weakly dominated by the others:

$$\begin{aligned} I(x) &= \left\{x' \in \mathbb{R}^{n\ell}_+ \mid \forall i = 1, ..., n,\ x_i\ I_i\ x'_i\right\}, \\ D &= \left\{x \in \mathbb{R}^{n\ell}_+ \mid \exists i, \forall j \neq i,\ x_j \geq x_i\right\}. \end{aligned}$$

Take the SOF $\widetilde{R}^2$ defined by:

$$x\ \widetilde{R}^2\ y \Leftrightarrow V^2(x) \geq V^2(y),$$

with

$$V^2(x) = \min\left\{\lambda \in \mathbb{R}_+ \mid \forall z \in I(x) \cap D,\ \lambda\Omega \geq \overline{z}\right\}.$$

Dropping Hansson Independence. Take the SOF $\widetilde{R}^3$ which coincides with R_{PS} if there are agents with identical preferences, and with $\widetilde{R}^2$ otherwise.

Dropping Scale Independence. Let $Z = (z_\alpha)_{\alpha \in \mathbb{R}_+}$ be a monotonic path (i.e. for all $\alpha > \alpha'$, $z_\alpha \gg z_{\alpha'}$) in $\mathbb{R}^\ell_+$ such that

$$Z \cap \left\{q \in \mathbb{R}^\ell_+ \mid \exists\lambda \geq 0,\ q = \lambda\Omega\right\} = \{0, \Omega/n\}.$$

For all i, let w_i be a utility function representing R_i and defined by $x_i\ I_i\ z_{w_i(x_i)}$. Take the SOF $\widetilde{R}^4$ defined by

$$x\ \widetilde{R}^4\ y \Leftrightarrow \min_i w_i(x_i) \geq \min_i w_i(y_i).$$

Dropping Support-Mean Dominance I. Take R_W. ■

Proof of Theorem 2:

Let x and y be two allocations such that $x \; P_W \; y$. Suppose that, contrary to the result, one has $y \; R \; x$. The latter fact, by Weak Pareto, implies the existence of some i_0 such that $y_{i_0} \; R_{i_0} \; x_{i_0}$.

Let

$$p_x \in \arg \max_p \min_i u_i(x_i, p),$$

and $\alpha, \beta, \gamma, \delta, \lambda, \varsigma$ and ω be such that

$$\begin{array}{l} \max_p \min_i u_i(x_i, p) > \alpha > \beta > \gamma > \delta > \lambda > \max_p \min_i u_i(y_i, p), \\ \omega > \varsigma > \max \{\max_i v_i(x_i), \max_i v_i(y_i)\}. \end{array}$$

Let R'_{i_0} be such that

$$C(R'_{i_0}, x_{i_0}) = C(R_{i_0}, x_{i_0}), \; C(R'_{i_0}, y_{i_0}) = C(R_{i_0}, y_{i_0}),$$

for all $0 \leq \theta \leq \alpha$,

$$C(R'_{i_0}, \theta\Omega) = \left\{ q \in \mathbb{R}^\ell_+ \mid p_x q \geq \theta p_x \Omega \right\},$$

and

$$C(R'_{i_0}, \omega\Omega) = \left\{ q \in \mathbb{R}^\ell_+ \mid q \geq \omega\Omega \right\}.$$

For $i \neq i_0$, let R'_i be such that

$$C(R'_i, x_i) = C(R_i, x_i), \; C(R'_i, y_i) = C(R_i, y_i),$$

and

$$C(R'_i, \omega\Omega) = \left\{ q \in \mathbb{R}^\ell_+ \mid q \geq \omega\Omega \right\}.$$

Let R' be the social ordering for the profile $(R'_1, ..., R'_n)$. By Hansson Independence, $y \; R' \; x$.

In order to reduce notational complexity, it is assumed that all constructed allocations below are feasible. Using homothetic reductions of preferences and allocations and invoking Scale Independence always makes it possible (see the proof of Th. 1 for a rigorous treatment of this issue). Let x' be such that for all $i \neq i_0$, $x_i \; P_i \; x'_i$ and

$$C(R_i, x'_i) \subset \left\{ q \in \mathbb{R}^\ell_+ \mid p_x q > \alpha p_x \Omega \right\},$$

whereas $x'_{i_0} = \alpha\Omega$. By Weak Pareto, $x \; P' \; x'$, so that $y \; P' \; x'$.

Let x'' be such that for all $i \neq i_0$, $x''_i = \omega\Omega$, while $x''_{i_0} = \beta\Omega$. Since

$$c(R'_{i_0}, x'_{i_0}) \supset \bigcup_{i \neq i_0} C(R'_i, x'_i),$$

by Lemma 2 one has $x' \; P' \; x''$, and therefore $y \; P' \; x''$.

Since

$$\max_p \min_i u_i(y_i, p) < \lambda,$$

there exists an allocation y' and positive integers $a_1, ..., a_n$ such that $y'_i \ P'_i \ y_i$ for all i, and

$$\frac{1}{\sum_i a_i} \sum_i a_i y'_i \ll \lambda\Omega.$$

Let $k = \sum_i a_i$, and consider the k-replicate of the economy, with related social ordering $R'^{(k)}$. Let the agents' labels in the replicate be denoted it for the tth version of agent i, with $t = 1, ..., k$ and $i = 1, ..., n$. Consider the allocation y'' such that $y''_{i1} = y'_i$, and $y''_{it} = \omega\Omega$ for all $t = 2, ..., k$. By Weak Pareto, $y'' \ P'^{(k)} \ y$, and therefore, by Replication Independence and transitivity, $y'' \ P'^{(k)} \ x''^{(k)}$.

Let the allocation x''' be defined by $x'''_{i_0 1} = \delta\Omega$, $x'''_{i_0 t} = \gamma\Omega$ for all $t = 2, ..., k$, and $x'''_{it} = \varsigma\Omega$ for all $i \neq i_0$. By Weak Pareto, $x''^{(k)} \ P'^{(k)} \ x'''$. Let the allocation x^* be defined by $x^*_{i_0 1} = \lambda\Omega$ and $x^*_{it} = \omega\Omega$ for all $(i, t) \neq (i_0, 1)$. By Lemma 2, $x''' \ P'^{(k)} \ x^*$, so that $y'' \ P'^{(k)} \ x^*$.

Now, consider a new profile $(R''_{11}, ..., R''_{1k}, ..., R''_{n1}, ..., R''_{nk})$ defined by: $R''_{i1} = R'_i$ for all i, and among the $(k-1)n$ remaining agents (who all have $\omega\Omega$ in both y'' and x^*), $a_i n - 1$ of them have a preference relation equal to R'_i. By reordering the agents, and letting $s_i = a_i n$ for $i = 1, ..., n$, the new profile may be denoted

$$(R''_{11}, ..., R''_{1s_1}, ..., R''_{n1}, ..., R''_{ns_2}),$$

so as to have $R''_{ij} = R''_{im}$ for all j, m in $\{1, ..., s_i\}$.

Let R'' denote the related social ordering. Since for all i, j,

$$C(R''_{ij}, y''_{ij}) = C(R'_{ij}, y''_{ij}), \ C(R''_{ij}, x^*_{ij}) = C(R'_{ij}, x^*_{ij}),$$

by Hansson Independence, one has $y'' \ P'' \ x^*$.

Close to y', in the initial economy with n agents there exist two allocations y''' and y^* such that for all i,

$$\omega\Omega \ P'_i \ y^*_i \gg y'''_i \gg y'_i,$$

and

$$\frac{1}{\sum_i a_i} \sum_i a_i y^*_i \ll \lambda\Omega.$$

In the replicated economy, let y^{**} be the allocation defined by $y^{**}_{i1} = y'''_i$ for all $i = 1, ..., n$, and $y^{**}_{ij} = y^*_i$ for all $i = 1, ..., n$ and all $j = 2, ..., s_i$. By Lemma 2, $y^{**} \ P'' \ y''$, so that $y^{**} \ P'' \ x^*$.

But x^* is such that for all i, j,

$$x^*_{ij} \geq \lambda\Omega \gg \frac{1}{\sum_i a_i} \sum_i a_i y^*_i \geq \overline{y^{**}}.$$

Moreover, by a suitable reduction-expansion of allocations and preferences (and invoking Scale Independence), one can obtain a situation in which the expanded-reduced version of x^* is efficient (see the proof of Th. 1 for a detailed treatment). This entails a contradiction with Support-Mean Dominance II.

Finally, we have to show that no axiom is redundant.

Dropping Weak Pareto. Take $\widetilde{R}^1$.

Dropping Transfer Principle. Take the SOF $\widetilde{R}^5$ defined by:

$$x \; \widetilde{R}^5 \; y \Leftrightarrow \max_p \sum_i u_i(x_i, p) \geq \max_p \sum_i u_i(y_i, p).$$

Dropping Hansson Independence. Take the SOF $\widetilde{R}^6$ which coincides with R_W if there are agents with identical preferences, and with $\widetilde{R}^5$ otherwise.

Dropping Scale Independence. Take the SOF $\widetilde{R}^7$ defined by

$$x \; \widetilde{R}^7 \; y \Leftrightarrow x \; R_W \; y \text{ or } x \text{ is fair-equivalent},$$

the latter meaning that x is efficient and there exists an envy-free allocation in $I(x)$ (set of allocations which are Pareto-indifferent to x, see definition of $\widetilde{R}^2$ above).

Dropping Replication Independence. Take the SOF $\widetilde{R}^8$ defined by:

$$x \; \widetilde{R}^8 \; y \Leftrightarrow V^8(x) \geq V^8(y),$$

with

$$V^8(x) = \min \left\{ \lambda \in \mathbb{R} \mid \lambda \Omega \geq \sum_{i=1}^n a_i q_i; \; \sum_{i=1}^n a_i = n; \; \forall i, \; a_i \in \mathbb{Z}_+, \; q_i R_i x_i \right\}.$$

Dropping Support-Mean Dominance II. Take R_{PS}. ■

Proof of Theorem 3:

Let x and y be two allocations such that $x \; P_{PS} \; y$. Suppose that, contrary to the desired result, one has $y \; R \; x$.

Let $\alpha, \beta, \gamma, \delta, \lambda, \mu$ and ω be such that

$$\begin{aligned}
\min_i v_i(x_i) &> \alpha > \beta > \gamma > \delta > \lambda > \mu > \min_i v_i(y_i), \\
\delta &> \frac{1}{n} \left((n-1)\beta + \lambda \right), \\
\omega &> \max \left\{ \max_i v_i(x_i), \max_i v_i(y_i) \right\},
\end{aligned}$$

and let i_0 be an agent such that $\min_i v_i(y_i) = v_{i_0}(y_{i_0})$.

Let $(R_1^1, ..., R_n^1)$ be a profile such that for all i,

$$\begin{aligned}
C(R_i^1, x_i) &= C(R_i, x_i) \text{ and } C(R_i^1, y_i) = C(R_i, y_i), \\
C(R_i^1, \omega\Omega) &= \left\{ q \in \mathbb{R}_+^\ell \mid q \geq \omega\Omega \right\},
\end{aligned}$$

and let R^1 denote the related social ordering.

As in the proof of Th. 2, it is assumed that all constructed allocations below are feasible. Using homothetic reductions of preferences and allocations and invoking Scale

Independence always makes it possible (see the proof of Th. 1 for a rigorous treatment). Let y' be an allocation defined by $y'_{i_0} = \mu\Omega$, and for all $i \neq i_0$,

$$y'_i = \omega\Omega.$$

By Weak Pareto, $y' \; P^1 \; y$. Since by Hansson Independence, $y \; R^1 \; x$, one has $y' \; P^1 \; x$.

Let $(R^2_1, ..., R^2_n)$ be a profile such that for all i,

$$C(R^2_i, x_i) = C(R^1_i, x_i) \text{ and } C(R^2_i, y'_i) = C(R^1_i, y'_i),$$

and for some $i_1 \neq i_0$,

$$c(R^2_{i_1}, \beta\Omega) \supset \bigcup_{i \neq i_1} C(R^2_i, \alpha\Omega).$$

Let R^2 denote the related social ordering.

Let x' be an allocation defined by $x'_{i_1} = \beta\Omega$ and for all $i \neq i_1$,

$$x'_i = \alpha\Omega.$$

By Weak Pareto, $x \; P^2 \; x'$. Since by Hansson Independence, $y' \; P^2 \; x$, one obtains $y' \; P^2 \; x'$.

Let x'' be an allocation defined by $x''_{i_1} = \gamma\Omega$ and for all $i \neq i_1$,

$$x''_i = \omega\Omega.$$

By Lemma 2, one has $x' \; P^2 \; x''$, and therefore $y' \; P^2 \; x''$.

Let $(R^3_1, ..., R^3_n)$ be a profile such that for all i,

$$\begin{aligned} C(R^3_i, x''_i) &= C(R^2_i, x''_i) \text{ and } C(R^3_i, y'_i) = C(R^2_i, y'_i), \\ C(R^3_i, \beta\Omega) &= \left\{ q \in \mathbb{R}^\ell_+ \mid q \geq \beta\Omega \right\} \end{aligned}$$

and for all $i \neq i_1$,

$$C(R^3_i, \delta\Omega) = \left\{ q \in \mathbb{R}^\ell_+ \mid q \geq \delta\Omega \right\}.$$

Let x^* be an allocation defined by $x^*_i = \delta\Omega$ for all i. By Weak Pareto, $x'' \; P^3 \; x^*$. Since by Hansson Independence, $y' \; P^3 \; x''$, one then has $y' \; P^3 \; x^*$.

Let y'' be an allocation defined by $y''_{i_0} = \lambda\Omega$ and for all $i \neq i_0$,

$$y''_i = \beta\Omega.$$

Notice that by construction

$$c(R^3_{i_0}, y''_{i_0}) \supset \bigcup_{i \neq i_0} C(R^3_i, y''_i).$$

Therefore, by Lemma 2, $y'' \; P^3 \; y'$, so that by transitivity, $y'' \; P^3 \; x^*$.

But x^* is egalitarian, and since $n\delta > (n-1)\beta + \lambda$, one has for all i,

$$x^*_i = \delta\Omega \gg \overline{y''} = \frac{1}{n}\left[(n-1)\beta + \lambda\right]\Omega \gg \min y'' = \lambda\Omega.$$

Moreover, by a suitable reduction-expansion of allocations and preferences (and invoking Scale Independence), one can obtain a situation in which the expanded-reduced version of x^* is efficient for the profile $(R_1^3, ..., R_n^3)$ (see the proof of Th. 1 for a detailed treatment). Therefore, by Minimal Preference for Equality, one should have $x^* \ R^3 \ y''$, a contradiction.

Finally, we check that no axiom is redundant. The counter-examples are the same as for Th. 1 (replacing Support-Mean Dominance I by Minimal Preference for Equality). ■