

# Axiomatic Foundations for Satisficing Behavior\*

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## Abstract

A theory of decision making is proposed that supplies an axiomatic basis for the concept of “satisficing” postulated by Herbert Simon. After a detailed review of classical results that characterize several varieties of preference-maximizing choice behavior, the axiomatization proceeds by weakening the inter-menu contraction consistency condition involved in these characterizations. This exercise is shown to be logically equivalent to dropping the usual cognitive assumption that the decision maker fully perceives his preferences among available alternatives, and requiring instead merely that his ability to perceive a given preference be weakly decreasing with respect to the relative complexity (indicated by set inclusion) of the choice problem at hand. A version of Simon’s hypothesis then emerges when the notion of “perceived preference” is endowed with sufficiently strong ordering properties, and the axiomatization leads as well to a constraint on the form of satisficing that the decision maker may legitimately employ.

**JEL classification codes:** D01, D71, D81.

**Keywords:** choice function, cognition, revealed preference, threshold.

## 1. INTRODUCTION

*Many writers have felt that the assumption of rationality, in the sense of a one-dimensional ordering of all possible alternatives, is absolutely necessary for economic theorizing. . . . There seems to be no logical necessity for this viewpoint; we could just as well build up our economic theory on other assumptions as to the structure of choice functions if the facts seemed to call for it.*

— KENNETH J. ARROW (1951).

HALF A CENTURY AGO, Herbert Simon published the first [32] of several early articles challenging the models of decision making then and now dominant in economic analysis. “[T]he task,” he wrote [p. 99], “is to replace the global rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in

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\*This paper contains material from Chapters 2–3 of the author’s PhD thesis [38].

the kinds of environments in which such organisms exist.” In Simon’s view, cognitive and information-processing constraints on the capabilities of economic agents, together with the complexity of their environment (see [33]), render optimal decision making an unattainable ideal. Rather than attempting a summary of the full argument — spread over his sixty-odd years of work in the behavioral and cognitive sciences — we refer the interested reader to [34] and [35], as well as to the three volumes [36] of Simon’s collected writings on the subject.

Optimal decision making is ordinarily implemented in economic models by means of the *maximizing criterion*

$$f(x) \geq \max f[A], \tag{1}$$

which requires the chosen alternative  $x$  to achieve a utility (returned by the function  $f$ ) no less than the maximum obtainable from the menu  $A$  of available options. After offering a terse review of the basic tools of axiomatic choice theory, Section 2 establishes versions of the classical results on preference-based choice (namely, Theorems 1–4) that provide a behavioral foundation for this criterion. While the exposition of these results may have some intrinsic value as a synthesis of widely scattered contributions, the primary purpose of this section is to build up the classical theory in a manner adaptable to the construction of the alternative theory to follow.

Having dismissed the idea that human decision makers exhibit “global rationality,” Simon suggests that they in fact engage in “satisficing”<sup>1</sup> — defined in [36, v. 3, p. 295] as “choos[ing] an alternative that meets or exceeds specified criteria, but that is not guaranteed to be either unique or in any sense the best.” Formulating this hypothesis in utility space leads naturally to the *satisficing criterion*

$$f(x) \geq \theta(A), \tag{2}$$

in which the threshold utility  $\theta(A)$  for acceptability of an alternative can take on any value less than or equal to the maximum in Equation 1. As this phrasing makes clear, maximizing is then a special case of satisficing, and it follows that any choice-theoretic basis for the latter will be logically weaker than the classical basis for the former.

Our objective, therefore, is to develop an axiomatic foundation for satisficing behavior by diluting the conditions that underpin utility maximization. This task is carried out in Section 3, where Simon’s cognitive and information-processing constraints are imagined to prevent the decision maker from fully perceiving his (strict) preferences among the available alternatives.<sup>2</sup> Under the maintained “nestedness” assumption that a preference perceived in choice problem  $B$  is also perceived in each problem  $A \subset B$  in which it is relevant, a formal analysis closely paralleling the classical theory leads to a series of new results (Theorems 5–7) that identify the restrictions on behavior implied by the imposition of different sets of ordering properties (such as acyclicity and transitivity) on the concept of “perceived preference.” These results uncover a correspondence between failures of perception and violations of the classical contraction consistency axiom, which states that acceptability of an alternative in choice problem  $B$  together with its availability in problem  $A \subset B$  should imply its acceptability in  $A$ . And when sufficiently strong ordering

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<sup>1</sup>Although Simon [36, v. 2, p. 415] identifies this word as Scottish in origin, the *O.E.D.* finds its earliest recorded use in the Swiss theologian Henry Bullinger’s [7] comment — presumably about the Romans — *That their founders were nourished by suckynge of a wolfe: so haue all that people wolues mindes, neuer satisfised with blood, euer greedy of dominion and hungryng after riches. . . .*

<sup>2</sup>Extensive discussion of the rationale for this response to Simon’s critique can be found in Chapter 1 of [38].

$[wxyz] \mapsto [xyz]$ $xPw \ \& \ yPw \ \& \ zPw$ $\theta([wxyz]) = 1$					
$[wxy] \mapsto [xy]$ $xPw \ \& \ yPw$ $\theta([wxy]) = 1$		$[wxz] \mapsto [z]$ $xPw \ \& \ zPw \ \& \ zPx$ $\theta([wxz]) = 3$		$[wyz] \mapsto [yz]$ $yPw \ \& \ zPw$ $\theta([wyz]) = 2$	
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$[wx] \mapsto [x]$ $xPw$ $\theta([wx]) = 1$	$[wy] \mapsto [y]$ $yPw$ $\theta([wy]) = 2$	$[wz] \mapsto [z]$ $zPw$ $\theta([wz]) = 3$	$[xy] \mapsto [xy]$ $-$ $\theta([xy]) = 1$	$[xz] \mapsto [z]$ $zPx$ $\theta([xz]) = 3$	$[yz] \mapsto [z]$ $zPy$ $\theta([yz]) = 3$

$$f(w) = 0$$

$$f(x) = 1$$

$$f(y) = 2$$

$$f(z) = 3$$

Figure 1: Choice behavior consistent with the satisficing criterion. A menu is a subset of the space  $[wxyz]$ ; the binary relation  $P$  indicates strict preference; the function  $f$  assigns utility values to alternatives; and the function  $\theta$  assigns threshold utilities to menus. Within the cells are displayed the mapping from menus to subsets of acceptable alternatives (e.g.,  $[wxy] \mapsto [xy]$ ), the preferences perceived in particular choice problems (e.g.,  $xPw$  and  $yPw$  in problem  $[wxy]$ ), and the values of  $\theta$  (e.g.,  $\theta([wxy]) = 1$ ).

properties are imposed on the preferences themselves (i.e., on the objects of perception or non-perception) through the requirement of acyclicity of the base relation revealed by binary choice data, a series of modified results (Theorems 8–10) are obtained of which the last (Theorem 10) supplies the desired foundation for satisficing.

The main features of our theory are illustrated by the example depicted in Figure 1. Here one cell is allocated to each nontrivial choice problem drawn from the four-element space  $[wxyz]$  (note the multiplicative notation for enumerated sets), and the upper entry in each shows the subset of acceptable alternatives associated with the menu in question. The alternatives deemed acceptable are those that are maximal with respect to the perceived preferences shown in the middle entry in the cell (e.g., the perception of  $xPw$  makes  $w$  unacceptable in problem  $[wxz]$ ); or, alternatively, those with utility values no smaller than the threshold shown in the lower entry (e.g.,  $[v \in [wyz] : f(v) \geq 2] = [yz]$ ). The system of perceived preferences satisfies the nestedness assumption (e.g., the preference  $yPw$  perceived in problem  $[wxyz]$  is also perceived in problem  $[wxy] \subset [wxyz]$ ). And the decision maker's behavior violates contraction consistency (e.g., alternative  $x$  is deemed acceptable in problem  $[wxyz]$  but not in problem  $[xyz] \subset [wxyz]$ ).

It is in permitting the latter type of violation that our theory departs from classical models of decision making, and the last of the above parenthetical examples thus merits closer scrutiny. As noted, alternative  $x$  is deemed unacceptable in problem  $[xyz]$ , a fact attributable to the preference  $zPx$  being perceived in this context. Classical assumptions would then require that this preference be perceived, and hence that  $x$  be deemed unacceptable, in problem  $[wxyz]$  as well. But neither of these requirements follows from our nestedness assumption (since  $[wxyz] \not\subset [xyz]$ ), one which therefore does not imply contraction consistency of the decision maker's behavior.

The purpose of this paper is to demonstrate the relationships among the three different types of constructs illustrated in Figure 1. Specifically, we determine the restrictions on behavior that characterize maximization of a nested system of perceived preferences — our model of decision making under cognitive and information-processing constraints — as well as the further restrictions needed for consistency with the satisficing criterion. Thus, in addition to providing a choice-theoretic axiomatization of Herbert Simon’s hypothesis, our analysis also offers a cognitive interpretation of it; an answer to “the crucial question of why people have the aspiration or satisfaction levels they have” (posed by Elster [9, pp. 26–27] as a challenge to satisficing theory).

This enterprise follows in the long tradition of axiomatic weakenings of the standard economic model of decision making delineated by Savage [23, Chapters 2–5]; a tradition exemplified by the contributions of Aumann [4] and Bewley [6] removing the completeness axiom (part of P1 in [23, p. 18]), by that of Machina and Schmeidler [19] abandoning the Sure-Thing Principle (P2 in [23, p. 23]), and by those of Schmeidler [24] and Gilboa and Schmeidler [11] effectively doing away with various comparative probability axioms (including P4 in [23, p. 31] and P4\* in [19, p. 761]). Like Kreps [17], who considers agents exhibiting a “preference for flexibility,” we suppress the usual state-space formulation of uncertainty and focus attention on one of Savage’s implicit assumptions; namely, that the decision maker’s behavior is maximal with respect to a fixed preference relation and therefore satisfies contraction consistency.

A number of more recent papers also relate to one or another aspect of this essay. Baigent and Gaertner [5] characterize a form of “polite” decision making that bears some resemblance to satisficing. Kalai et al. [16] allow for menu-dependence of the preference relation (which can then incorporate “multiple rationales”), focusing on the question of how much of this variation is needed to rationalize a given pattern of behavior. Gul and Pesendorfer [12] consider a decision maker who, as a result of temptation rather than of constraints on cognition, may suffer from the provision of extra alternatives. Sheshinski [31] (following Mirrlees [20]) investigates the implications for public policy of choice behavior that fails to reliably maximize the agent’s welfare. And Iyengar and Lepper [14] (among others) examine the psychological effects of decision complexity.

## 2. CLASSICAL CHOICE THEORY

### 2.1. Choice functions

At a high level of abstraction, axiomatic choice theory expresses the decision making environment as a pair  $\langle X, \mathbb{A} \rangle$ , where  $X$  denotes an arbitrary nonempty set and  $\mathbb{A}$  a collection of subsets of  $X$ . In this *choice space* formulation, a set  $A \in \mathbb{A}$  is a menu of mutually-exclusive alternatives,  $\mathbb{A}$  itself a list of the choice problems (menus) of interest, and  $X$  a full catalog of the alternatives potentially available. The mathematical primitive of the theory is then a *choice function*  $C : \mathbb{A} \rightarrow 2^X$  associating with each  $A \in \mathbb{A}$  a so-called *choice set*  $C(A)$  properly interpreted as the set of alternatives whose selection from this menu cannot be ruled out by the theorist.

Under the suggested interpretation, two conditions on  $C$  are unobjectionable enough to be considered part of the definition of a choice function.

**Postulate 1 (Availability)**  $C(A) \subset A$ .

**Postulate 2 (Decisiveness)**  $A \neq \emptyset \implies |C(A)| \geq 1$ .

A third condition has rather more content, requiring the specification of a choice set for each subset of the catalog  $X$ , but will also be taken to hold tacitly throughout.

**Postulate 3 (Universality)**  $\mathbb{A} = 2^X$ .<sup>3</sup>

## 2.2. Binary relations and orderings

We next review some elementary definitions and facts about binary relations, which play a central role in axiomatic choice theory.

A (binary) relation on the set  $X$  is a subset of  $X \times X$ . For example, the equality relation is  $E = [\langle x, y \rangle : x = y]$ . We write  $xRy$  for  $\langle x, y \rangle \in R$ , the statement “ $x$  bears the relation  $R$  to  $y$ .” Given a relation  $R$ , we can define its converse  $R' = [\langle x, y \rangle : yRx]$ , complement  $\bar{R} = [\langle x, y \rangle : \neg(xRy)]$ , and symmetric residue  $R^\circ = \bar{R} \cap \bar{R}'$ . Given a second relation  $Q$ , the composition of  $Q$  with  $R$  is  $QR = [\langle x, z \rangle : (\exists y) xQy \ \& \ yRz]$ . The  $n$ th power of  $R$  is then defined inductively via  $R^n = RR^{n-1}$  (with  $R^1 = R$ ), and the associated ancestral relation is  $R^* = \bigcup_{n=1}^{\infty} R^n$ .

Table 1 lists ten properties that a binary relation may or may not exhibit, while Table 2 records certain (easily verifiable) logical relationships among them. Inspection of the second table reveals two families of properties, each with a hierarchical structure. From weakest to strongest, irreflexivity, asymmetry, and acyclicity are *antireflexivity* properties; each forbidding elements of  $X$  from standing in some relation ( $R$ ,  $R^2$ , and  $R^*$ , respectively) to themselves. And similarly, residual, cross, and negative transitivity are (for want of a better name) *extratransitivity* properties; each requiring a product of two relations to be included in a third.

Table 3 lists five classes of binary relations together with their defining properties. In light of the implications shown in Table 2, we see that the four classes of orderings listed also have a hierarchical relationship: A proto order is a relation exhibiting the antireflexivity properties, a partial order is a proto order exhibiting transitivity, a weak order is a partial order exhibiting the extratransitivity properties, and a linear order is a weak order exhibiting weak connectedness. (The class of equivalences is of course logically independent of these four classes of orderings.)

## 2.3. Revealed preference relations

Let us write  $P$  for the binary relation on  $X$  that encodes our decision maker’s strict preferences among the alternatives, so that the preference-maximal members of a given menu  $A$  are contained in the set  $P\uparrow(A) = [x \in A : (\forall y \in A) y\bar{P}x]$ . The hypothesis that the chosen alternative will be preference maximal then implies that  $C(A) \subset P\uparrow(A)$ , and when this inclusion holds for each  $A \in \mathbb{A}$  we shall write  $C \subset P\uparrow$  and say that  $P$  supplies an *upper bound* for the choice function. In the absence of any further hypotheses we have also the analogous *lower bound* inclusions, indicated by  $P\uparrow \subset C$ , and hence the equality  $C = P\uparrow$ . When the latter holds we shall say that  $P$  *generates* the choice function.

<sup>3</sup>Herzberger [13, p. 192] dubs this the property of “full extension,” and suggests that any theory incompatible with it is inherently deficient.

[E]ssentially non-extended choice functions must be ones that satisfy certain rationality conditions solely by grace of “gaps” at critical points in their domain. Holding the rationality conditions constant, these are gaps that cannot be filled. . . . [A] theory of rationality would be better to safeguard itself against such gerrymandered satisfaction of its requirements.

Arrow [2] and Sen [27] have also defended Universality in the archetypical setting of consumer demand theory.

reflexivity:	$E \subset R$	transitivity:	$R^2 \subset R$
irreflexivity:	$R \subset \bar{E}$	residual transitivity:	$(R^\circ)^2 \subset R^\circ$
symmetry:	$R \subset R'$	cross transitivity:	$RR^\circ \cup R^\circ R \subset R$
asymmetry:	$R^2 \subset \bar{E}$	negative transitivity:	$\bar{R}^2 \subset \bar{R}$
acyclicity:	$R^* \subset \bar{E}$	weak connectedness:	$R^\circ \subset E$

Table 1: Properties of binary relations. Here  $R$  denotes an arbitrary relation,  $R'$  its converse,  $\bar{R}$  its complement,  $R^\circ$  its symmetric residue,  $R^*$  the associated ancestral relation, and  $E$  the equality relation.

antecedent properties	consequent property
asymmetry	irreflexivity
acyclicity	asymmetry
irreflexivity, transitivity	acyclicity
acyclicity, weak connectedness	transitivity
asymmetry, negative transitivity	transitivity
cross transitivity	residual transitivity
irreflexivity, weak connectedness	residual transitivity
negative transitivity	cross transitivity
transitivity, residual transitivity	negative transitivity

Table 2: Logical relationships among properties. A relation exhibits the antecedent properties only if it also exhibits the consequent property.

class of relations	defining properties
proto orders	acyclicity
partial orders	irreflexivity, transitivity
weak orders	asymmetry, negative transitivity
linear orders	acyclicity, weak connectedness
equivalences	reflexivity, symmetry, transitivity

Table 3: Classes of binary relations. A relation belongs to a class if and only if it exhibits the indicated properties.

While this construction illustrates how a hypothesized preference relation can be used to place restrictions on the choice function, it is also possible to take this function as given and to construct from it notions of “revealed” preference. Of the many such notions that have been proposed,<sup>4</sup> we shall require only a few.

**Definition 1** The *global relation*  $P^g$  is defined by  $xP^gy$  if and only if for each menu  $A$  with  $x \in A$  we have  $y \notin C(A)$ . The *base relation*  $P^b$  is defined by  $xP^by$  if and only if  $y \notin C([xy])$ . The *separation relation*  $P^s$  is defined by  $xP^sy$  if and only if there exists a menu  $A$  such that both  $x \in C(A)$  and  $y \in A \setminus C(A)$ .

**Proposition 1**  $P^g \subset P^b \subset P^s$ .  $P^g$  is acyclic.  $P^b$  is asymmetric.  $P^s$  is irreflexive.

An alternative  $x$  bears the global relation to a second alternative  $y$  when we know that  $y$  will be rejected in any choice problem in which  $x$  is available, while  $x$  bears the base relation to  $y$  when we know simply that  $y$  will be rejected in a pairwise choice between the two. Weaker still,  $x$  bears the separation relation to  $y$  when there exists some choice problem (not necessarily the pairwise choice) from which we can rule out  $y$  but not  $x$ .

The base relation has a certain salience as an indicator of preference, since the datum  $xP^by$  indicates that  $y$  will be rejected and hence  $x$  chosen when no other alternatives are present to complicate matters. Moreover, the base relation alone would appear to be sufficient for our purposes in this section in light of Arrow’s [3, p. 16] observation that within the classical theory “the choice in any environment can be determined by a knowledge of the choices in two-element environments.” But in fact the base and global relations will turn out to coincide within the classical theory, and it will facilitate comparison with later results if we now focus attention on  $P^g$ .

The following two conditions demarcate the class of choice functions generated by their respective global relations.

**Condition 1 (Global Upper Bound)**  $C \subset P^g \uparrow$ .

**Condition 2 (Global Lower Bound)**  $P^g \uparrow \subset C$ .

By the definition of  $P^g$ , the first of these is a tautology that places no restrictions on  $C$ .

**Proposition 2** *Global Upper Bound holds for any choice function.*

Note also that  $P^g$  includes any other binary relation that supplies an upper bound for  $C$ .

**Proposition 3** *If  $C \subset R \uparrow$ , then  $R \subset P^g$ .*

This implies that if a relation  $R$  generates the choice function, then  $P^g \uparrow \subset R \uparrow = C \subset P^g \uparrow$  and hence  $P^g$  does as well.

**Proposition 4** *A choice function is generated by a relation if and only if it is generated by  $P^g$ .*

And finally, combining Propositions 2 and 4 shows that Global Lower Bound is necessary and sufficient for the choice function to be consistent with the preference maximization hypothesis (a property known as [13, p. 203] “binariness” or [26, p. 309] “normality” of  $C$ ).

**Corollary 1** *A choice function is generated by a relation if and only if it satisfies Global Lower Bound.*

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<sup>4</sup>See, for example, the bestiary of revealed preference relations discussed in Herzberger [13].

#### 2.4. Proto preference orders

In the present context, the antireflexivity properties are undoubtedly the most appealing among those listed in Table 1. Indeed, for a relation encoding preference assessments, a violation of irreflexivity can only be described as nonsensical, one of asymmetry as contradictory, and one of acyclicity as [13, p. 195] “extremely pathological.” These assertions are supported by the following result, which shows that the choice function cannot be consistent with the preference maximization hypothesis unless the maximized relation exhibits all three of the antireflexivity properties.

**Proposition 5** *Any relation that supplies an upper bound for  $C$  is a proto order.*<sup>5</sup>

This fact enables us to strengthen slightly the characterization in Corollary 1.

**Proposition 6** *A choice function is generated by a proto order if and only if it satisfies Global Lower Bound.*

Global Lower Bound can be cast in a more familiar form as the conjunction of two conditions with long histories in axiomatic choice theory.

**Condition 3 (Contraction)**  $A \subset B \implies C(B) \cap A \subset C(A)$ .<sup>6</sup>

**Condition 4 (Weak Expansion)**  $\bigcap_k C(A_k) \subset C(\bigcup_k A_k)$ .<sup>7</sup>

**Proposition 7** *Contraction and Weak Expansion together are logically equivalent to Global Lower Bound.*

The first of these conditions requires that if an alternative  $x$  is in the choice set associated with a particular menu (i.e.,  $B$ ), then it must also be in the choice sets associated with a collection of “contracted” menus (i.e., each  $A$  satisfying  $x \in A \subset B$ ). Conversely, the second condition requires that if  $x$  is in the choice sets associated with a collection of menus (i.e.,  $A_k$  for each  $k$  in some index set), then it must also be in the choice set associated with a particular “expanded” menu (i.e.,  $\bigcup_k A_k$ ). Since Global Lower Bound amounts to a combination of these requirements, Proposition 6 can be rephrased as a characterization of the proto order maximization hypothesis in terms of contraction and expansion consistency.

**Theorem 1** *A choice function is generated by a proto order if and only if it satisfies Contraction and Weak Expansion.*<sup>8</sup>

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<sup>5</sup>Cf. Jamison and Lau’s [15, p. 903] Theorem 1.

<sup>6</sup>This is Sen’s [25, p. 384] Property  $\alpha$ , or [29, p. 500] “basic contraction consistency.” The condition seems first to have appeared as Chernoff’s [8, p. 429] Postulate 4, although Nash [21, p. 159] employs a precursor.

<sup>7</sup>This is Sen’s [26, p. 314] Property  $\gamma$ , or [29, p. 500] “basic expansion consistency.”

<sup>8</sup>This is Sen’s [26, p. 314] T.9. Incidentally, the theorem answers a question posed by Kreps [18, p. 15].



## 2.5. Partial preference orders

If the suitability of the antireflexivity properties is difficult to dispute, transitivity at least offers “impressive credentials” [13, p. 194] as a criterion of consistency for preference assessments; and imposing this property in conjunction with the maximization hypothesis entails a further restriction on the choice function.<sup>9</sup>

**Condition 5 (Adjunct Expansion)**  $C(B) \subset A \subset B \implies C(A) \subset C(B)$ .

This condition dictates that when a menu expands to include new options, the incumbent members of the choice set must retain this membership as long as no new alternative attains it.

**Lemma 1** *Adjunct Expansion implies that  $P^g$  is a partial order. A choice function is generated by a partial order only if it satisfies Adjunct Expansion.*

**Theorem 2** *A choice function is generated by a partial order if and only if it satisfies Contraction, Weak Expansion, and Adjunct Expansion.*<sup>10</sup>

## 2.6. Weak preference orders

Once transitivity is admitted as a consistency criterion, Table 2 shows that to admit any one of the extratransitivity properties is to admit them all. In the present context, these new properties lack the immediate intuitive appeal of those previously considered, and to understand their interpretation it is useful to consider for a moment the symmetric residue  $P^\circ = \bar{P} \cap P'$  of the decision maker’s preference relation  $P$ . Now in general, one alternative bearing  $P^\circ$  to a second need mean nothing more than that neither alternative is definitely preferred to the other. When the extratransitivity properties are imposed on  $P$ , however, its symmetric residue inherits the properties of an equivalence — those same properties that an expression of *positive indifference* would be expected to satisfy. And since in this case the relations  $P$ ,  $P'$ , and  $P^\circ$  partition the space of alternative-pairs, admitting the extratransitivity properties amounts to asserting that our decision maker should be able, given any two alternatives, to affirm either a definite preference for one over the other or his indifference between them.

This further strengthening of the preference maximization hypothesis entails a restriction on the choice function that subsumes both of the expansion consistency conditions thus far introduced.

**Condition 6 (Strong Expansion)**  $A \subset B \ \& \ C(B) \cap A \neq \emptyset \implies C(A) \subset C(B)$ .<sup>11</sup>

**Proposition 8** *Strong Expansion implies both Weak Expansion and Adjunct Expansion.*

This condition insists that when a menu expands, the incumbent members of the choice set must retain this membership as long as any incumbent alternative attains it.

**Lemma 2** *Strong Expansion implies that  $P^g$  is a weak order. A choice function is generated by a weak order only if it satisfies Strong Expansion.*

<sup>9</sup>Anand [1] discusses philosophical aspects of transitivity.

<sup>10</sup>Cf. Sen’s [26, p. 315] T.10 and Jamison and Lau’s [15, p. 904] Theorem 2.

<sup>11</sup>This is Sen’s [28, p. 66] Property  $\beta(+)$ .

**Theorem 3** *A choice function is generated by a weak order if and only if it satisfies Contraction and Strong Expansion.*<sup>12</sup>

Incidentally, the conditions that appear in Theorem 3 can be joined together to form the better-known Weak Axiom of Revealed Preference.

**Condition 7 (Weak Axiom)**  $P^s \subset P^g$ .<sup>13</sup>

**Proposition 9** *Contraction and Strong Expansion together are logically equivalent to the Weak Axiom.*

Note also, recalling Proposition 1, that  $P^g = P^b = P^s$  when the Weak Axiom holds.

### 2.7. Linear preference orders

Because of its close connection to the machinery of utility maximization (see Section 2.8), the assumption that weak preference orders guide human choice behavior is widespread in economic analysis. But occasionally it is useful to adopt the stronger assumption that the guiding relation is a *linear* order, thereby requiring that the decision maker be able to affirm a definite preference between any two distinct alternatives. The incremental restriction on the choice function that captures this new requirement — a strengthening of the Decisiveness postulate — demands that each (nonempty) choice set contain a single element.

**Condition 8 (Univalence)**  $A \neq \emptyset \implies |C(A)| = 1$ .<sup>14</sup>

While this is, per se, neither a contraction nor an expansion consistency axiom, when paired with a condition of either type it can serve in the complementary capacity.

**Proposition 10** *Contraction and Strong Expansion are logically equivalent in the presence of Univalence.*

**Lemma 3** *Strong Expansion and Univalence jointly imply that  $P^g$  is a linear order. A choice function is generated by a linear order only if it satisfies Univalence.*

**Theorem 4** *A choice function is generated by a linear order if and only if it satisfies Strong Expansion and Univalence.*

Figure 2 illustrates the principal logical implications among the (non-tautological) conditions on the choice function introduced in this paper. The reader may find it useful at this point to review Theorems 1–4 and to locate in the figure the set of conditions that hold in each instance.

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<sup>12</sup>Cf. Sen's [25, p. 385] Corollary 1.

<sup>13</sup>This is Arrow's [2, p. 123] C5 and Sen's [26, p. 309] Weak Congruence Axiom. A version of the condition first appeared as Samuelson's [22, p. 65] Postulate III.

<sup>14</sup>This is Herzberger's [13, p. 212] Condition 8.

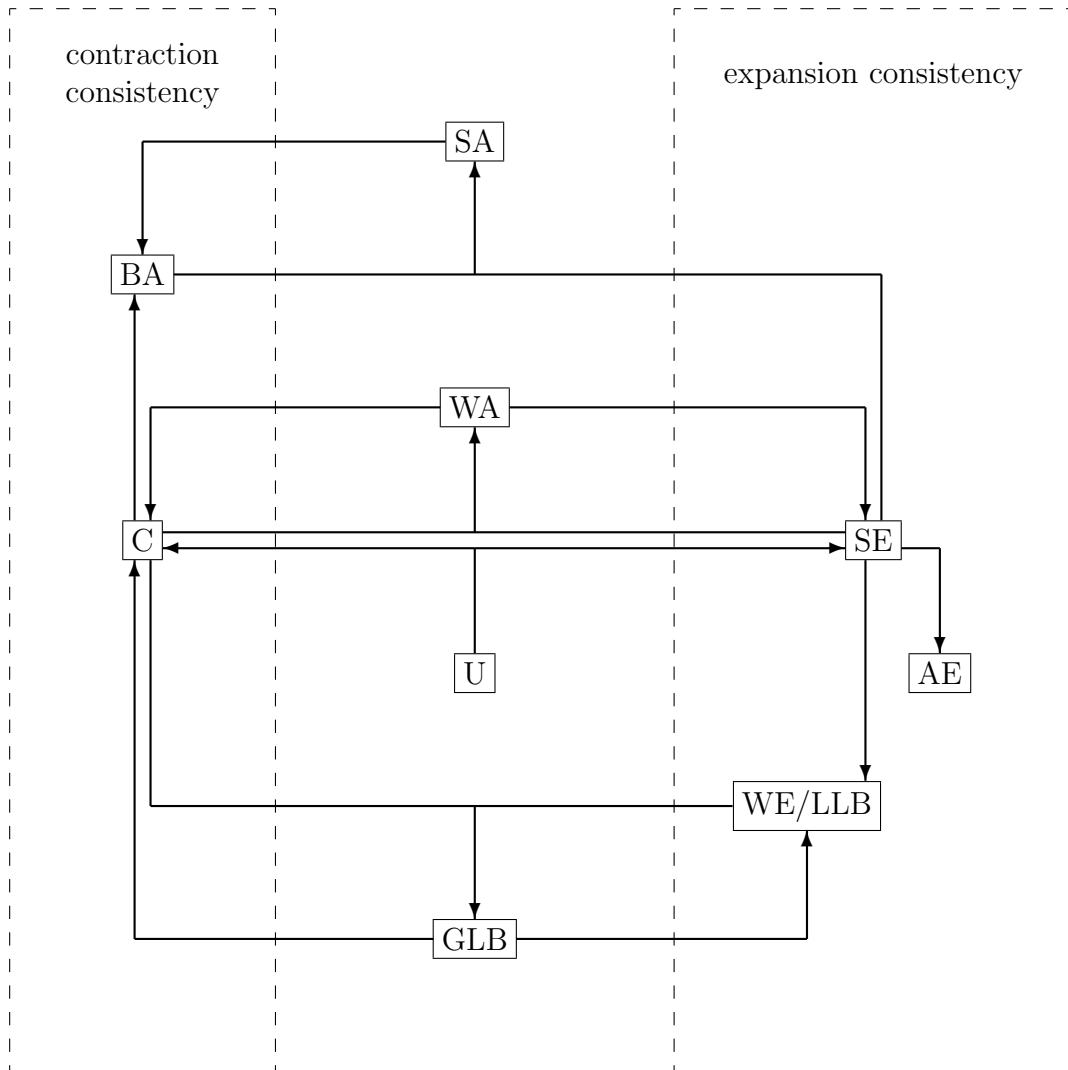


Figure 2: Logical implications among (non-tautological) conditions on the choice function. A condition is indicated by its initials and an implication by a directed edge. Intersecting edges indicate a joint hypothesis.

## 2.8. Utility representations

An important historical and practical concern of economic theory has been assessing the plausibility of and the axiomatic basis for the utility-maximization model of decision making. This model posits the existence of a function  $f : X \rightarrow \mathfrak{R}$  that encodes the preference relation in the sense that  $xPy$  if and only if  $f(x) > f(y)$ , and an obvious consequence of this encoding is that the maximization operators (applied to menus) associated with  $P$  and  $f$  coincide. If, moreover, the relation  $P$  generates the choice function, then for each nonempty menu  $A$  we have

$$C(A) = [x \in A : f(x) \geq \max f[A]], \quad (3)$$

which is to say that the members of a choice set are those available alternatives that meet the maximizing criterion. When this is so, we shall say that the function  $f$  provides a *utility representation* of the choice function, and we shall call the representation *injective* when  $f$  is one-to-one.

Since a preference relation that can be encoded in a function taking real values will inherit from the relation  $>$  on  $\mathfrak{R}$  the properties of a weak order, the utility-maximization model of decision making clearly falls under the purview of Theorem 3 above. But well-known counterexamples show that not every weak order can be thus encoded, with an additional order-topological property being needed to ensure encodability.<sup>15</sup> So as to avoid becoming preoccupied with this issue — one that is irrelevant to the main point of this essay — we provide here an exact characterization of the class of choice functions admitting utility representations only for the very simple case of a finite choice space (i.e., a finite catalog  $X$ ).

**Proposition 11** *A choice function on a finite choice space admits a utility representation (resp., an injective utility representation) if and only if it satisfies Contraction and Strong Expansion (resp., Strong Expansion and Univalence).*

## 3. CHOICE THEORY WITHOUT CONTRACTION CONSISTENCY

### 3.1. Relation systems and nestedness

If our decision maker cannot be relied upon to perceive all of his preferences, then we shall require a model of his cognitive capabilities more elaborate than that embodied in the relation  $P$ . In this case, our being told of the existence of a preference for one alternative over another does not enable us to conclude that the second alternative will never be chosen when the first is available, since this preference might be imperceptible in the context of some particularly complex choice problem in which it is relevant. But if the *perceptibility* as well as the existence of preferences is to be called into question, then we can certainly devise a formalism that describes the former in the same way that a binary relation describes the latter. Given a menu  $A$ , let us write  $P_A$  for the relation on  $A$  that encodes the preferences that the decision maker perceives when faced with this choice problem. Now, allowing the menu to vary, we collect the associated perceived preference relations in a vector  $\mathbf{P} = \langle P_A \rangle_{A \in \mathfrak{A}}$  to be referred to as the *preference system*. The perceived-preference-maximal members of the menu  $A$  are then contained in the set  $\mathbf{P} \uparrow (A) = [x \in A : (\forall y \in A) y \bar{P}_A x]$ , and when  $C \subset \mathbf{P} \uparrow$  (resp.,  $\mathbf{P} \uparrow \subset C$ ) — adopting a notation analogous to that in Section 2.3 — we shall say that  $\mathbf{P}$  supplies an upper bound

<sup>15</sup>Here Fishburn [10, p. 27] is a valuable reference for both counterexamples and positive results.

(resp., a lower bound) for the choice function. When  $C = \mathbf{P} \uparrow$ , we shall of course say that  $\mathbf{P}$  generates the choice function.

Let us call an arbitrary vector  $\mathbf{R}$  of relations indexed by  $\mathbb{A}$  a *relation system*. The *projection* of  $\mathbf{R}$  is the union  $\bigcup \mathbf{R} = \bigcup_{A \in \mathbb{A}} R_A \subset X \times X$  of its component relations. A relation system will be said to exhibit a property normally ascribed to a binary relation (e.g., reflexivity) when each of its component relations exhibits the property, and we can form derived relation systems (e.g., the system of converse relations) in the obvious fashion. Furthermore, we shall refer to a relation system whose components each belong to a particular class of relations by appending the name of the class (e.g., a system of proto orders).

It will not have escaped the reader that every choice function is generated by a relation system, and so the hypothesis that  $\mathbf{P}$  generates  $C$  excludes no logical possibilities. There is, however, a natural intercomponent restriction on the generating system that *does* constrain the choice function, and that interacts with various sets of intracomponent (i.e., ordering) restrictions in a manner that will soon become apparent. By way of introducing this restriction, let us suppose that when facing the menu  $B$  our decision maker perceives a preference for one alternative over another. Then, when confronted with a different menu  $A$  that contains the two alternatives related by the preference and that is in some sense no more complex than  $B$ , we might reasonably expect the decision maker again to perceive this relationship on the grounds that only an increase in the complexity of the problem could have rendered it imperceptible. Although we have not specified what it means for one choice problem to be more or less complex than another, we can treat the set inclusion relation as being demonstrative of weak comparative complexity under the modest assumption that adding new alternatives to a problem cannot make it any simpler. We thus define a relation system  $\mathbf{R}$  to be *nested* if for any  $x, y \in A \subset B$  we have  $xR_B y$  only if  $xR_A y$  (cf. Anand [1, p. 339]), thereby formalizing what we shall take to be a basic property of our decision maker's preference system.<sup>16</sup>

### 3.2. The local relation system

In order to characterize choice behavior governed by preference systems, we shall require a suitable indicator of perceived preference assembled (like the relations in Section 2.3) from choice function data. Constructing such an indicator poses no small difficulty, since it will consist not of a single binary relation, but rather of an entire vector of relations indexed by  $\mathbb{A}$ . Nevertheless, we can define an appropriate relation system by exploiting the presumed nestedness of  $\mathbf{P}$ .

**Definition 2** *The local relation system  $\mathbf{P}^l$  is defined by  $xP_B^l y$  if and only if  $x, y \in B$  and for each menu  $A \subset B$  with  $x \in A$  we have  $y \notin C(A)$ .*

**Proposition 12**  $\mathbf{P}_X^l = \mathbf{P}^g$ .  $\bigcup \mathbf{P}^l = \mathbf{P}^b$ .  $\mathbf{P}^l$  is nested and acyclic.

---

<sup>16</sup>As always with blanket statements in abstract settings, one can attempt to concoct counterexamples. A choice, say, between execution by firing squad or by electrocution might well be more complex, in the sense that it is more difficult to reach a decision, than a choice among these two modes of execution and dinner with the Queen at Buckingham Palace. Note, however, that this scenario (envisioned by Yossi Feinberg) will not violate the nestedness restriction unless, for example, a preference for electrocution over the firing squad is perceived when dinner is available but not when it isn't.

For a case in which nestedness clearly *is* violated, see Sen's [30, p. 753] "Tea or heroin?" example (pointed out by Nageeb Ali), which illustrates the idea that a menu can have "epistemic importance."

Since an alternative  $x$  bears the relation  $P_B^1$  to a second alternative  $y$  whenever  $y$  will be rejected in any choice problem included in  $B$  and containing  $x$ , we can view  $P_B^1$  as a global relation that searches only the subsets of its subscript.

The role of  $P^1$  in this section will mirror that of  $P^g$  in Section 2, and therefore our first task is to identify the class of choice functions generated by their respective local relation systems.

**Condition 9 (Local Upper Bound)**  $C \subset P^1 \uparrow$ .

**Condition 10 (Local Lower Bound)**  $P^1 \uparrow \subset C$ .

Like its analog in the classical theory, the first of these two conditions is a tautology.

**Proposition 13** *Local Upper Bound holds for any choice function.*

It is also true that  $P^1$  includes componentwise any other nested relation system that supplies an upper bound for the choice function — a formal expression of the intuitive notion that the local relation system “generously” declares a perceived preference to exist whenever no contradictory evidence is forthcoming.

**Proposition 14** *If  $R$  is nested and  $C \subset R \uparrow$ , then for each menu  $A$  we have  $R_A \subset P_A^1$ .*

This implies that if a nested relation system  $R$  generates  $C$ , then  $P^1 \uparrow \subset R \uparrow = C \subset P^1 \uparrow$  and hence  $P^1$  does as well.

**Proposition 15** *A choice function is generated by a nested relation system if and only if it is generated by  $P^1$ .*

And finally, combining Propositions 13 and 15 shows that Local Lower Bound is necessary and sufficient for the choice function to arise from maximization of a nested preference system.

**Corollary 2** *A choice function is generated by a nested relation system if and only if it satisfies Local Lower Bound.*

### 3.3. Preference systems of proto orders

Continuing to proceed in parallel with the classical theory, we now strengthen Corollary 2 by establishing the following analogs to Propositions 5 and 6.

**Proposition 16** *Any nested relation system that supplies an upper bound for  $C$  is a system of proto orders.*

**Proposition 17** *A choice function is generated by a nested system of proto orders if and only if it satisfies Local Lower Bound.*

Despite its arcane appearance, the condition used in this characterization is one that we have already encountered in a different form.

**Proposition 18** *Local Lower Bound is logically equivalent to Weak Expansion.*

Thus we obtain an alternative characterization that excises contraction consistency from Theorem 1.

**Theorem 5** *A choice function is generated by a nested system of proto orders if and only if it satisfies Weak Expansion.*

It is worth noting that Theorem 5 links Weak *Expansion* to the nestedness requirement that perceived preferences be preserved under *contraction* of the menu of alternatives. Similarly, *Contraction* can be linked to the requirement that perceived preferences be preserved under *expansion* of the menu. Here the terminological inversion results from the inverse relationship between preference and choice: A (perceived) preference for one alternative over another is a reason not to choose the second alternative — but not in itself a reason to choose the first.

### 3.4. Preference systems of partial orders

The credentials of transitivity as a consistency criterion are surely no less impressive in the case of perceived preferences than they are in the case of ordinary preference assessments. Once again, imposing this property further constrains the choice function; in fact, it entails precisely the same incremental restriction on  $C$  as before.

**Lemma 4** *Adjunct Expansion implies that  $\mathbf{P}^1$  is a system of partial orders. A choice function is generated by a nested system of partial orders only if it satisfies Adjunct Expansion.*

**Theorem 6** *A choice function is generated by a nested system of partial orders if and only if it satisfies Weak Expansion and Adjunct Expansion.*

When Weak Expansion is supplemented with Adjunct Expansion, the consequences for the latent preference system are actually somewhat subtle. As we have seen, Weak Expansion alone suffices to make  $\mathbf{P}^1$  a nested system of proto orders that generates the choice function, and the astute reader will have observed that forming the transitive closure  $(\mathbf{P}^1)^*$  then creates a system of partial orders that generates  $C$  as well. This artifice fails to invalidate Theorem 6, however, because forming the system of ancestral relations does not in general preserve nestedness. The latter point should make it clear that our intercomponent nestedness and intracomponent ordering assumptions on  $\mathbf{P}$  do not operate independently: On the contrary, without nestedness only very strong (linear) ordering assumptions place any restriction whatsoever on the choice function.

### 3.5. Preference systems of weak orders

Admitting the extratransitivity properties as consistency criteria for perceived preferences also entails the same incremental restriction on the choice function as in the classical theory.

**Lemma 5** *Strong Expansion implies that  $\mathbf{P}^1$  is a system of weak orders. A choice function is generated by a nested system of weak orders only if it satisfies Strong Expansion.*

**Theorem 7** *A choice function is generated by a nested system of weak orders if and only if it satisfies Strong Expansion.*

While in the case of the preference relation  $\mathbf{P}$  we could parse the extratransitivity properties in terms of its symmetric residue  $\mathbf{P}^\circ$  taking on the characteristics of an indifference relation (see Section 2.6), in the case of the preference system  $\mathbf{P}$  these properties are not so readily interpretable. We can of course construct the system  $\mathbf{P}^\circ$  of symmetric residues, but we cannot then interpret an expression of the form  $x\mathbf{P}_A^\circ y$  as an assertion of “perceived indifference” since it does not preclude a contradictory perceived preference assessment of the form  $x\mathbf{P}_B y$  (except, under nestedness, when  $A \subset B$ ). And conversely, without the indifference interpretation of  $\mathbf{P}^\circ$  we lack an obvious justification for imposing the extratransitivity properties on  $\mathbf{P}$ .<sup>17</sup>

A more illuminating perspective on the characterization in Theorem 7 focuses instead on the negative transitivity of the preference system; the requirement, given  $x, z \in A$  such that  $x\mathbf{P}_A z$ , that each  $y \in A$  satisfy either  $x\mathbf{P}_A y$  or  $y\mathbf{P}_A z$ . This can be phrased as a demand that the decision maker be able to place any available alternative somewhere on the scale of value created by a particular perceived preference — to judge it either worse than the better alternative or better than the worse alternative. A nested preference system of weak orders is therefore the structure appropriate to a decision maker who can always fully resolve his opinions at some level of precision, though his ability to discriminate among alternatives may diminish as the menu expands on which they appear.

### 3.6. Preference systems of linear orders

Imposing weak connectedness on  $\mathbf{P}$  once more leads to a familiar restriction on the choice function.

**Lemma 6** *Strong Expansion and Univalence jointly imply that  $\mathbf{P}^1$  is a system of linear orders. A choice function is generated by a system of linear orders only if it satisfies Univalence.*

But according to Theorem 4, any choice function satisfying both Strong Expansion and Univalence is generated simply by a linear order, and this implication brings us to the following conclusion.

**Proposition 19** *A choice function is generated by a nested system of linear orders if and only if it is generated by a linear order.*

The logic of this result is most easily understood in the light of our commentary on Theorem 7 above. As we have seen, a decision maker possessing a preference system of weak orders is one who can always fully resolve his opinions, though his discriminatory capabilities may depend upon the menu he faces. But if the components of  $\mathbf{P}$  are *linear* orders then these capabilities *cannot* in fact depend upon the menu, since a weakly connected relation necessarily discriminates between any two distinct alternatives. The requirement of nestedness then ensures that the components are all drawn from a single linear order on  $X$ , and the latter will be certain to generate  $C$  whenever  $\mathbf{P}$  does so.

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<sup>17</sup>As Herzberger [13, p. 201] would have it, “Indifference relations representing preferential matching [i.e., indifference] ought to be held subject to quite different rationality conditions from those appropriate to amalgamated mutual nonpreference [i.e., symmetric residue] relations,” and no doubt the same can be said of relations for which we are driven to the unfortunate locution “amalgamated mutual absence of perceived preference.”



### 3.7. Rehabilitating the preference relation

While Theorems 5–7 succeed in characterizing choice behavior governed by preference systems that exhibit various sets of ordering properties, these results place no restrictions on the preference assessments that the decision maker either does or does not perceive. This is because, as our analysis has shown, the various sets of expansion consistency conditions actually impose the corresponding ordering properties on the revealed preference system  $\mathbf{P}^1$ , and it is only in the classical world of full perception (i.e., under Contraction) that these properties are inherited by the revealed preference relation  $\mathbf{P}^g$ . Yet we may wish to insist that the decision maker’s preference assessments satisfy certain consistency criteria quite apart from any question of cognition, since the assumptions we have made in this section about his *powers* of perception (i.e., about  $\mathbf{P}$ ) are logically distinguishable from the assumptions made in Section 2 about the *objects* of perception (i.e., about  $\mathbf{P}$ ).

If we call a relation  $\mathbf{Q}$  a *foundation* for the relation system  $\mathbf{R}$  whenever  $\bigcup \mathbf{R} \subset \mathbf{Q}$ , then we can reimpose a particular ordering property on  $\mathbf{P}$  simply by requiring  $\mathbf{P}$  to admit a foundation exhibiting this property. (Here there is of course an implicit assumption that  $\bigcup \mathbf{P} \subset \mathbf{P}$ ; i.e., that perceived preferences are in fact preferences.) This task is simplified by the following result, which implies that the projection of each nested relation system that generates  $C$  is the familiar base relation.

**Proposition 20** *If  $\mathbf{R}$  is nested and for each  $x, y \in X$  we have  $C([xy]) = \mathbf{R}\uparrow([xy])$ , then  $\bigcup \mathbf{R} = \mathbf{P}^b$ .*

At a minimum, we shall want to reimpose the antireflexivity properties on the preference relation. If a relation system  $\mathbf{R}$  is nested, generates the choice function, and admits a proto order foundation  $\mathbf{Q}$ ; then we have that  $\mathbf{P}^b = \bigcup \mathbf{R} \subset \mathbf{Q}$  (by Proposition 20), that  $(\mathbf{P}^b)^* \subset \mathbf{Q}^* \subset \bar{\mathbf{E}}$ , and hence that the following acyclicity condition is satisfied.

**Condition 11 (Base Acyclicity)**  $(\mathbf{P}^b)^* \subset \bar{\mathbf{E}}$ .

**Proposition 21** *Contraction implies Base Acyclicity.*

On the other hand, if the condition holds then the relation  $(\mathbf{P}^b)^*$  is a partial order and by Szpilrajn’s [37] Embedding Theorem can be strengthened to a linear order  $\mathbf{Q}$ . We then have  $\bigcup \mathbf{P}^1 = \mathbf{P}^b \subset (\mathbf{P}^b)^* \subset \mathbf{Q}$  (using Proposition 12), which is to say that  $\mathbf{Q}$  is a foundation for  $\mathbf{P}^1$ .

**Lemma 7** *Base Acyclicity implies that  $\mathbf{P}^1$  admits a linear order foundation. A choice function is generated by a nested relation system that admits a proto order foundation only if it satisfies Base Acyclicity.*

Lemma 7 demonstrates that we can use Base Acyclicity to reimpose ordering properties on our decision maker’s preference relation. But since this condition is both necessary for the preference system to admit a proto order foundation and sufficient for it to admit a linear order foundation, preference relations belonging to the different classes of orderings cannot be distinguished (absent full perception) on the basis of choice function data. In other words, once we have insisted that the preference relation exhibit the antireflexivity properties, any stronger ordering properties will have no empirical content.

We are now in a position to provide modifications of Theorems 5–7 that rehabilitate the preference relation in the sense described above.

	Contraction	—	Base Acyclicity
Weak Expansion	a proto order	a nested system of proto orders	a nested system of proto orders that admits a linear order foundation
Weak Expansion and Adjunct Expansion	a partial order	a nested system of partial orders	a nested system of partial orders that admits a linear order foundation
Strong Expansion	a weak order	a nested system of weak orders	a nested system of weak orders that admits a linear order foundation
Strong Expansion and Univalence	a linear order		

Table 4: Summary of Theorems 1–10. A choice function is generated by the indicated structure if and only if it satisfies the marginal conditions. Note that both Contraction and Base Acyclicity are implied by the conjunction of Strong Expansion and Univalence.

**Theorem 8** *A choice function is generated by a nested system of proto orders that admits a linear order foundation if and only if it satisfies Weak Expansion and Base Acyclicity.*

**Theorem 9** *A choice function is generated by a nested system of partial orders that admits a linear order foundation if and only if it satisfies Weak Expansion, Adjunct Expansion, and Base Acyclicity.*

**Theorem 10** *A choice function is generated by a nested system of weak orders that admits a linear order foundation if and only if it satisfies Strong Expansion and Base Acyclicity.*

Note that although these results are phrased in terms of linear order foundations for the generating systems, in each case the listed conditions would remain necessary if we were to require only a foundation belonging to one of the weaker classes of orderings.

Theorems 1–10, our main choice-theoretic characterization results, are summarized in Table 4. Here the row and column headings indicate different sets of conditions on  $C$ , while the cells contain the structures thereby characterized. Juxtaposition of the first and second columns of cells reveals the parallel features of choice theory with and without contraction consistency: Deleting Contraction from the set of conditions identified in a classical characterization corresponds to relaxing the assumption that the preference relation is fully perceived and assuming merely that the preference system is nested, while at the same time transferring any ordering properties from the one (P) to the other (P). Similarly, juxtaposition of the second and third columns of cells shows that restoring Base Acyclicity (an implication of Contraction) to a set of conditions then corresponds to reimposing the linear ordering properties on the preference relation. Note that any attempt to delete Contraction from or to restore Base Acyclicity to the set of conditions identified in Theorem 4 will have no effect, since both of these conditions are implied by the conjunction of Strong Expansion and Univalence.

Examples of choice functions illustrating Theorems 1–10 are furnished in Table 5.

menu:	$[wx]$	$[wy]$	$[wz]$	$[xy]$	$[xz]$	$[yz]$	$[wxy]$	$[wxz]$	$[wyz]$	$[xyz]$	$[wxyz]$
1.	$[w]$	$[wy]$	$[wz]$	$[x]$	$[xz]$	$[y]$	$[w]$	$[wz]$	$[wy]$	$[x]$	$[w]$
	satisfies GLB, C, WE/LLB, BA; violates AE, SE, WA, U, SA										
2.	$[w]$	$[wy]$	$[wz]$	$[xy]$	$[xz]$	$[y]$	$[wy]$	$[wz]$	$[wy]$	$[xy]$	$[wy]$
	satisfies GLB, C, WE/LLB, AE, BA; violates SE, WA, U, SA										
3.	$[wx]$	$[w]$	$[w]$	$[x]$	$[x]$	$[yz]$	$[wx]$	$[wx]$	$[w]$	$[x]$	$[wx]$
	satisfies GLB, C, WE/LLB, AE, SE, WA, BA, SA; violates U										
4.	$[w]$	$[w]$	$[w]$	$[x]$	$[x]$	$[y]$	$[w]$	$[w]$	$[w]$	$[x]$	$[w]$
	satisfies GLB, C, WE/LLB, AE, SE, WA, U, BA, SA										
5.	$[w]$	$[y]$	$[wz]$	$[x]$	$[z]$	$[y]$	$[x]$	$[wxz]$	$[yz]$	$[y]$	$[xyz]$
	satisfies WE/LLB; violates GLB, C, AE, SE, WA, U, BA, SA										
6.	$[w]$	$[y]$	$[w]$	$[x]$	$[x]$	$[yz]$	$[xy]$	$[w]$	$[yz]$	$[x]$	$[xy]$
	satisfies WE/LLB, AE; violates GLB, C, SE, WA, U, BA, SA										
7.	$[x]$	$[y]$	$[z]$	$[x]$	$[z]$	$[y]$	$[xy]$	$[xz]$	$[yz]$	$[xyz]$	$[xyz]$
	satisfies WE/LLB, AE, SE; violates GLB, C, WA, U, BA, SA										
8.	$[w]$	$[y]$	$[wz]$	$[xy]$	$[z]$	$[y]$	$[xy]$	$[wxz]$	$[yz]$	$[y]$	$[xyz]$
	satisfies WE/LLB, BA; violates GLB, C, AE, SE, WA, U, SA										
9.	$[x]$	$[y]$	$[z]$	$[x]$	$[x]$	$[y]$	$[xy]$	$[wx]$	$[yz]$	$[xy]$	$[xy]$
	satisfies WE/LLB, AE, BA; violates GLB, C, SE, WA, U, SA										
10.	$[x]$	$[y]$	$[z]$	$[x]$	$[xz]$	$[y]$	$[xy]$	$[xz]$	$[yz]$	$[xyz]$	$[xyz]$
	satisfies WE/LLB, AE, SE, BA, SA; violates GLB, C, WA, U										

Table 5: Examples illustrating Theorems 1–10. Choice functions on the four-element space  $X = [wxyz]$  appear as rows numbered according to the relevant theorem. Cells contain the choice sets associated with the menus that serve as the corresponding column headings. Below each row of data the conditions depicted in Figure 2 are registered as being either satisfied or violated by the choice function.

### 3.8. Threshold utility representations

Proposition 11 establishes that a choice function (on a finite choice space) can admit a utility representation only if it satisfies Contraction, and thus relaxing this condition allows our decision maker to behave inconsistently with the utility-maximization model. The targeted generalization of this model posits the existence of functions  $f : X \rightarrow \mathfrak{R}$  and  $\theta : \mathbb{A} \rightarrow \mathfrak{R}$  such that for each menu  $A$  we have

$$C(A) = [x \in A : f(x) \geq \theta(A)]; \quad (4)$$

i.e., such that the members of a choice set are those available alternatives that meet the satisficing criterion. In this case we shall say that the vector  $\langle f, \theta \rangle$  provides a *threshold utility representation* of the choice function, and we shall again call the representation injective when the function  $f$  is one-to-one.

If the choice function admits a threshold utility representation  $\langle f, \theta \rangle$ , then  $xP^s y$  implies that  $f(x) \geq \theta(A) > f(y)$  for some menu  $A$ ; and the following acyclicity condition is an obvious consequence of this implication.

**Condition 12 (Separation Acyclicity)**  $(P^s)^* \subset \bar{E}$ .

On the other hand, if the condition holds then the relation  $(P^s)^*$  is a partial order and by the Embedding Theorem can be strengthened to a linear order  $Q$ . Assuming a finite choice space (as in Section 2.8), this  $Q$  will be encodable in a one-to-one function  $f$  that together with the mapping  $\theta$  defined by

$$\theta(A) = \begin{cases} \min f[C(A)] & \text{for } A \neq \emptyset \\ 0 & \text{for } A = \emptyset \end{cases} \quad (5)$$

will provide a threshold utility representation for  $C$ .<sup>18</sup>

**Proposition 22** *A choice function on a finite choice space admits an injective threshold utility representation if and only if it satisfies Separation Acyclicity.*

The logical strength of Separation Acyclicity vis-à-vis earlier conditions is measured by the following result, which supplies the final implication illustrated in Figure 2.

**Proposition 23** *Separation Acyclicity implies Base Acyclicity. Strong Expansion and Base Acyclicity jointly imply Separation Acyclicity.*

The last two propositions demonstrate that the conditions identified in Theorem 10 are sufficient for the choice function to admit a threshold utility representation  $\langle f, \theta \rangle$ . Strong Expansion is not necessary for the existence of such a representation, however, and in order to derive the further constraint implied by this condition let us imagine nonempty menus  $A \subset B$  such that  $\theta(B) \leq \max f[A]$ . Choosing  $x \in A \subset B$  such that  $f(x) = \max f[A] \geq \theta(B)$ , we have that  $x \in C(B) \cap A$ . Strong Expansion then implies that  $C(A) \subset C(B)$ , and hence

$$\theta(A) = \min f[C(A)] \geq \min f[C(B)] = \theta(B). \quad (6)$$

We can capture this additional property of our construction by calling a threshold utility representation  $\langle f, \theta \rangle$  *dichotomous* whenever, given  $\emptyset \neq A \subset B$ , we have either

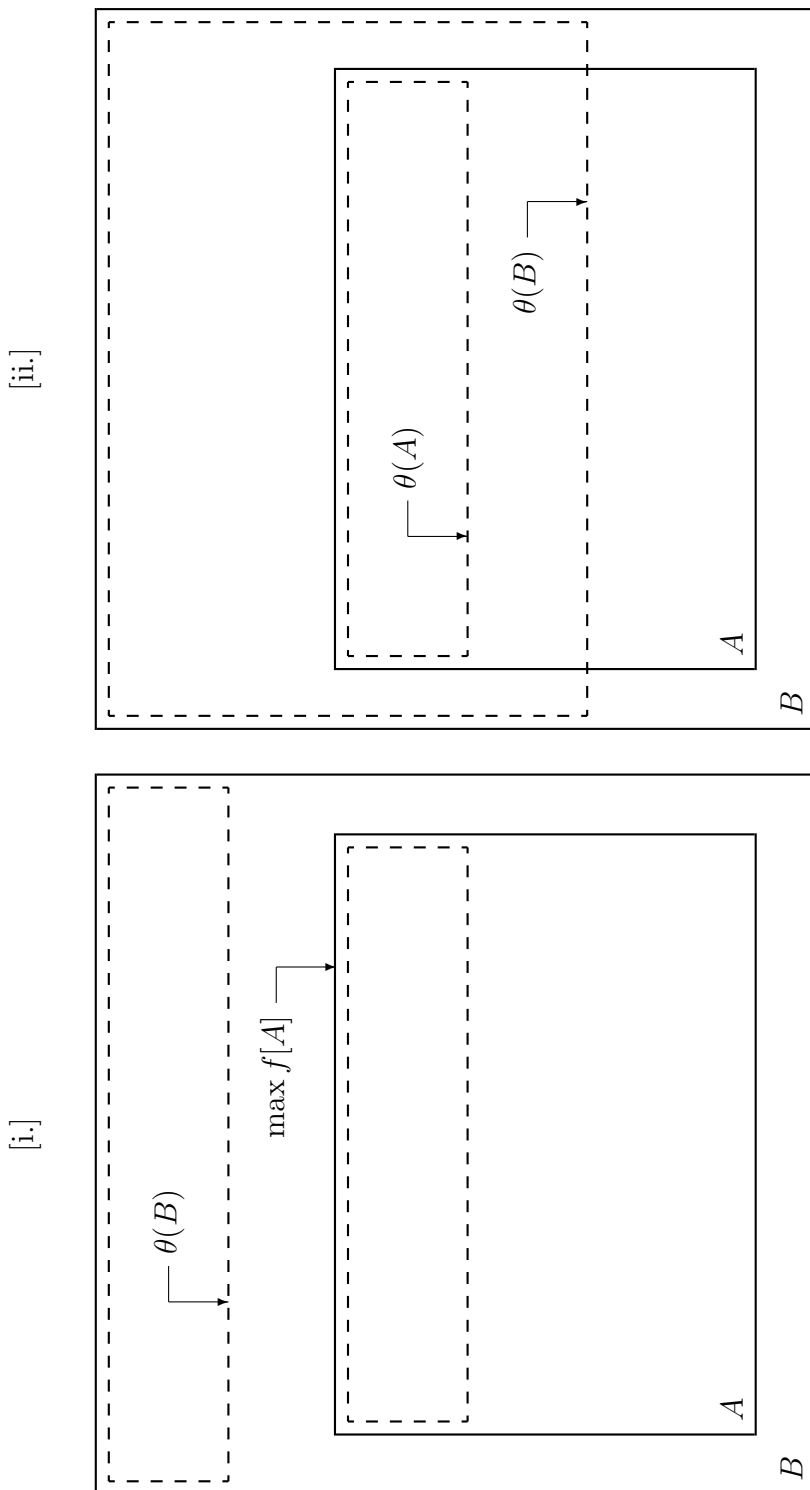


Figure 3: The property of dichotomousness. A menu is indicated by a solid box and a choice set by a dashed box. The real-valued function  $f$  increases towards the top of the figure. The threshold utility representation  $\langle f, \theta \rangle$  of  $C$  is dichotomous whenever, given  $\emptyset \neq A \subset B$ , we have either [i.]  $\theta(B) > \max f[A]$  or [ii.]  $\theta(A) \geq \theta(B)$ .

$\theta(B) > \max f[A]$  or  $\theta(A) \geq \theta(B)$ . (See Figure 3.) And our final result characterizes the class of choice functions that admit representations with this feature.

**Proposition 24** *A choice function on a finite choice space admits an injective, dichotomous threshold utility representation if and only if it satisfies Strong Expansion and Base Acyclicity.*

Note that when a nonempty menu  $A \subset B$  expands to include the alternatives in  $B \setminus A$ , we can be sure that the decision maker will derive benefit from the added flexibility only when  $\theta(B) > \max f[A]$  and hence

$$\min f[C(B)] \geq \theta(B) > \max f[A] = \max f[C(A)]; \quad (7)$$

i.e., only on one side of the identified dichotomy. On the other side, when  $\theta(A) \geq \theta(B)$ , we cannot in general determine whether the provision of extra alternatives will lead to an increase or a decrease in welfare, since when there are multiple alternatives that meet the satisficing criterion (i.e., multiple members of the choice set) our analysis does not tell us which one the decision maker will select. This should be seen as an advantageous feature of the theory, as it leaves room for aspects of the environment not modelled here to have an impact on the decision maker's well-being. For example, marketing activities such as merchandizing and non-informative advertising can be viewed as means of inducing the consumer to break in a particular direction the pseudo-indifference that results from his inability to perceive his preferences among roughly comparable products; and these aspects of the retail environment could be introduced into a market model populated by the sort of cognitively-constrained agents whose behavior we have been studying.

## A. PROOFS

Here we prove many of the results stated in Section 3.

**Proposition 12:** The nontrivial assertions follow from Propositions 13, 16, and 20.

**Proposition 14:** Let  $\mathbf{R}$  be nested and  $C \subset \mathbf{R} \uparrow$ . Given  $x, y \in A$ , if  $x \bar{\mathbf{P}}_A^1 y$  then there exists a  $B \subset A$  such that  $x \in B$  and  $y \in C(B) \subset \mathbf{R} \uparrow(B)$ . It follows that  $x \bar{\mathbf{R}}_B y$  and hence  $x \bar{\mathbf{R}}_A y$  since  $\mathbf{R}$  is nested. By contraposition,  $\mathbf{R}_A \subset \mathbf{P}_A^1$ .

**Proposition 16:** Let  $\mathbf{R}$  be nested and  $C \subset \mathbf{R} \uparrow$ . If  $\mathbf{R}$  is not acyclic, then there exist a menu  $B$  and alternatives  $x_1, x_2, \dots, x_n \in B$  such that  $x_1 \mathbf{R}_B x_2 \mathbf{R}_B \dots \mathbf{R}_B x_n \mathbf{R}_B x_1$ . Letting  $A = [x_1 x_2 \dots x_n] \subset B$ , it follows that  $x_1 \mathbf{R}_A x_2 \mathbf{R}_A \dots \mathbf{R}_A x_n \mathbf{R}_A x_1$  since  $\mathbf{R}$  is nested. But then  $C(A) \subset \mathbf{R} \uparrow(A) = \emptyset$ , contradicting Decisiveness.

**Proposition 18:** If Local Lower Bound holds, then  $\bigcap_k C(A_k) \subset \bigcap_k \mathbf{P}^1 \uparrow(A_k) \subset \mathbf{P}^1 \uparrow(\bigcup_k A_k) \subset C(\bigcup_k A_k)$  (the first inclusion by Proposition 13, the second by the definition of  $\mathbf{P}^1$ , and the third by Local Lower Bound) and thus Weak Expansion holds. To show the converse, suppose that Weak Expansion holds. For each  $x, y \in B$  such that  $x \in \mathbf{P}^1 \uparrow(B)$  there exists a menu  $A_y \subset B$  such that  $y \in A_y$  and  $x \in C(A_y)$ . We then have  $x \in \bigcap_{y \in B} C(A_y) \subset C(\bigcup_{y \in B} A_y) = C(B)$  (the inclusion by Weak Expansion) and thus Local Lower Bound holds.

**Lemma 4:** If  $\mathbf{P}^1$  is not a system of partial orders, then it cannot be transitive (being intrinsically acyclic) and so there must exist alternatives  $x, y, z \in D$  such that  $x \mathbf{P}_D^1 y \mathbf{P}_D^1 z$  and  $x \bar{\mathbf{P}}_D^1 z$ . The latter implies that there exists a menu  $A \subset D$  such that  $x \in A$  and  $z \in C(A)$ , and letting  $B = A \cup [y] \subset D$  we have that  $x \mathbf{P}_B^1 y \mathbf{P}_B^1 z$  since  $\mathbf{P}^1$  is nested. By Proposition 13 we have  $C \subset \mathbf{P}^1 \uparrow$

<sup>18</sup>That  $C(A) \subset [x \in A : f(x) \geq \theta(A)]$  is immediate. Conversely, for any  $A \neq \emptyset$  there must exist some  $y \in C(A)$  that achieves  $\min f[C(A)]$ ; and given  $x \in A$  such that  $f(x) \geq \theta(A) = f(y)$  we then have that  $y \bar{\mathbf{P}}^s x$  (since  $f$  encodes  $\mathbf{Q} \supset (\mathbf{P}^s)^* \supset \mathbf{P}^s$ ) and hence that  $x \in C(A)$ .

and therefore  $y, z \notin C(B)$ . But then  $C(B) \subset A \subset B$  and  $z \in C(A) \setminus C(B)$ , and thus Adjunct Expansion fails.

Now let  $\mathbf{R}$  be a nested system of partial orders that generates  $C$ . If Adjunct Expansion fails, then there must exist both menus  $C(B) \subset A \subset B$  and an alternative  $x \in C(A) \setminus C(B)$ , and since  $C = \mathbf{R} \uparrow$  the latter implies both that  $x \in \mathbf{R} \uparrow(A)$  and that there exists a  $y_1 \in B$  such that  $y_1 \mathbf{R}_B x$ .

[*Inductive step begins.*] Let  $y_k \in B$  be such that  $y_k \mathbf{R}_B x$ . If  $y_k \in A$  then  $y_k \mathbf{R}_A x$  since  $\mathbf{R}$  is nested, contradicting  $x \in \mathbf{R} \uparrow(A)$ . Alternatively, if  $y_k \in B \setminus A$  then  $y_k \notin C(B) = \mathbf{R} \uparrow(B)$  since  $C(B) \subset A$ . But then there exists a  $y_{k+1} \in B$  such that  $y_{k+1} \mathbf{R}_B y_k \mathbf{R}_B x$ , and therefore  $y_{k+1} \mathbf{R}_B x$  since  $\mathbf{R}$  is transitive. [*Inductive step ends.*]

Using induction, we can construct a set  $D = [y_1 y_2 \dots] \subset B$  with the property that  $y_{k+1} \mathbf{R}_B y_k$  and hence (since  $\mathbf{R}$  is nested)  $y_{k+1} \mathbf{R}_D y_k$  for each  $k \geq 1$ . But then  $C(D) = \mathbf{R} \uparrow(D) = \emptyset$ , contradicting Decisiveness.

**Lemma 5:** If  $\mathbf{P}^1$  is not a system of weak orders, then it cannot be negatively transitive (being intrinsically asymmetric) and so there must exist alternatives  $x, y, z \in D$  such that  $x \bar{\mathbf{P}}_D^1 y \bar{\mathbf{P}}_D^1 z$  and  $x \mathbf{P}_D^1 z$ . The former implies that there exist both a menu  $A \subset D$  such that  $x \in A$  and  $y \in C(A)$  and a menu  $B \subset D$  such that  $y \in B$  and  $z \in C(B)$ . Decisiveness ensures that there exists an alternative  $w \in C(A \cup B)$ , and since both  $x \in A \cup B \subset D$  and  $x \mathbf{P}_D^1 z$  we have also  $z \notin C(A \cup B)$ . If either  $w \in B$  or  $y \in C(A \cup B)$  then we have both  $C(A \cup B) \cap B \neq \emptyset$  and  $z \in C(B) \setminus C(A \cup B)$ , and thus Strong Expansion fails. Alternatively, if both  $w \notin B$  and  $y \notin C(A \cup B)$  then we have both  $w \in C(A \cup B) \cap A$  and  $y \in C(A) \setminus C(A \cup B)$ , and again Strong Expansion fails.

Now let  $\mathbf{R}$  be a nested system of weak orders that generates  $C$ . If Strong Expansion fails, then there must exist both menus  $A \subset B$  and alternatives  $x \in C(B) \cap A$  and  $y \in C(A) \setminus C(B)$ , and since  $C = \mathbf{R} \uparrow$  we have  $y \mathbf{R}_B x$  and  $x \mathbf{R}_A y$  and thus  $x \mathbf{R}_B y$  (since  $\mathbf{R}$  is nested). Moreover, there must also exist an alternative  $z \in B$  such that  $z \mathbf{R}_B y \mathbf{R}_B^\circ x$ , and since  $\mathbf{R}$  is cross transitive it follows that  $z \mathbf{R}_B x$  and hence that  $x \notin C(B)$ , contradicting  $x \in C(B)$ .

**Lemma 6:** If  $\mathbf{P}^1$  is not a system of linear orders, then it cannot be weakly connected (being intrinsically acyclic) and so there must exist distinct alternatives  $x, y \in D$  such that  $x (\mathbf{P}_D^1)^\circ y$ . This implies that there exist both a menu  $A \subset D$  such that  $x \in A$  and  $y \in C(A)$  and a menu  $B \subset D$  such that  $y \in B$  and  $x \in C(B)$ . If  $x \in C([xy])$ , then we have  $x \in C(A)$  by Strong Expansion and thus Univalence fails. Alternatively, if  $y \in C([xy])$  then we have  $y \in C(B)$  by Strong Expansion and again Univalence fails.

Now let  $\mathbf{R}$  be a system of linear orders that generates  $C$ . If Univalence fails, then there must exist distinct alternatives  $x, y \in A$  such that  $x, y \in C(A) = \mathbf{R} \uparrow(A)$ . But this implies that  $x (\mathbf{R}_A)^\circ y$ , contradicting the weak connectedness of  $\mathbf{R}$ .

**Proposition 20:** Let  $\mathbf{R}$  be nested and for each  $x, y \in X$  let  $C([xy]) = \mathbf{R} \uparrow([xy])$ . The latter implies that  $x \mathbf{P}^b y$  if and only if  $x \mathbf{R}_{[xy]} y$ , and it follows that  $\mathbf{P}^b = \bigcup_{x, y \in X} \mathbf{R}_{[xy]} \subset \bigcup_{A \in \mathbb{A}} \mathbf{R}_A = \bigcup \mathbf{R}$ . Conversely, since  $\mathbf{R}$  is nested we have  $x \mathbf{R}_A y$  only if  $x \mathbf{R}_{[xy]} y$  and therefore only if  $x \mathbf{P}^b y$ , and it follows that  $\bigcup \mathbf{R} = \bigcup_{A \in \mathbb{A}} \mathbf{R}_A \subset \mathbf{P}^b$ .

**Proposition 21:** Let Contraction hold. Given  $x, y \in X$ , if  $x \mathbf{P}^b y$  then  $y \notin C([xy])$  and for any menu  $A \supset [xy]$  we have  $y \notin C(A)$  by Contraction, which is to say that  $x \mathbf{P}^s y$ . But then  $\mathbf{P}^b \subset \mathbf{P}^s$  and  $(\mathbf{P}^b)^* \subset (\mathbf{P}^s)^* \subset \bar{\mathbf{E}}$  since  $\mathbf{P}^s$  is acyclic, and thus Base Acyclicity holds.

**Proposition 23:** Let Separation Acyclicity hold. Since  $\mathbf{P}^b \subset \mathbf{P}^s$  by Proposition 1, we have  $(\mathbf{P}^b)^* \subset (\mathbf{P}^s)^* \subset \bar{\mathbf{E}}$  by Separation Acyclicity, and thus Base Acyclicity holds.

Now let Strong Expansion and Base Acyclicity hold. Given  $x, y \in X$ , if  $x \mathbf{P}^s y$  then there exists a menu  $A$  such that  $x \in C(A)$  and  $y \in A \setminus C(A)$ . Since both  $[xy] \subset A$  and  $x \in C(A) \cap [xy]$ , Strong Expansion implies that  $C([xy]) \subset C(A)$  and hence that  $y \notin C([xy])$ , which is to say that  $x \mathbf{P}^b y$ . But then  $\mathbf{P}^s \subset \mathbf{P}^b$  and  $(\mathbf{P}^s)^* \subset (\mathbf{P}^b)^* \subset \bar{\mathbf{E}}$  by Base Acyclicity, and thus Separation Acyclicity holds.

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