Estimating Quadratic Variation When Quoted Prices Jump by a Constant Increment

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Abstract

Financial assets' quoted prices normally change through frequent revisions, or jumps. For markets where quotes are almost always revised by the minimum price tick, this paper proposes a new estimator of Quadratic Variation which is robust to microstructure effects. It compares the number of *alternations*, where quotes are revised back to their previous price, to the number of other jumps. Many markets exhibit a lack of autocorrelation in their quotes' alternation pattern. Under quite general *no leverage* assumptions, whenever this is so the proposed statistic is consistent as the intensity of jumps increases without bound. After an empirical implementation, some useful corollaries of this are given.

JEL classification: C10; C22; C80

Keywords: Realized Volatility; Realized Variance; Quadratic Variation; Market Microstructure; High-Frequency Data; Pure Jump Process.

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1 Introduction

There is widespread evidence of persistence in financial assets' volatility. Therefore, estimating recent volatility furthers the desirable goal of forecasting volatility in the near future. In practice, it is more convenient to estimate a price's recent quadratic variation (QV) rather than its volatility, although the two are closely related, for QV is the integral of the squared volatility over a specified period such as a trading day. Andersen, Bollerslev, and Diebold (2005) provide a survey of the literature on this topic.

The availability of rich second-by-second price data has encouraged high-frequency sampling when estimating QV. However, consistent estimation is significantly complicated at the highest frequencies by market microstructure effects. This paper points out features in many markets' microstructure which, when tested for positively, can be used as structural restrictions to control for this interference. Ultimately, this leads to a new estimator for QV.

Specifically, the paper points to the fact that on many markets agents almost always revise quoted prices by exactly the minimum admissible price increment, i.e. by one price tick at a time. Furthermore, the direction of these revisions, or jumps, often has a non-correlation property in its reversal pattern, called *Uncorrelated Alternation*. Both these features are, for example, found on the Chicago Board of Trade's (CBOT's) market for 10-Year US Treasury Bond Futures, as well as the London Stock Exchange's (LSE's) market in Vodafone, which was its most active equity on a number of measures in early 2004.

Under theoretical assumptions that rule out leverage effects, the testable features imply that a simple estimator of QV is consistent as the intensity of jumps grows large, namely:

$$nk^2\frac{c}{a}, \tag{1}$$

where $n \in \mathbb{N}$ is the number of jumps in the quoted price sequence, the constant $k > 0$ is the price tick, and $a \leq n$ is the number of *alternations*, i.e. jumps whose direction is a reversal of the last jump. Engle and Russell (1998) studies these. Jumps which do not alternate are *continuations*, and number $c = (n - a)$. The term nk^2 is the QV of the observed price. This is an inconsistent, and normally an upwardly biased, estimate of underlying QV because of microstructure effects. However the upwards bias implies an excess of alternation, and in fact multiplying by the fraction c/a compensates consistently.

The modelling framework necessitates a new double asymptotic limit theory reflecting the high-frequency microstructure it describes. In this theory, the intensity of jumping grows without limit, but the variance of each jump diminishes at the same rate (see Delattre and Jacod (1997) for a related theory). The statistic then converges to the truth at rate $n^{\frac{1}{2}}$. This rate is faster than the statistic reported in Zhang, Mykland, and Aït-Sahalia (2003), whose rate is $n^{\frac{1}{6}}$, or the alternatives in Zhang (2004) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004), where the rate is $n^{\frac{1}{4}}$. The added asymptotic efficiency is bought at a cost, however, because the necessary testable features do not apply on all markets, and some markets exhibit leverage effects. With the same proviso, the statistic is consistent regardless of the short-run order flow dynamics, and is robust to a range of microstructure effects including the bid-ask bounce, informed trading (see Kyle (1985), Glosten and Milgrom (1985) and Hasbrouck (1991)) , volume-related effects (see Engle and Lange (2001)), and resiliency dynamics (see Biais, Hillion, and Spatt (1995), Coppejans, Domowitz, and Madhavan (2003) and Degryse, de Jong, van Ravenswaaij, and Wuyts (2003)).

The paper proceeds as follows: Section 2 presents the main theorem of the paper, as well as the relevant asymptotic limit theory. Section 3 then outlines the theorem's proof (detailed derivations are left to the Appendix). Section 4 applies the method to Vodafone equity data and treasury bond futures prices at CBOT. Section 5 discusses corollaries and extensions, while Section 6 concludes.

2 The model and main result

This section first prepares the ground for the main result, given in Section 2.3. Consider a probability space $(\Omega, \mathcal{F}, P)$. Let it have a filtration $\{\mathcal{F}_t \;: t \in \mathbb{R}^+\}$, which, as the sigma-field $\mathcal{F}$, is generated by three stochastic processes on $\mathbb{R}^+$: X and Y, which are prices, and σ, a volatility process. For some $T > 0$, only $\{Y_t : 0 \leq t \leq T\}$ is observed. Y has a random initial value. Let

$$X_t = W_{[X]_t} \tag{2}$$

where W is a standard Brownian motion and

$$[X]_t = \int_0^t \sigma_u^2 du. \tag{3}$$

Therefore X is a time-changed Brownian motion, with stochastic volatility σ.[1] A short-hand for X is

$$X = W \circ [X], \tag{4}$$

where $\circ$ denotes composition on a time-change. The quantity to be estimated is the QV of X over the observed period, or $[X]_T$. Let σ be piecewise smooth, càdlàg and everywhere positive with probability one. Hence $[X]$ is invertible with probability one. It is possible that σ is constant, and σ may have arbitrary serial dependence. X has no leverage effects, in fact[2]

$$W \perp\!\!\!\perp \sigma, \tag{5}$$

where $\perp\!\!\!\perp$ indicates independence. As X is not observed, nor is $[X]_T$. Instead, Y is observed. But Y may jump and deviates from X by a microstructure effect, called the process ϵ:

$$\epsilon := Y - X. \tag{6}$$

Definition The microstructure, ϵ, has no leverage effects if

$$\epsilon \circ [X]^{-1} \perp\!\!\!\perp \sigma. \tag{7}$$

This means that viewed in "business time" the microstructure effect is independent of current volatility. Operationally, this will mean that the *frequency* (not the magnitude) of quote revisions grows with increased volatility – a realistic stance given the tightness of spreads on the markets later introduced.[3]

Definition The microstructure, ϵ is stationary in business time if

$$\epsilon \circ [X]^{-1} \text{ is stationary.} \tag{9}$$

[1]For more on stochastic volatility, see, for example, the reviews in Ghysels, Harvey, and Renault (1996) and Shephard (2005, Ch 1).

[2]Barndorff-Nielsen and Shephard (2004c) treat in detail time-change Lévy processes with no leverage.

[3]An alternative definition for there to be no leverage effects in the market microstructure, not adopted here, would be one in "calendar time" :

$$\left\{ \int_0^t \frac{\epsilon_u}{\sigma_u} du \right\}_{t=0}^{\infty} \perp\!\!\!\perp \sigma. \tag{8}$$

This would imply that the normalized increments in ϵ were independent of current volatility, meaning that, for example, the mean magnitude of revisions in quoted prices would grow with increased volatility.

Assume that for all realizations of σ, eg. $\bar{\sigma}$, the conditional probability measure

$$P^{\bar{\sigma}}(\cdot) = P(\cdot|\ \sigma = \bar{\sigma}) \tag{10}$$

exists.

2.1 The observed price as a pure jump process

The observed price, Y is a pure jump process specified by

$$Y_t = Y_0 + \int_0^t G_u dN_u, \tag{11}$$

where N is a simple[4] counting process with an intensity. Its intensity and the process G are adapted to the filtration generated by the process $|\epsilon|$.[5] They therefore depend on both X and Y, but taking the absolute value of ϵ implies that they respond to buying and selling symmetrically. N deviates from 0 with positive probability over $[0, T]$. The QV of Y is clearly

$$[Y]_t = \int_0^t G_u^2 dN_u, \tag{12}$$

a random variable.

Decompose the process N by

$$N = A + C, \tag{13}$$

where A and C are counting processes. The *alternation* process, A, counts the jumps in Y which have opposite sign to the one before, and the *continuation* process C counts jumps that continue in the same direction as the one before. Both are adapted to Y. Notice that as N is simple, arrivals of A and C at the same time are not possible, so the decomposition is unique. View the first jump in Y as an alternation. For all $i \in \mathbb{N}$ let t_i be the time of the i'th jump in Y. Define the random sequence $Q = \{dA_{t_i} - dC_{t_i} : i \in \mathbb{N}\}$. So Q records $+1$ for an alternation and -1 for a continuation.

Definition Y has Uncorrelated Alternation if Q has zero first-order autocorrelation.

[4]I.e. the probability of observing two or more events in a small period of time, when divided by the probability of observing one event, is second order.

[5]For richer dynamics in Y, this filtration (and the full filtration) can be arbitrarily augmented by some stochastic, real vector-valued initial condition independent of the sign of Y_0.

Identification Assumption Given two events observable before any jumping time t_i, $H_1 \in \mathcal{F}_{t_i-}$ and $H_2 \subset H_1$,

$$\{ \ E(Y_{t_i}|H_1) = E(Y_{t_i}|H_2) \ \} \leftrightarrow \{ \ E(X_{t_i}|H_1) = E(X_{t_i}|H_2) \ \}. \tag{14}$$

Thus, if H_2 adds (no) new information to H_1 concerning the likely direction of Y's next jump, it adds something (nothing) new about the level of X.

2.2 Asymptotic limit theory

A long sample invites invites the time series econometrician to suppose that the sample were of "infinite" length. This is behind much sampling theory used in macroeconomics and financial economics. Of course, in practice the data is finite and so these asymptotics provide an approximation, whose accuracy can be assessed through the simulation of realistic cases.

Similarly, high frequency market microstructure data invites the asymptotic thought experiment that, given an underlying price process, the microstructure had evolved "infinitely" fast, with "infinitely" small jumps. Delattre and Jacod (1997) exemplify such an approach. This implies a double asymptotic theory with a relative rate of convergence between the increase in jumping intensity and the decline in jumping magnitude. The rate is here proposed so that $[Y]_T$ can, in line with observed data, have a non-zero limit in probability. The next part formalizes this intuitive idea in terms of a scaling constant, $\alpha \in \mathbb{R}^+$, which converges to zero from above.

2.2.1 Formal asymptotic theory

As σ is measurable with respect to X, $P^{\bar{\sigma}}$ may be factorized into a marginal and a conditional distribution by

$$P^{\bar{\sigma}}(Y, X) = P(Y|X)P^{\bar{\sigma}}(X). \tag{15}$$

Define the process X^α by

$$X_t^\alpha := \frac{1}{\alpha} X_{\alpha^2 t}, \tag{16}$$

So, for $\alpha < 1$, the functional $X \rightarrow X^\alpha$ slows X down, but normalizes it so that its spot volatility is preserved. Holding $P^{\bar{\sigma}}(X)$ fixed, define a new conditional probability measure $P_\alpha(Y|X)$ by:

$$P_\alpha(Y|X) := P(Y^{1/\alpha}|X^\alpha). \tag{17}$$

The asymptotic theory studies

$$\lim_{\alpha \to 0} P_\alpha(Y|X) P^{\bar{\sigma}}(X), \tag{18}$$

as an approximation to (15).

Intuition Suppose that $\alpha < 1$. The conditional measure P_α can be understood as the result of a three stage process: first the latent X is slowed and scaled up. Second, Y evolves stochastically according to the model, conditional on the slowed and expanded X. Third, Y (and, implicitly, X) is speeded back up, and appropriately shrunk.

Therefore the measure $P_\alpha(Y|X)$ makes more likely, for given X, realizations of Y with more jumps, of smaller magnitude. Indeed, with probability 1, $N_T \to \infty$ as $\alpha \to 0$. Note that this asymptotic formalization is also applicable in cases when Y is not simply a pure jump process. This manipulation preserves the stationarity of ϵ in business time.

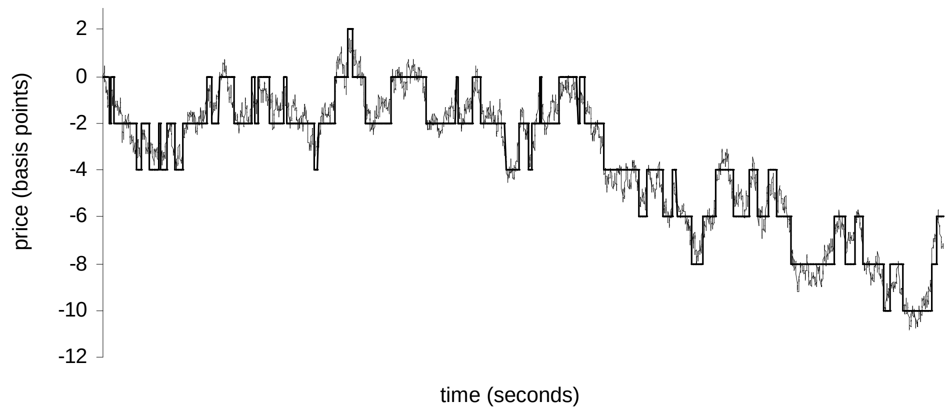

Figure 1: A simulation of this paper's proposed model. It shows an asset's observed price, which jumps, and its continuous underlying price, here a scaled Brownian motion. The observed price has a propensity to alternate.

2.3 The main result

Theorem 2.1 *Suppose that*

(A) Y has Uncorrelated Alternation,

(B) Y always jumps by a constant $\pm k$,

(C) ϵ has no leverage effects, is stationary in business time, is ergodic, and $E(\epsilon) = 0$.

(D) Y always jumps towards X, and

(E) The Identification Assumption holds.

Then

$$[Y]_T \frac{C_T}{A_T} \tag{19}$$

is a consistent estimator of $[X]_T$ as $\alpha \to 0$. Conditional on σ, the limiting distribution as $\alpha \to 0$ of

$$\sqrt{N_T}\left[[Y]_T \frac{C_T}{A_T} - [X]_T\right] \tag{20}$$

exists and is normal. Its variance depends on σ and the market's short-run order dynamics, and is detailed later in Proposition 3.5.

Proof. Section 3 provides the proof of this Theorem through a series of propositions, whose detailed derivations are left to the appendices. ■

Figure 1 shows a process satisfying the Theorem's assumptions.

Discussion of the result The result is semi-parametric because it does not refer to the dynamics or the intensity of N. The proposed estimator is easy to calculate. It multiplies together two components: $[Y]_T$, which is equal to $N_T k^2$, and C_T/A_T (a positive ratio which in practice tends to be less than 0.5, though the theory does not require this). Many jumps are indicative of high stochastic volatility unless most of them are alternations, a possibility which the observed proportion of alternations to continuations provides a means to account for.

Discussion of the assumptions Note that (B) and (D) are jointly equivalent to

$$G \equiv -k \ \mathrm{sign}(\epsilon). \tag{21}$$

Assumptions (A) and (B) can be tested empirically. (A) states that the likelihood that a jump is an alternation does not depend on whether the last jump was. It may be tested via a regression of Q on itself lagged. (B), which assumes a constant jump magnitude, is true of many markets, including both markets studied in the empirical section of the paper.

The assumptions (C), (D) and (E) cannot be tested. (C) states that viewed in "business time" the microstructure effect is ergodic and independent of current volatility. Thus, while at times it evolves fast, these are exactly the times when X also evolves fast. (D) states that Y always jumps towards X, meaning that $[X]_T$ includes the effects of noise traders on prices at the microstructure level. Finally, (E) is innocuous.

Relationship to the existing literature The availability of rich second-by-second data has encouraged high-frequency sampling of prices when measuring their QV. The benchmark case is to compute the observed price's Realized Variance (RV) at some high frequency. This (in calendar time) is calculated by breaking up a period of time, e.g. a trading day, into many intervals of equal length, then squaring the observed returns over these intervals, and adding them up. Barndorff-Nielsen and Shephard (2002a) and (2002b) provide an asymptotic limit theory for RV as it approaches QV but before market microstructure becomes a central concern. In the current framework, at the highest frequency RV is $[Y]_T$.

However, researchers have found that a price's RV at high frequencies typically deviates significantly from RV at low frequencies, see Zhou (1996), Andreou and Ghysels (2002) and Oomen (2002). This has been attributed to microstructure noise, i.e. ϵ, in Zhang, Mykland, and Aït-Sahalia (2003) and Barndorff-Nielsen, Hansen, Lunde, and

Shephard (2004). This paper follows these in making a correction to RV to account for microstructure effects.

The idea that observed prices are pure jump processes, which deviate from a fundamental price, is already present in Oomen (2004) and Zeng (2003). This perspective explains two interrelated puzzles. First, if prices are pure jump processes, then the observed asymptotic behavior of bipower variation, the influential statistic introduced in Barndorff-Nielsen and Shephard (2004b), which converge to zero with finer sampling, is explicable. Second, when studying quotes data, Hansen and Lunde (2004) find that at high frequencies RV is at times a downwards-biased estimator of QV. They show that this result implies a negative covariation between efficient returns and the error due to microstructure effects. They also document time-dependence in the error, a phenomenon that has been abstracted from in much of the literature. A pure jump process can account for these features: the local infrequency of jumps implies both serial correlation in the error and instantaneously negative covariation between the error and efficient price.

The use of a time-change argument in the proof of the Theorem relates to Barndorff-Nielsen and Shephard (2004c).

3 Proof of Theorem 2.1

Proposition 3.1 *Suppose that Assumptions (B),(C) and (D) of Theorem 2.1 hold. Let R be the ratio $\frac{[X]_T}{E\ [Y]_T}$. The error just before the i'th jump is ϵ_{t_i-}. Taking the ergodic expectation, for all i*

$$E[\mid \epsilon_{t_i-} \mid] = \frac{k}{2}\left[R+1\right]. \tag{22}$$

Proof. See Appendix B. ■

Discussion Note $E\ [Y]_T$ exists because N is simple and σ is bounded over the compact set $[0,T]$. This proposition implies that the expected magnitude of the error, measured just before a jump, is an increasing affine function of $R = \frac{[X]_T}{E\ [Y]_T}$. Its intercept is $\frac{k}{2}$, so that if X is constant, then Y simply jumps between $X \pm \frac{k}{2}$. As $[X]_T$ increases, the expected error magnitude increases.

Proposition 3.1 provides a unbiased estimate of $\mid \epsilon_{t_i-} \mid$ while the direction of the jump at t_i gives the sign of ϵ_{t_i-}. Combining these, an unbiased estimate of ϵ_{t_i-} itself is available. Equally, X_{t_i} can be estimated without bias by adding or subtracting $E[\mid \epsilon_{t_i-} \mid]$ to/from

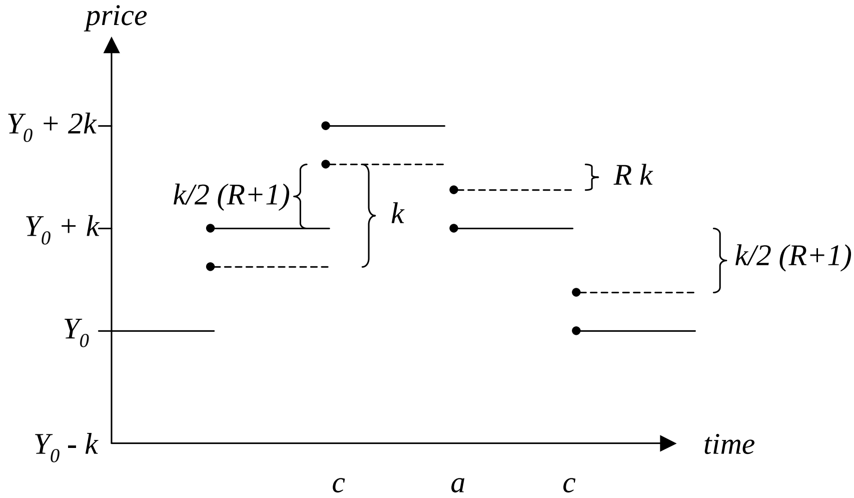

Figure 2: The solid line shows Y, while the dashed line shows Z. The letters on the time axis indicate if the jump is an alternation or a continuation. The diagram is of an example illustrating the relative contribution to the QV of Z by alternations and continuations.

Y_{t_i-}, depending on the direction Y jumps at t_i. This is illustrated in Figure 2. Since X is a martingale, this estimate remains valid until Y next jumps at $t_{(i+1)}$, and is clearly an improvement on the estimate just given by Y. The next definition gives the name Z to this conditional estimation process.

Definition For each of Y's jumping times, t_i, define Z_{t_i} by

$$Z_{t_i} = E[X_{t_i} \mid Y_{t_i}\ , G_{t_i}\ , R\ ]. \tag{23}$$

(Recall that $G_{t_i} = \pm k$ is the jump in Y at t_i.) Extend the sequence $\{Z_{t_i} : i \in \mathbb{N}\}$ rightwards to a càdlàg pure jump process Z. Note that Z is not observed because R is not observed. The evolution of Z is described in Figure 2, which also illustrates the following lemma.

Lemma 3.2 *The Quadratic Variation process for Z, denoted $[Z]$, is a linear combination of the processes A and C given by*

$$[Z] = k^2(C + R^2 A). \tag{24}$$

Proof. When Y jumps by continuing in the same direction as the last jump, Z jumps by k. When Y jumps by alternating in direction, Z jumps by Rk. This follows from simple calculation, and is easily seen in Figure 2. The QV of Z is the sum of its squared jumps. The lemma now follows. ■

Definition A process S, adapted to $\{\mathcal{F}_t\}$, has Ideal Error if conditional on any σ,

$$E\ [S]_T\ = [X]_T, \tag{25}$$

where the expectation is ergodic.

Proposition 3.3 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1, as well as the Identification Assumption, hold. Uncorrelated Alternation then implies that Z has Ideal Error.*

Proof. See Appendix C. ■

Uncorrelated Alternation may be tested simply by regressing Q linearly on itself lagged, and making the usual F test that the regressor is significant.

Proposition 3.4 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then,*

$$E[A_T R - C_T] = 0, \tag{26}$$

and R has the Method of Moments estimator

$$\hat{R} = \frac{C_T}{A_T}. \tag{27}$$

(Define $\hat{R} = 0$ if $C_T = A_T = 0$).

Proof. See Appendix D. The main case is when Y does not have Ideal Error. In a sketch, as Z has Ideal Error, and $[X]_T = R\ E[k^2 N_T]$,

$$E[k^2(C_T + A_T R^2)] = RE[k^2 N_T]. \tag{28}$$

Thus the expectation of a quadratic in R is 0:

$$E[A_T R^2 - N_T R + C_T] = 0. \tag{29}$$

The quadratic has roots at 1 and C_T/A_T. But $R \neq 1$ since Y does not have Ideal Error. Hence (26) is true. ■

So, recalling $[X]_T = R\ E[Y]_T$, the proposed estimator of $[X]_T$ is

$$\hat{R}[Y]_T. \tag{30}$$

Denoting by $\hat{Z}$ the estimate of the process Z constructed by replacing R with $\hat{R}$ in (23),

$$\hat{R}[Y]_T = [\hat{Z}]_T. \tag{31}$$

The final proposition in this section provides the asymptotic limit theory for this estimator, proving its consistency.

Proposition 3.5 *Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then conditionally on σ, the following limit theory applies:*

$$\lim_{\alpha \to 0} \sqrt{N_T} \left(\frac{\hat{R}[Y]_T}{[X]_T} - 1 \right) \sim N(0, UVU'), \tag{32}$$

where U is the pair $(1, \frac{(1+R)^2}{R})$ and V is the long-run variance matrix of the stationary time series of pairs, Π:

$$\Pi = \left\{ \left(\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})} \ , \ \frac{Q_i + 1}{2} \right) \ : i \in \mathbb{N} \right\}. \tag{33}$$

The left hand term here is the elapsed QV in X between the $(i-1)$th and ith jumps in Y, once de-averaged. As previously defined, Q_i takes value $+1$ if the ith jump in Y is an alternation, and -1 if it is a continuation.

Proof. See Appendix E. ■

This asymptotic limit theory is infeasible because the elapsed QV is not directly observed, ruling out estimates of Π and so of V.

Proof of Theorem 2.1 Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. (Then Z may be constructed as in Figure 2.) If in addition Assumptions (A) and (E) hold, then Proposition 3.3 shows that Z has Ideal Error. Therefore Propositions 3.4 and 3.5 apply and the Theorem follows.

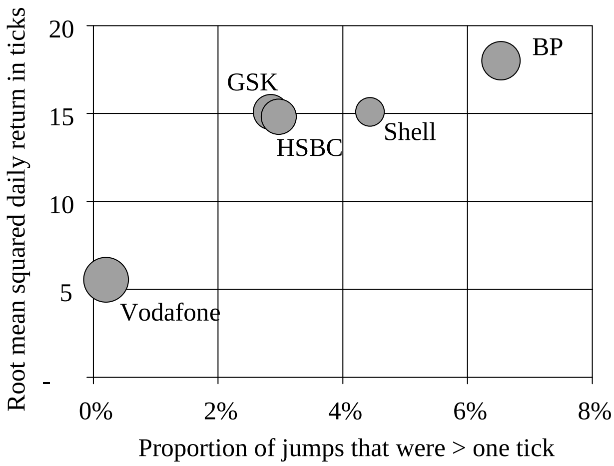

Figure 3: A representation of the five most traded stocks on the London Stock Exchange in January 2004, relating the price tick to (on the vertical axis) daily estimated volatility (excluding overnight returns) and (on the horizonal axis) the size of quote revisions. The diameters of the disks are proportional to the value of shares traded in the month.

4 Empirical implementation

4.1 Vodafone on the London Stock Exchange

This part implements the proposed estimator for Vodafone stock traded on the LSE's electronic limit order book, called SETS. Vodafone was the LSE's most heavily traded stock (by dollar value) in January 2004. This month was a period of 21 trading days running from 8:00am to 4:30pm. Figure 3 compares Vodafone to the next four most traded equities in the period.[6] It shows that Vodafone's price tick was on average larger as a proportion of the average daily squared return (excluding overnight returns), and that Vodafone had many fewer quote revisions of magnitude greater than the price tick. In line with, for example, Engle (2000), the first 15 minutes of the trading day were excluded. Quotes were timed to the nearest second. Vodafone's best offer was revised 4,580 times over the sampled interval, on average 218 times per day. Of these, 8 revisions (i.e. jumps), or 0.2 per cent, were of magnitude greater than the price tick.

Andersen, Bollerslev, Diebold, and Labys (2000) introduced a graphical technique

[6]More than 1 per cent of the quote revisions in BP, Shell, GSK and HSBC stock were by more than one price tick. This makes them somewhat less applicable, given Assumption (B) of Theorem 2.1. A more detailed tabulation is given later in Figure 11.

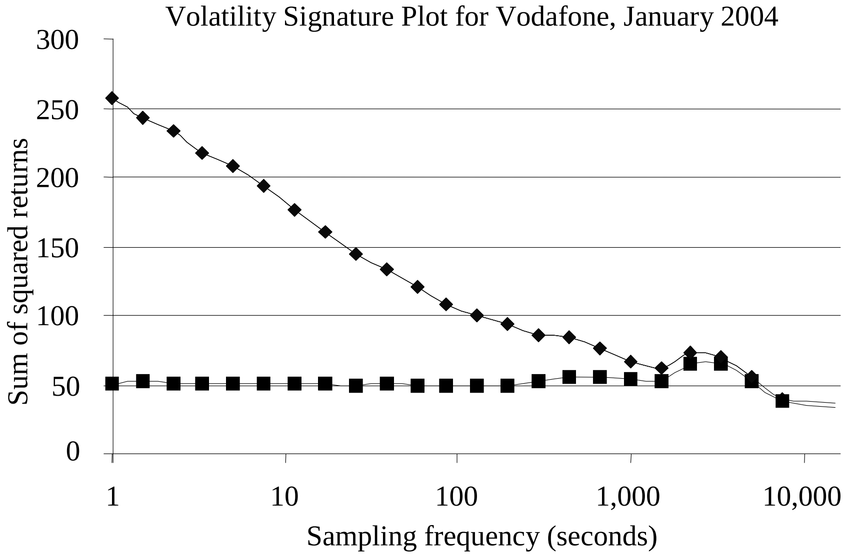

Figure 4: Volatility signature plots for Vodafone's best ask price. The diamonds show the estimated RV of the quote at various observation frequencies (the unit is pence). The squares show the estimated RV of the transformed quote, $\hat{Z}$, at various frequencies.

known as the volatility signature plot. This plots the Realized Variance of a price process as a function of the frequency at which it is sampled. Legibility is enhanced by plotting sampling frequency on a log-scale. Finally, by analyzing a long enough time series, it is hoped that (even at its right-hand end) the shape of the schedule will be reasonably stable due to large number effects. Figure 4 shows for the current data, volatility signature plots of Y, and the process $\hat{Z}$, which approximates Z and is constructed using $\hat{R} = C_T/A_T$.

Under the assumptions of Proposition 3.3, passing the Uncorrelated Alternation test implies that Z has Ideal Error. Therefore loosely, the test can be interpreted as a sufficient test for the hypothesis that the volatility signature plot of Z is flat. Inspection of Figure 4 suggests this may at least be so of $\hat{Z}$. But more formally, for each trading day, the sequence of alternations / continuations, Q, was regressed on itself lagged. For each of the 21 days, the significance of the regressor could be assessed with the usual F test. On 17 days the null hypothesis of *no first order autocorrelation* was accepted at a confidence level of 5 per cent. On two further days, it was accepted at 2 per cent. On the two days where it was still rejected, the lower of the p-values was 0.3 per cent. Having passed specification testing, QV estimation was performed day by day. Its results are reported

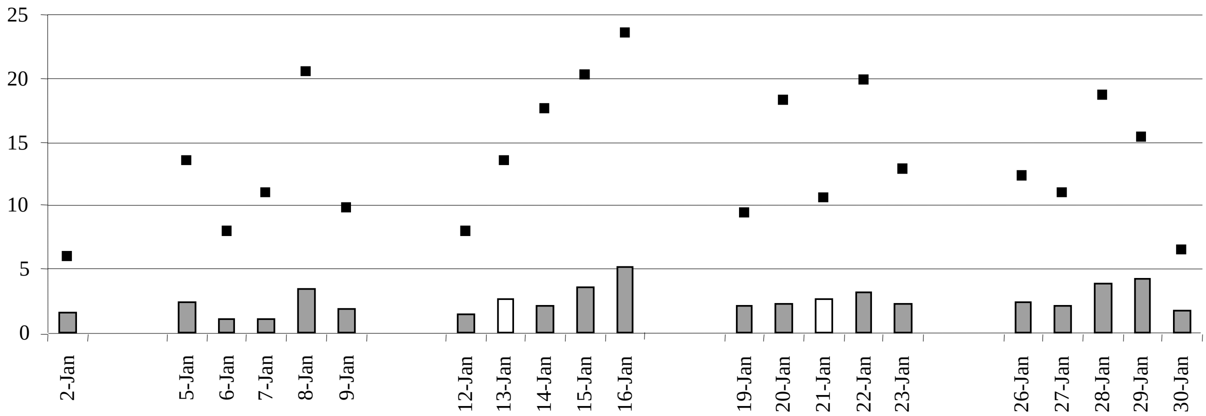

Figure 5: The estimated QV for Vodafone on each trading day in January 2004. The columns show the estimated underlying QV (the unshaded columns represent days where specification testing failed), while the black squares show the QV of the observed price process. The vertical axis is in pence squared.

in Figure 5, and are tabulated in detail in Figure 10.

The initially surprising lack of correlation in alternation may be due to the fact that the Vodafone market is deep, with a bid-ask spread that is at or adjacent to its regulatory minimum most of the time, meaning temporary resiliency effects do not play a large role. Furthermore, concentrating on its quoted prices, not transaction prices, eliminates Vodafone's bid-ask bounce. These features are also present on the 10-Year Treasury Bond Future at CBOT.

4.2 CBOT Treasury Bond Future

The data contains the prices of the 10-Year Treasury Bond Future on 29 July, 30 July and 2 August 2004 (the dates span a weekend), as displayed in Figure 7. Open trading was observed on the electronic limit order book from 7:30am to 4:00pm on each day.[7] The outstanding best price at the bid was studied, which is quoted in quantities representing $\frac{1}{12{,}800}$ of the contract's nominal value, \$100,000. The price level on this market was approximately 14,000 at this time. The minimum admissible price increment was two. The best bid was revised 2,432 times over the sampled interval, or on average 811 times per day. Of these, 21 jumps, or 0.86 per cent, were of magnitude greater than the price

[7] Trading begins at 7:00am.

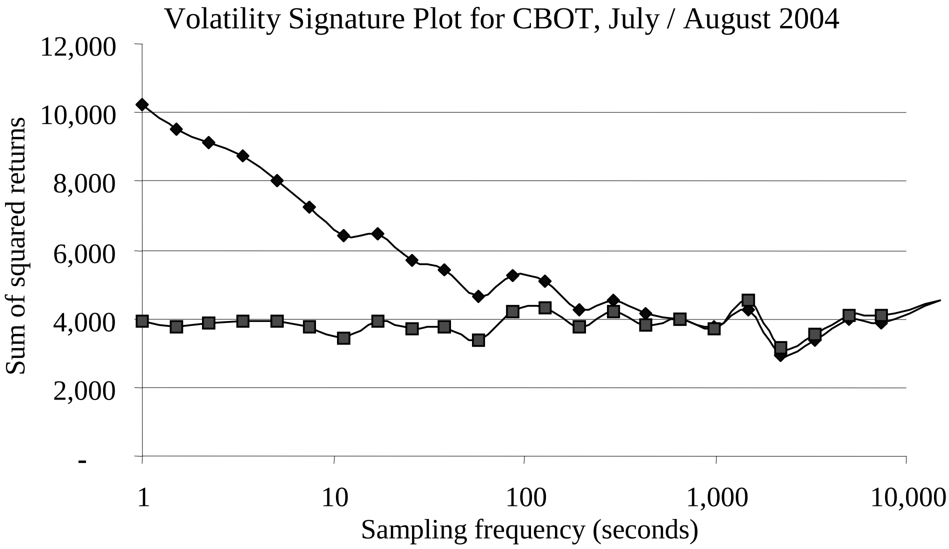

Figure 6: Volatility signature plots for the best bid on the CBOT limit order book for the 10-Year Treasury Bond Future. The diamonds show the estimated RV of the quote at various observation frequencies. The squares show the estimated RV of the transformed quote, $\hat{Z}$, at various frequencies.

tick.

Each day was divided into the period before 9:00am, and the period after. Small intervals containing jumps greater than the price tick were excised, principally a 25-minute period around 9:00am on 30 July. At this time there was a sudden period of very high volatility, perhaps due to a public announcement (See Figure 7).[8] For three of the six periods the null hypothesis of *no first order autocorrelation* was accepted at a confidence level of 5 per cent. For the other three periods, it was accepted at 2 per cent. The data's volatility signature plot, as well as the one for $\hat{Z}$, are presented in Figure 6. The latter's flatness provides further corroboration of the hypothesis that Z has Ideal Error. QV estimation results for CBOT are reported in Figure 8.

5 Extensions and corollaries

This section first discusses a closely related filtering technique. It then shows how when certain further restrictions can be imposed on a well-specified model, the central limit theory presented in Proposition 3.5 becomes feasible, permitting inference about the estimated QV. This is followed by a discussion of an extension to the bivariate case

[8]In addition, a few seconds' trading around 7:30am on 29 and 30 July were excluded.

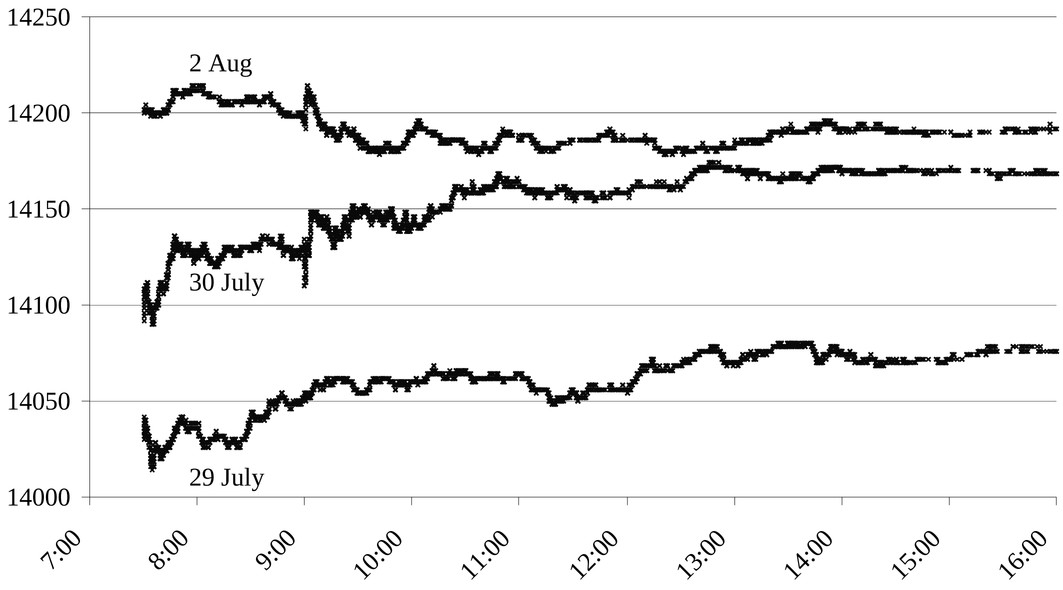

Figure 7: The price-path taken by the best bid for the 10-Year Treasury Bond Future on 29, 30 July and 2 August 2004.

where two price processes have correlated returns.

5.1 A filtering technique

Under the assumptions of Theorem 2.1, X may be estimated by $\hat{Z}$. For further econometric analysis, first constructing $\hat{Z}$ provides a better estimate of X, the underlying price, than using Y directly. In this sense, $\hat{Z}$ thus constructed is a useful filter for Y.

When estimating the QV $[X]_T$, $[\hat{Z}]_T$ may be preferable to the proposed statistic, $k^2 N_T C_T / A_T$ if the data contains short episodes of model mis-specification with large quote revisions or swings due to public announcements. The two are identical if there is no mis-specification. The CBOT data provides an example of this. Its price path is presented in Figure 7. It is clear that around 9am on July 30 2004, there was a brief substantial increase in volatility: $[\hat{Z}]_T$ would include a contribution to volatility from this time, but $k^2 N_T C_T / A_T$ would not. This discrepancy is reported for CBOT in Figure 8.

	A_T	N_T	$\hat{R}$	$N_T k^2 C_T / A_T$	$[\hat{Z}]_T$
29-Jul	487	672	38 %	1,021	1,021
30-Jul	834	1,163	39 %	1,835	2,134
02-Aug	456	597	31 %	738	759
Total	1,777	2,432	37 %	3,586	3,914

Figure 8: Estimation results for the 10-Year Treasury Bond Future on 29, 30 July and 2 August 2004. The discrepancy in some values of the two right-hand columns is due to the short episodes of mis-specification described in the text.

5.2 A feasible limit theory when volatility is constant

The asymptotic limit theory of Proposition 3.5 is infeasible because Π is not observed, and therefore its long-run variance matrix, V, cannot be estimated. However, circumstances may exist where the econometrician may reasonably suppose the process σ to be constant. Then the elapsed QV in X between jumps at t_i and t_{i-1} is given by

$$[X]_{t_i} - [X]_{t_{(i-1)}} = \sigma^2(t_i - t_{i-1}), \tag{34}$$

and, de-averaged,

$$\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})} = \frac{(t_i - t_{i-1})}{T} E(N_T). \tag{35}$$

Substituting N_T for $E(N_T)$ gives an estimate $\hat{\Pi}$, on which the Newey and West (1987) method, and related long-run variance estimation techniques, can be used to estimate V.

5.3 A feasible limit theory when Y follows Sluggish Rounding

Where the assumption of constant volatility is untenable, nevertheless inference may still be feasible by assuming a specific dynamic structure for the order flow. This approach is exemplified by Sluggish Rounding, which is introduced below.

When prices jump by a constant increment, it would seem reasonable to view them as resulting from rounding off a continuous process to the nearest penny, half-penny etc. This approach is taken (with the inclusion of trading costs) in Hasbrouck (1998). It is also present in Zeng (2003) (where the underlying price is corrupted by noise, then rounded). An appropriate central limit theory is provided in this context by Delattre and Jacod (1997). In these papers, prices are discretely sampled. However, in the current

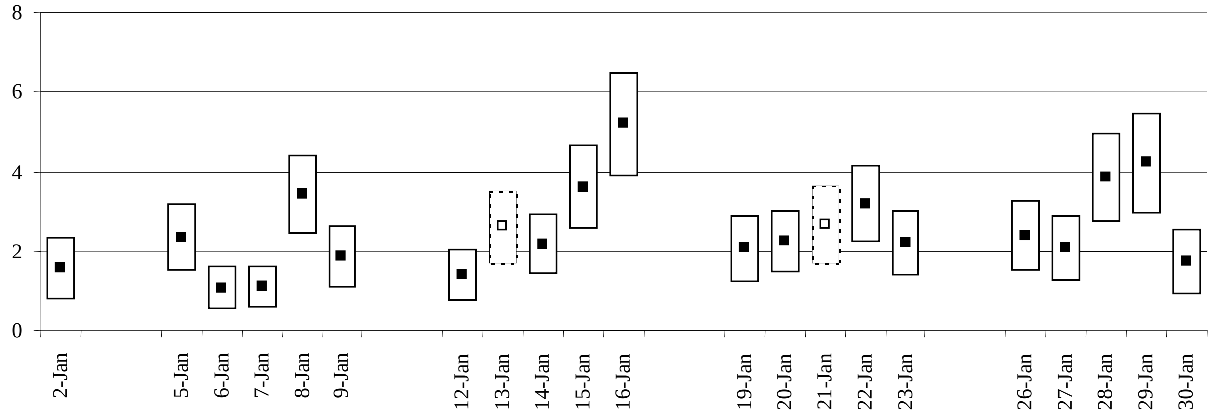

Figure 9: The estimated QV for Vodafone on each trading day in January 2004. The floating columns estimate ± 2 standard deviations' error, under an assumption that prices evolved according to Sluggish Rounding. The vertical axis is in pence squared. The dashed columns represent days where specification testing failed.

setting of continuous sampling, rounded-off Itô processes must have QV either of zero, or of ∞, the latter with positive probability (provided prices are unbounded). This is because when an Itô process crosses a rounding threshold, with probability 1 it does so infinitely more times in the next instant.

While retaining continuous sampling, this problem can be avoided by introducing a "sluggishness",[9] whereby any threshold for rounding observed prices up by one increment exceeds by a small margin the threshold for rounding them back down. Formally, suppose that there exists $\rho > 0$ such that Y moves towards X by amount k whenever $|X - Y| = \rho$. For this to have finite activity, it must be that

$$\rho > \frac{k}{2}. \tag{36}$$

Otherwise, any single jump would precipitate an infinite flurry.[10]

Proposition 5.1 *Suppose that Y evolves according to Sluggish Rounding. Then it has Uncorrelated Alternation. Furthermore, Π is an i.i.d. sequence and*

$$UVU' = \frac{2}{3R}(1 + 4R + 2R^2). \tag{37}$$

Therefore UVU' may be estimated consistently by replacing R in (37) with $\hat{R} = C_T/A_T$.

[9] I am grateful to Peter Hansen for introducing me to this idea and suggesting the term *sluggish*.

[10] In the case where $\rho = k$, Y jumps to exactly the value of X whenever X reaches $(Y \pm k)$.

Proof. See Appendix F. ■

This method is implemented in Figure 9 for Vodafone, January 2004, assuming that the observed price evolves according to Sluggish Rounding. A simulation of Sluggish Rounding 10,000 times based on $R = 0.4$ and $[X]_T = 1296$, estimates which are in line with the CBOT data, is suggestive of good small sample properties. In particular, the truth was rejected at 1% on 1.0% of runs, at 5% on 4.7% of runs, and at 10% on 9.3% of runs (standard errors were estimated one run at a time using Proposition 5.1). Logarithms were taken at the lower tail to mitigate the influence of the lower bound at zero.

5.4 Estimating bivariate covariation

It seems plausible that where the returns of two financial assets are positively correlated, the value of a portfolio containing both might have disproportionately many continuations. This intuition is supported by the following formalization. Let (X_1, Y_1) and (X_2, Y_2) be the models of two asset prices, satisfying the assumptions of Theorem 2.1, and let (X_1, X_2) be a bivariate Itô process. Note that without loss of generality they can be scaled so Y_1 and Y_2 have the same jump size, k. The quantity of interest is the covariation of X_1 and X_2, written

$$[X_1, X_2]_T = \text{p-lim} \sum_{j=1}^{M} [X_1(t_j) - X_1(t_{j-1})][X_2(t_j) - X_2(t_{j-1})], \tag{38}$$

where $\{0 = t_0, t_1, \; ... \; t_M = T\}$ is a lattice on $[0, T]$ whose mesh tends to zero in the limit, and p-lim denotes that limit in probability, as studied in Barndorff-Nielsen and Shephard (2004a).

Corollary 5.2 *Suppose that the models $(X_1 + X_2, Y_1 + Y_2)$ and $(X_1 - X_2, Y_1 - Y_2)$ each satisfy the Assumptions of Theorem 2.1 and, conditional on (X_1, X_2), Y_1 and Y_2 are independent. Write C^+ (A^+) for the number of observed continuations (alternations) in $Y_1 + Y_2$. Define C^- and A^- analogously for $Y_1 - Y_2$. Then*

$$\frac{1}{4}\left(\frac{C^+}{A^+} - \frac{C^-}{A^-}\right)\left(\; [Y_1]_T + [Y_2]_T \;\right) \tag{39}$$

is a consistent estimator of $[X_1, X_2]_T$.

Proof. See Appendix H. ■

The corollary has the interpretation that, asymptotically, the covariation is positive if and only if there are more continuations in the summed price processes than in the differenced prices. However, the assumptions of this corollary are strong, and it is not expected that they would hold of all data.

6 Conclusion

This paper views the observed price as a pure jump process whose deviations from an underlying stochastic process are stationary. Noting that on many markets the magnitude of jumping is constant and equal to the market's minimum admissible price increment, it proposes a new estimator for QV which down-weights the QV of the observed quoted price process by a factor that takes into account its propensity to alternate. Provided that alternation is uncorrelated at the first order, and given a set of theoretical assumptions, this estimator is consistent and converges at a rate proportional to the square root of the number of observed jumps. Some extensions and corollaries are discussed, and the proposed technique is implemented for a UK equity and a US future.

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A An equivalence with the constant volatility case

Lemma A.1 *Suppose that X and ϵ have no leverage effects. Let $\bar{\sigma}$ be any realization of σ. Let*

$$\bar{s}(t) = \int_0^t \bar{\sigma}_u^2 du, \tag{40}$$

and for any t let

$$\bar{f}^t : \mathcal{F}_t \mapsto \mathcal{F}_{\bar{s}(t)}. \tag{41}$$

be the natural transformation of events induced by the time-change, $\bar{s}$. Let

$$\bar{f} = \cup_{t \in \mathbf{R}^+} \bar{f}^t. \tag{42}$$

Then $\bar{f} : \mathcal{F} \mapsto \mathcal{F}$ is invertible, and for any $A \in \mathcal{F}$,

$$P^{\bar{\sigma}}\left(\bar{f}^{-1}(A)\right) = P^{\mathbf{1}}(A). \tag{43}$$

Proof. Suppose that $\sigma = \bar{\sigma}$. Note that $\bar{s} = [X]$. As $\bar{\sigma}$ is piecewise smooth, càdlàg and everywhere positive, it follows that $\bar{s}^{-1} = [X]^{-1}$ exists. So, $\bar{f}$ is invertible. As X and ϵ have no leverage effects,

$$(X, Y) \circ [X]^{-1} \perp\!\!\!\perp \sigma. \tag{44}$$

Let $\underline{\sigma}$ be another realization of σ. Since $\mathcal{F}$ is generated by X and Y, for any A in $\mathcal{F}$, (44) implies that, with the obvious notation,

$$P^{\bar{\sigma}}\left(\bar{f}^{-1}(A)\right) = P^{\underline{\sigma}}\left(\underline{f}^{-1}(A)\right). \tag{45}$$

The proposition follows by putting $\underline{\sigma} \equiv \mathbf{1}$. ■

Consequently, where Assumption (C) of Theorem 2.1 holds, any properties of the conditional model when $\sigma \equiv 1$, will also hold for σ equal to arbitrary $\bar{\sigma}$, when its events are transformed by the function $\bar{f}$, which is induced by $\bar{s} = [X]$. For analytical clarity, proofs in the Appendix do not restrict σ to be 1, but rather begin by restricting σ to be a positive constant. This restriction is removed in a final step.

B Proof of Proposition 3.1

The proposition states: Suppose that if Assumptions (B),(C) and (D) of Theorem 2.1 hold. Let R be the ratio $\frac{[X]_T}{E\ [Y]_T}$. The error just before the ith jump is ϵ_{t_i-}. Taking the ergodic expectation, for all i

$$E[\ |\ \epsilon_{t_i-}\ |\ ] = \frac{k}{2}\left[R+1\right]. \tag{46}$$

Proof Assume that σ is constant. The proof proceeds by equating the ergodic variances of ϵ at (i.e. just after) two subsequent jumps, at random times, say the second and third, at times t_2 and t_3. For all t, $Var(\epsilon_t) = E(\epsilon_t^2)$. Define $\bar{\lambda}$ as the ergodic or average intensity of N. Define $\{x_i\}$ and $\{y_i\}$ as the respective sets of random increments in X and Y:

$$x_i = X_{t_i} - X_{t_{i-1}}, \tag{47}$$

$$y_i = Y_{t_i} - Y_{t_{i-1}}, \tag{48}$$

so that y_3 is equal to the jump at t_3, i.e. $\pm k$, and $\{x_i\}$ are i.i.d. of known variance:

$$x_i \sim N(\ 0\ ,\ \sigma^2(t_i - t_{i-1})\ ). \tag{49}$$

It then follows that

$$\epsilon_{t_i} = \epsilon_{t_{(i-1)}} + y_i - x_i. \tag{50}$$

So,

$$E[\epsilon_{t_3}^2] = E[(\epsilon_{t_2} + y_3 - x_3)^2] \tag{51}$$

$$= E[\epsilon_{t_2}^2 + y_3^2 + x_3^2 - 2x_3\epsilon_{t_2} + 2y_3\epsilon_{t_2} - 2y_3x_3] \tag{52}$$

$$= E[\epsilon_{t_2}^2 + y_3^2 + x_3^2 - 2x_3\epsilon_{t_2} + 2y_3(\epsilon_{t_2} - x_3)]. \tag{53}$$

But by Assumptions (B) and (D), y_3 is $-k\ sign(\epsilon_{t_2} - x_3)$. Furthermore, $E(x_i^2)$ is $\sigma^2/\bar{\lambda}$. Further, as X is a martingale, $E[x_3|\epsilon_{t_2}] = 0$, and so $E[x_3\epsilon_{t_2}] = 0$. So, (53) is

$$E[\epsilon_{t_2}^2] + k^2 + \sigma^2/\bar{\lambda} - 2kE[\ |\ \epsilon_{t_2} - x_3\ |\ ]. \tag{54}$$

Moreover, $(\epsilon_{t_2} - x_3)$ is $\epsilon_{t_3}^-$, the right limit of ϵ before the jump at t_3. As ϵ is stationary, we may equate $E[\epsilon_{t_2}^2]$ and $E[\epsilon_{t_3}^2]$ to obtain the equality

$$E[\mid \epsilon_{t_3}^- \mid] = \frac{k}{2}\left[\frac{\sigma^2}{k^2\bar{\lambda}} + 1\right]. \tag{55}$$

But, $[X]_T = T\sigma^2$ and $E\ [Y]_T = Tk^2\bar{\lambda}$. As one could equally have looked at any two successive jumps (not only the second and third), the proposition follows in the case of constant volatility. By Lemma A.1 it also follows, conditional on σ, in the case of stochastic volatility.

C Proof of Proposition 3.3

The proposition states: Suppose that Assumptions (B), (C) and (D) of Theorem 2.1, as well as the Identification Assumption, hold. Uncorrelated Alternation then implies that Z has Ideal Error.

Proof First suppose that Y has Ideal Error. Then $Z = Y$ has Ideal Error trivially. Now, and for the rest of the proof, assume that Y does not have Ideal Error. If Q has first lag autocorrelation of zero then it is easily checked that

$$E(G_{t_{i+1}}|G_{t_i}) = E(G_{t_{i+1}}|G_{t_i}, G_{t_{(i-1)}}). \tag{56}$$

Therefore, by the Identification Assumption, and the fact that between jumps ϵ is a martingale,

$$E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{(i-1)}}). \tag{57}$$

The Proposition now follows from Corollary C.1.

Corollary C.1 *Assume Assumptions (B), (C) and (D) of Theorem 2.1, and that Y does not have Ideal Error. Then Z has Ideal Error iff at each jump, timed t_i, $i > 1$,*

$$E(\epsilon_{t_i}|G_{t_i}) = E(\epsilon_{t_i}|G_{t_i}, G_{t_{(i-1)}}). \tag{58}$$

Proof. If Z has Ideal Error, then by Lemma C.2, for all t,

$$E\left(Z_t - X_t \middle| \text{ the last two jumps in Y went up, then down }\right) = 0. \tag{59}$$

So, conditional on the two jumps in Y before t going up, then down

$$E\left(Y_t - X_t\right) = Y_t - Z_t. \tag{60}$$

So,

$$E\left(\epsilon_t | \text{ last 2 jumps in Y went up, then down}\right) = E\left(\epsilon_t | \text{ last jump in Y went down}\right). \tag{61}$$

The proposition now follows by the up-down symmetry of the model, considering exhaustively the four cases where prior to t:

the last 2 jumps in Y went up, then down,

the last 2 jumps in Y went up, then up,

the last 2 jumps in Y went down, then up, and

the last 2 jumps in Y went down, then down. ∎

So Z has Ideal Error whenever conditioning not only on the last jump, but also on the last-but-one jump does not improve the best ergodic estimate of X_t given Y_t.

Lemma C.2 *Assume Assumptions (B), (C) and (D) of Theorem 2.1. If Y does not have Ideal Error then for any t,*

$$\frac{E[Z]_T - [X]_T}{E[Y]_T - [X]_T} = 2p_A E\left(\frac{Z_t - X_t}{k} | \textit{ the last two jumps in } Y \textit{ went up, then down}\right), \tag{62}$$

where p_A is the probability that a jump is an alternation.

Proof. See Appendix G. ∎

D Proof of Proposition 3.4

The Proposition states: Suppose that Assumptions (B),(C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Then,

$$E[A_T R - C_T] = 0, \tag{63}$$

and R has the Method of Moments estimator

$$\hat{R} = \frac{C_T}{A_T}. \tag{64}$$

(Define $\hat{R} = 0$ if $C_T = A_T = 0$).

The proof studies two cases in turn. The second case, where Y does not have Ideal Error, contains an important argument.

Case where Y has Ideal Error Then $R = 1$. By Proposition 3.1, the expected absolute value of ϵ_t just before a jump is k. Therefore, the expected value of ϵ_t conditional on Y just having jumped upwards is 0. The Identification Assumption implies that Y has equal probability of jumping up as down after this upwards jump. Given buy-sell symmetry, and as Q is uncorrelated, the ergodic probability that any given jump is an alternation is 0.5. Hence

$$E[\ A_T - C_T\ ] = 0. \tag{65}$$

Case where Y does not have Ideal Error Z has Ideal Error if

$$\begin{aligned} [X]_T &= E([Z]_T) \qquad (66) \\ &= E(k^2(C_T + A_T R^2)). \qquad (67) \end{aligned}$$

Also, R is defined by

$$\begin{aligned} [X]_T &= RE([Y]_T) \qquad (68) \\ &= E(k^2 R(C_T + A_T)). \qquad (69) \end{aligned}$$

Subtracting and dividing by k^2, we therefore have the moment condition,

$$E[\ \ (C_T + A_T R^2) - R(C_T + A_T)\ \ ] = 0. \tag{70}$$

Or, factorizing,

$$(R-1)\ E[\ \ (A_T R - C_T)\ \ ] = 0. \tag{71}$$

Since Y does not have Ideal Error, $R \neq 1$. Divide through by $(R-1)$:

$$E[\ A_T R - C_T\ ] = 0. \tag{72}$$

E Proof of Proposition 3.5

The proposition states : Suppose that Assumptions (B), (C) and (D) of Theorem 2.1 hold. Suppose that Z has Ideal Error. Conditionally on σ, the following limit theory applies:

$$\lim_{\alpha \to 0} \sqrt{N_T} \left(\frac{\hat{R}[Y]_T}{[X]_T} - 1 \right) \sim N(0, UVU'), \tag{73}$$

where U is the pair $(1, \frac{(1+R)^2}{R})$ and V is the long-run variance matrix of the stationary time series of pairs, Π:

$$\Pi = \left\{ \left(\frac{[X]_{t_i} - [X]_{t_{(i-1)}}}{E([X]_{t_i} - [X]_{t_{(i-1)}})} \; , \; \frac{1+Q_i}{2} \right) \; : i \in \mathbb{N} \right\}. \tag{74}$$

Left hand term is the de-averaged elapsed QV in X between the $(i-1)$th and ith jumps in Y.

Assume that σ is constant. Then Π reduces to

$$\Pi = \left\{ \left(\bar{\lambda}(t_i - t_{(i-1)}), \; \frac{1+Q_i}{2} \right) \; : i \in \mathbb{N} \right\}. \tag{75}$$

Note that the right-hand fraction takes the value 1 when Y alternates, and 0 when it continues. Suppose first that $T \to \infty$ so that $N_T \to \infty$ with probability 1. By a standard central limit theorem, as T is the sum of the durations between events, ignoring the time after the last jump in the sample,

$$\lim_{N_T \to \infty} \sqrt{N_T} \left(\begin{pmatrix} \frac{T\bar{\lambda}}{N_T} \\ \frac{A_T}{N_T} \end{pmatrix} - \begin{pmatrix} 1 \\ p_A \end{pmatrix} \right) \sim N(0, V), \tag{76}$$

where p_A is the probability that a jump is an alternation. Let

$$f : (x, y) \to (1-y)/xy. \tag{77}$$

Then f is differentiable in the positive quadrant and so by the Delta Method,

$$\lim_{N_T \to \infty} \sqrt{N_T} \left(\frac{N_T C_T}{T\bar{\lambda} A_T} - \frac{(1-p_A)}{p_A} \right) = N(0, df' V df), \tag{78}$$

where df is evaluated at $(1, p_A)'$. But the moment constraint in Proposition 3.4 implies that

$$\frac{(1-p_A)}{p_A} = R, \tag{79}$$

so, by simple calculation, the evaluated df is $-RU'$ and

$$\lim_{N_T \to \infty} \sqrt{N_T} \left(\frac{N_T C_T}{T\bar{\lambda} A_T} - R \right) = N(0, RUVU'R). \tag{80}$$

So,

$$\lim_{N_T \to \infty} \sqrt{N_T} \left(\frac{N_T k^2 C_T}{T k^2 \bar{\lambda} R A_T} - 1 \right) = N(0, UVU'). \tag{81}$$

But, $Tk^2\bar{\lambda}R = [X]_T$, while $N_T k^2 = [Y]_T$ and $C_T/A_T = \hat{R}$.

Now consider the asymptotic theory as $\alpha \to 0$. As $\alpha \to 0$, N_T, A_T and C_T all $\to \infty$ almost surely. However, due to the definition of X^α, $k^2\bar{\lambda}$ is invariant to α. Therefore, $R = \sigma^2/k^2\bar{\lambda}$ is also invariant to α. Thus, given the moment constraint, C_T/A_T converges almost surely to R as $\alpha \to 0$. As ϵ is stationary and independent of σ, the distribution of Π is invariant to α. Hence V is also invariant, and (76) is true as $\alpha \to 0$. The proposition now follows for constant σ. But, due to Lemma A.1, the proposition also holds, conditional on σ, for stochastic σ.

F Proof of Proposition 5.1

The proposition states : Suppose that Y evolves according to Sluggish Rounding. Then it has Uncorrelated Alternation. Furthermore, Π is an i.i.d. sequence and

$$UVU' = \frac{2}{3R}(1 + 4R + 2R^2). \tag{82}$$

Therefore UVU' may be estimated consistently by replacing R in (82) with $\hat{R} = C_T/A_T$.

Proof Suppose that σ is constant. The duration between jumps is the time taken for a scaled Brownian motion of volatility σ to exit the interval $(-k, Rk)$. A moment's thought reveals that Q and Π are then i.i.d. The probability that a Brownian motion of volatility σ starting at zero reaches the level Rk before the level $-k$ is known to be

$$\frac{k}{k + Rk}, \tag{83}$$

or $1/(1 + R)$. The expected time to the first hit is Rk^2/σ^2. This therefore also describes the probability that at a jump Z moves up (down) by Rk rather than down (up) by k, i.e. the probability that Y alternates, rather than continuing, and the expected duration before this jump. The following formulae are derived from results recorded in Borodin and Salminen (1996):

The variance of the time between jumps is

$$\frac{Rk^4(1 + R^2)}{3\sigma^4} \tag{84}$$

So the variance of the normalized time between jumps, $(t_i - t_{(i-1)})/\delta$, is

$$\frac{1 + R^2}{3R} \tag{85}$$

The expected time between jumps, conditional on them alternating, is

$$\frac{2Rk^2(2R+1)}{3(R+1)\sigma^2} \tag{86}$$

Therefore the covariance of Q_i and $\bar{\lambda}(t_i - t_{(i-1)})$ is

$$-\frac{1-R}{3(1+R)} \tag{87}$$

The variance of Q_i is

$$\frac{R}{(1+R)^2} \tag{88}$$

These give the components of the matrix V. UVU' is now easily calculated.

G Proof of Proposition C.2

The proposition states : Assume Assumptions (B), (C) and (D) of Theorem 2.1. If Y does not have Ideal Error then for any t,

$$\frac{E[Z]_T - [X]_T}{E[Y]_T - [X]_T} = 2p_A E\left(\frac{Z_t - X_t}{k} | \text{ the last two jumps in Y went up, then down }\right), \tag{89}$$

where p_A is the probability that a jump is an alternation.

Proof Suppose that σ is constant. The stationarity of ϵ implies that

$$E((\epsilon_t + d\epsilon_t)^2) = E(\epsilon_t^2). \tag{90}$$

Therefore,

$$E(\epsilon_t^2 + 2\epsilon_t d\epsilon_t + d\epsilon_t^2) = E(\epsilon_t^2). \tag{91}$$

And

$$-2E(\epsilon_t d\epsilon_t) = E(d\epsilon_t^2). \tag{92}$$

Or,

$$-2E(\epsilon_t(G_t dN_t - dX_t)) = E((G_t dN_t - dX_t)^2). \tag{93}$$

So

$$-2E(\epsilon_t G_t \lambda_t)dt = E(G_t^2 \lambda_t)dt + \sigma^2 dt. \tag{94}$$

Multiplying by $-T/dt$ and adding on a constant,

$$2TE(\epsilon_t G_t \lambda_t) + 2TE(G_t^2 \lambda_t) = TE(G_t^2 \lambda_t) - T\sigma^2. \tag{95}$$

Or,

$$2TE((\epsilon_t + G_t)G_t\lambda_t) = TE(G_t^2\lambda_t) - T\sigma^2. \tag{96}$$

But the left hand side of this is

$$2T\bar{\lambda}E(\epsilon_t G_t| \text{ jump at t }) \tag{97}$$

While the right hand side is

$$E[Y]_T - [X]_T. \tag{98}$$

Putting this together, given the up-down symmetry of ϵ,

$$E[Y]_T - [X]_T = 2T\bar{\lambda}E(\epsilon_t G_t| \; Y \text{ jumped up at t }). \tag{99}$$

G.1

Write

$$\eta_t = Z_t - X_t. \tag{100}$$

We can define $\tilde{G}$ so that η obeys

$$d\eta_t = \tilde{G}_t dN_t - dX_t. \tag{101}$$

This means $|\tilde{G}_t| = k$ if a jump at t would be a continuation, or $|\tilde{G}_t| = Rk$ if the jump would be an alternation. Then in the same way as in (99),

$$E[Z]_T - [X]_T = 2T\bar{\lambda}E(\eta_t\tilde{G}_t| \; Y \text{ jumped up at t }). \tag{102}$$

And

$$\frac{E[Z]_T - [X]_T}{E[Y]_T} = \frac{2}{k^2}E(\eta_t\tilde{G}_t| \; Y \text{ jumped up at t }). \tag{103}$$

So, writing p_A for the ergodic probability of alternation; and distinguishing the case of an alternation from that of a continuation in order to extract the magnitude of $\tilde{G}$ from the expectation, (103) is

$$\begin{aligned}
& \frac{2}{k^2}\left\{p_A RkE(\eta_t| \; Y \text{ alternated up at t }) + (1-p_A)kE(\eta_t| \; Y \text{ continued up at t })\right\} \\
= \; & \frac{2}{k}E(\eta_t| \; Y \text{ jumped up at t }) + \frac{2}{k}p_A(R-1)E(\eta_t| \; Y \text{ alternated up at t }) \\
= \; & 0 + \frac{2}{k}p_A(R-1)E(\eta_t| \; Y \text{ alternated up at t }).
\end{aligned}$$

Therefore,

$$\frac{E[Z]_T - [X]_T}{E[Y]_T(1-R)} = -\frac{2}{k} p_A E(\eta_t | \ Y \text{ alternated up at t }). \tag{104}$$

That is, reversing the up/down direction of the conditioning,

$$\frac{E[Z]_T - [X]_T}{E[Y]_T - [X]_T} = \frac{2}{k} p_A E(\eta_t | \ Y \text{ alternated down at t }). \tag{105}$$

H Proof of Corollary 5.2

The corollary states: suppose that the models $(X_1 + X_2, Y_1 + Y_2)$ and $(X_1 - X_2, Y_1 - Y_2)$ each satisfy the assumptions of Proposition 3.4 and, conditional on (X_1, X_2), Y_1 and Y_2 are independent. Write C^+ (A^+) for the number of continuations (alternations) in $Y_1 + Y_2$. Define C^- and A^- analogously for $Y_1 - Y_2$. Then

$$\frac{1}{4}\left(\frac{C^+}{A^+} - \frac{C^-}{A^-}\right)(\ [Y_1]_T + [Y_2]_T\ ) \tag{106}$$

is a consistent estimator of $[X_1, X_2]_T$.

Proof. Begin with the identity

$$[X_1, X_2]_T = \frac{1}{4}\left([X_1 + X_2]_T - [X_1 - X_2]_T\right). \tag{107}$$

Note that as Y_1 and Y_2 are conditionally independent pure jump processes,

$$[Y_1]_T + [Y_2]_T = [Y_1 + Y_2]_T = [Y_1 - Y_2]_T. \tag{108}$$

As $(X_1 + X_2, Y_1 + Y_2)$ and $(X_1 - X_2, Y_1 - Y_2)$ satisfy the assumptions of Proposition 3.4,

$$[Y_1 + Y_2]_T \frac{C^+}{A^+} \text{ and } [Y_1 - Y_2]_T \frac{C^-}{A^-} \tag{109}$$

are consistent estimators of

$$[X_1 + X_2]_T \text{ and } [X_1 - X_2]_T. \tag{110}$$

The corollary now follows by a straightforward substitution. ■

	# jumps	p-value	# alternations	QV	+ / - 95% error
2-Jan	95	0.29	75	1.58	0.76
5-Jan	216	0.88	184	2.35	0.83
6-Jan	127	0.86	112	1.06	0.53
7-Jan	176	0.02	160	1.10	0.51
8-Jan	328	0.14	281	3.43	0.99
9-Jan	156	0.08	131	1.86	0.75
12-Jan	128	0.13	109	1.39	0.64
13-Jan	216	**0.007**	181	2.61	0.90
14-Jan	282	0.12	251	2.18	0.74
15-Jan	323	0.85	274	3.61	1.03
16-Jan	377	0.19	309	5.19	1.31
19-Jan	150	0.46	123	2.06	0.82
20-Jan	293	0.04	261	2.25	0.75
21-Jan	169	**0.003**	135	2.66	0.97
22-Jan	318	0.37	274	3.19	0.95
23-Jan	206	0.84	176	2.19	0.80
26-Jan	196	0.91	164	2.39	0.86
27-Jan	175	0.16	147	2.08	0.80
28-Jan	299	0.18	248	3.84	1.11
29-Jan	246	0.88	193	4.22	1.26
30-Jan	103	0.14	81	1.75	0.81
	4579		3869	52.52	3.97

Figure 10: For each day of January 2004 (grouped in weeks), various statistics are recorded for Vodafone's best ask price. The p-value is for the test that Q has zero first-order autocorrelation. QV records the estimated quadratic variation, while the column to the right shows estimated 95% confidence intervals for this estimate, on the hypothesis that Y evolves according to Sluggish Rounding.

	Vodafone		GSK		HSBC	
	Bid	Ask	Bid	Ask		Ask
Price increment (pence)	0.25	0.25	1.0	1.0	0.50	0.50
Number of jumps	4,607	4,580	11,774	10,620	11,334	10,418
Number of alternations	3,890	3,869	9,076	8,034	8,707	7,945
Number of jumps of tick size >1	10	8	329	307	335	314
Percent of jumps that are of tick size >1	0.2%	0.2%	2.8%	2.9%	3.0%	3.0%
$[Y]_T$	288	286	11,774	10,620	2,834	2,605
Continuations / Alternations	0.18	0.18	0.30	0.32	0.30	0.31
$[Y]_T C_T / A_T$	53	53	3,500	3,418	855	811
Implied average volatility	1.6	1.5	13.2	13.1	6.5	6.4
Root mean squared daily return (leaving out overnight price movements)	1.4		15.1		7.4	
Value of stock traded in the month (£m)	12,901		8,219		8,326	

	Shell		BP	
	Bid	Ask		Ask
Price increment (pence)	0.25	0.25	0.25	0.25
Number of jumps	15,153	14,048	19,817	16,825
Number of alternations	11,759	10,848	15,123	12,559
Number of jumps of tick size >1	697	595	1,307	1,090
Percent of jumps that are of tick size >1	4.6%	4.2%	6.6%	6.5%
$[Y]_T$	947	878	1,239	1,052
Continuations / Alternations	0.29	0.29	0.31	0.34
$[Y]_T C_T / A_T$	273	259	384	357
Implied average volatility	3.7	3.6	4.4	4.2
Root mean squared daily return (leaving out overnight price movements)	3.8		4.5	
Value of stock traded in the month (£m)	5,966		9,808	

Figure 11: Summary Statistics for the five most actively traded stocks in January 2004 on the LSE.