Supplementary Appendices to 'Understanding Preferences: "Demand Types", and the Existence of Equilibrium with Indivisibilities'

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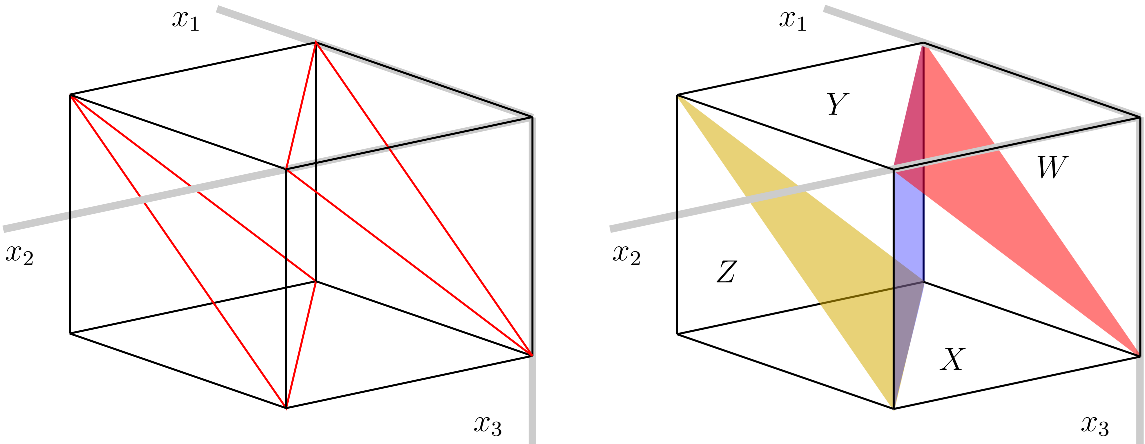
C Strong Substitute Valuations in 3 dimensions

Here we develop valuations for strong substitutes, in 3 dimensions, by using the Valuation-Complex Equivalence Theorem (Thm. 2.14), together with duality between the price complex and the LIP (Prop. 2.20).

Start with a domain of $\{0,1\}^3$. We consider possible demand complexes for strong substitute valuations. Such a demand complex is a subdivision of the cube $[0,1]^3$, such that the edges are all strong substitute vectors: they all are in directions $\mathbf{e}^i$ or $\mathbf{e}^i - \mathbf{e}^j$ where $i, j = 1, 2, 3$. Thus the full set of allowed 1-cells is the collection of red and black edges shown in Fig. 1a. A candidate subdivision is given in Fig. 1b. Three 2-cells,



(a) Possible 1-cells in the demand complex for a strong substitute valuation on $\{0,1\}^3$. Black lines are edges of the cube $[0,1]^3$, and must be 1-cells of the demand complex. The red lines may also be 1-cells of the demand complex.

(b) A polyhedral complex subdivision of $[0,1]^3$, with edges as shown in Fig. 1a. Letters W, X, Y, Z label the four 3-cells.

Figure 1: Developing a candidate demand complex.

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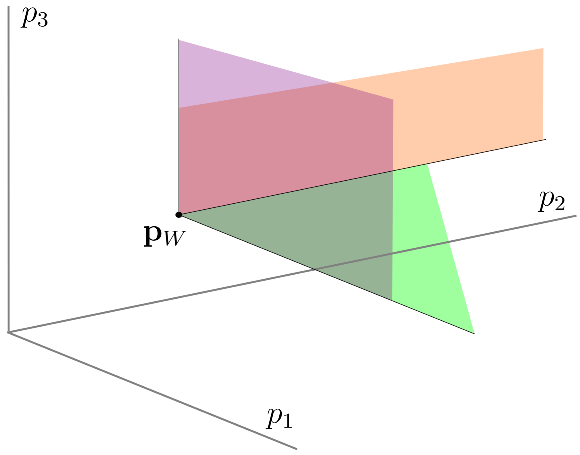
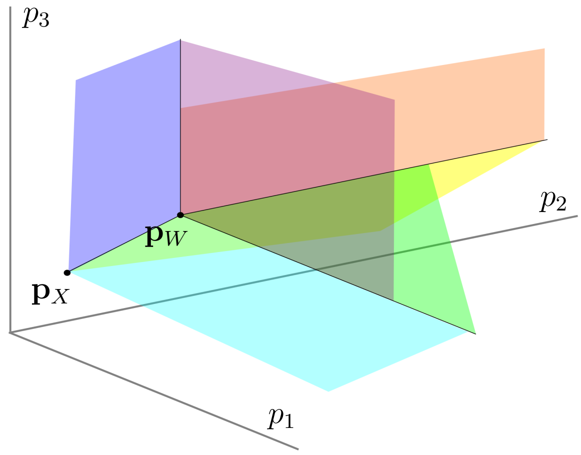
distinguished by being red, blue and yellow, separate the cube into four 3-cells. We label these 3-cells W, X, Y and Z, as shown. *If* this is the demand complex of a valuation, then that valuation is for strong substitutes. But recall that *not* every such subdivision is necessarily a demand complex of some valuation. To ascertain whether it is, we will develop the dual complex in price space. If we can develop a balanced weighted rational polyhedral complex in this way, then we can apply Thm. 2.14 to obtain a valuation.[1]

We thus plot out the $(n-1)$-cells of the dual in price space, that is, the facets. First identify that the demand complex 3-cell for the lowest quantities ("W") corresponds to a price 0-cell ($\mathbf{p}_W$) for "high" prices. There are 1-cells terminating in $\mathbf{p}_W$ coming in from even higher prices in each of the coordinate directions, corresponding to the three 2-cells of W that are in the boundary of the cube. Between each pair of these 1-cells is a facet; each is dual to one of the three edges in Fig 1b that lies along a coordinate axis. Thus we obtain Fig. 2a.



(a) Some cells of the complex dual to Fig. 1b: 0-cell $\mathbf{p}_W$ is dual to the 3-cell W in Fig. 1b; and each facet shown here is dual to an edge lying along a coordinate axis in Fig. 1b.



(b) Development of Fig. 2a with additional cells: 0-cell $\mathbf{p}_X$ is dual to the 3-cell X in Fig. 1b; and the three facets shown, that meet it, are dual to the edges of the red 2-cell in Fig. 1b.

Figure 2: First steps in developing the rational polyhedral complex in price space, dual to Fig. 1b. Facets are shown cut back, so that others lying "behind" can be seen. The complete complex is shown in Fig. 3.

Similar consideration of the red 2-cell in Fig. 1b allows us to develop our picture further: see Fig. 2b. The edges of this red 2-cell correspond to three further facets, all meeting along the 1-cell dual to the red 2-cell itself. This 1-cell runs from $\mathbf{p}_W$ (corresponding to the 3-cell W in Fig. 1b) to a new point $\mathbf{p}_X$ (corresponding to the 3-cell X in Fig. 1b).

The final result is Fig. 3: a 2-dimensional rational polyhedral complex in $\mathbb{R}^3$.

We give weight 1 to every facet of Fig. 3, as it is dual to an edge of "length" 1 in Fig 1b. This weighted LIP is balanced. To see this, consider the full set of facets meeting any 1-cell of Fig. 3. This configuration is dual to a 2-cell of Fig. 1b, taken together with

[1]If we could find a valuation directly, such that Fig. 1 is the demand complex, then we would not need to proceed to price space. This does not seem so easy.

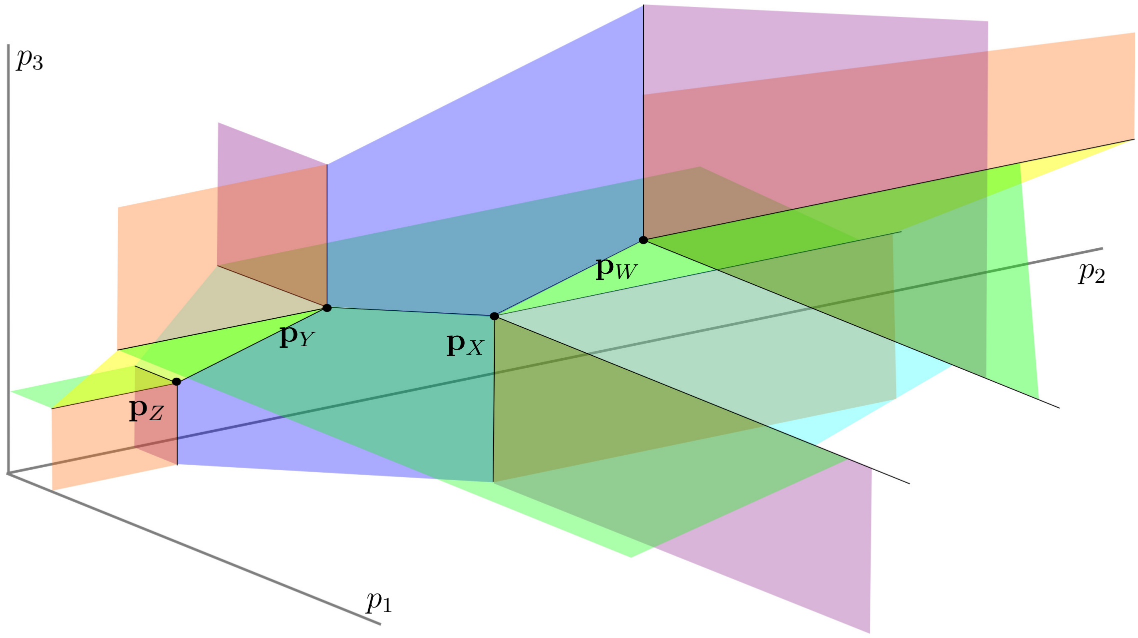


Figure 3: A rational polyhedral complex in price space, dual to Fig. 1b.

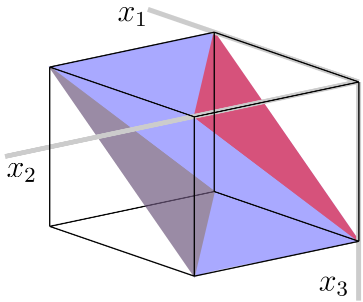
its edges. The vector sum of the edges, going once around the 2-cell, must be zero. But the edges are equal to the normal vectors to the facets (Prop. 2.20). Thus, an oriented weighted sum of the normal vectors to the facets in Fig. 3 is also zero.

Thus we may apply Thm. 2.14: there exists a valuation u whose LIP is depicted in Fig. 3. Indeed, since we did not yet specify any precise coordinate information, Fig. 3 represents an entire combinatorial type of valuations (as defined in Defn. 2.22). Moreover, we can see more combinatorial types immediately. Recall that we developed Fig. 3 on the basis of one subdivision of $[0,1]^3$ (namely Fig. 1b) that was consistent with Fig. 1a. We can flip Fig. 1b over, interchanging the second and third coordinates, and obtain Fig. 4a. This is, of course, also consistent with Fig. 1a. Now the blue face has normal $(1,0,1)$; in Fig. 1b, its normal vector was $(1,1,0)$. The dual LIP would then, of course, be the image of Fig. 3 under the same transformation. The final option, given in Fig. 4b, is when the normal vector to the blue facet is $(0,1,1)$. We obtain the dual LIP for this case by interchanging the first and third coordinates in Fig. 3.

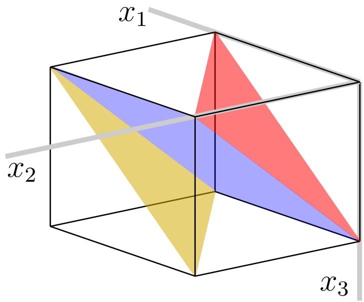
There are, of course, strong substitute valuations on $\{0,1\}^3$ with simpler demand complexes; the trivial subdivision, in which $[0,1]^3$ is itself a 3-cell, also represents a valuation for strong substitutes. But we can recover this demand complex from Fig. 1b by merging adjacent 3-cells. Doing so is equivalent, in price space, to bringing together two 0-cells at the end-points of a 1-cell. If we do so, then this 1-cell collapses into the 0-cells that we are bringing together. The facets adjoining the 1-cell similarly collapse onto 1-cells in their boundaries. This is the same limiting process as we described (in 2 dimensions) in Example B.2.

In fact, any strong substitute valuation on $\{0,1\}^3$ may be obtained in this way: it either is of the combinatorial type of Fig. 3; or is a transformation of this which interchanges the coordinate axes; or is the limit of one of these cases, in which some or all of the 0-cells have been brought together.

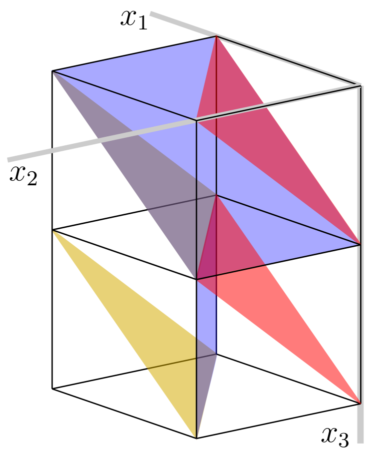
We now find, explicitly, a general form for any valuation of the combinatorial type



(a) Alternative polyhedral complex subdivision of $[0,1]^3$, also with edges as shown in Fig. 1a.



(b) The third possible maximal subdivision of $[0,1]^3$ whose edges are as shown in Fig. 1a.



(c) Demand complex for a valuation in which two units of good 3 are available, composed of Figs. 1b and 4a.

Figure 4: Additional demand complexes for strong substitutes.

shown in Fig. 3. First, we give coordinates to the labelled 0-cells, in such a way that forces consistency with geometry of the complex. That is, first set $\mathbf{p}_W = (a, b, c)$. Then there must exist $\alpha > 0$ such that $\mathbf{p}_X = (a-\alpha, b-\alpha, c-\alpha)$, because we know that the 1-cell connecting these points is in direction $(1,1,1)$ (it is dual to the red 1-cell in Fig. 1b). Similarly, there exists $\beta > 0$ such that $\mathbf{p}_Y = \mathbf{p}_X - \beta(1,1,0)$ for some $\beta > 0$, since the 1-cell connecting $\mathbf{p}_X$ and $\mathbf{p}_Y$ is dual to the blue 2-cell in Fig. 1b. So $\mathbf{p}_Y = (a-\alpha-\beta, b-\alpha-\beta, c-\alpha)$. Finally, since $\mathbf{p}_X$ lies below $\mathbf{p}_Y$ in direction $(1,1,1)$ again, there exists $\gamma > 0$ such that $\mathbf{p}_Z = (a-\alpha-\beta-\gamma, b-\alpha-\beta-\gamma, c-\alpha-\gamma)$. Note that any $a, b, c \in \mathbb{R}$ and any $\alpha, \beta, \gamma > 0$ are consistent with Fig. 3.

Now we know the coordinates of the facets, we may infer the valuation itself by following a simple rule. The rule is: $u(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} = u(\mathbf{y}) - \mathbf{p}\cdot\mathbf{y}$, where $\mathbf{p}$ is in a facet, and bundles $\mathbf{x}, \mathbf{y}$ are demanded on either side of that facet. See Table 5. For any $a, b, c \in \mathbb{R}$ and any $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$, this gives a strong substitute valuation of the same combinatorial type as Fig. 3. Conversely, any valuation of the combinatorial type Fig. 3 may be presented in this form.

x_1		$u(\mathbf{x})$;	
1	*0*	$x_3 = 0$	
a	0	*0*	x_2
$a+b-\alpha-\beta$	b	*1*	

x_1		$u(\mathbf{x})$;	
1	*0*	$x_3 = 1$	
$a+c-\alpha$	c	*0*	x_2
$a+b+c-2\alpha-\beta-\gamma$	$b+c-\alpha$	*1*	

Figure 5: The strong substitute valuation of Fig. 3, given in terms of parameters $a, b, c \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$.

We can go further: the process described above, of collapsing together two 0-cells which are the end-points of the same 1-cell, is the geometric counterpart of just letting one of α, β, γ relax to 0. So Table 5 also presents a strong substitute valuation if we assume only that $\alpha, \beta, \gamma \geq 0$, and by doing so we obtain further combinatorial types of valuations. For example, if $\alpha = \beta = \gamma = 0$ then the valuation is additively separable, the demand complex is the trivial case (one 3-cell consisting of $[0,1]^3$) and the LIP consists of three planes intersecting at (a,b,c). Additional cases correspond to only one or two of these parameters being zero.

Since the remaining combinatorial types are obtained by transforming Figs. 1 and 3 by just interchanging the coordinate axes, a task it is straightforward to replicate in Table 5, we conclude that we have in this way obtained *all* strong substitute valuations for at most one unit of three goods.

Moreover, we may consider the agent's value for additional units in the same way. Extend the example to make a second unit of good 3 available, and assume that the demand complex breaks down as one cube on top of another. We can keep our existing analysis and apply the same technique to the second cube. Let the demand complex now be that shown in Fig. 4c. The LIP is given in Fig. 6. The lower part is the same

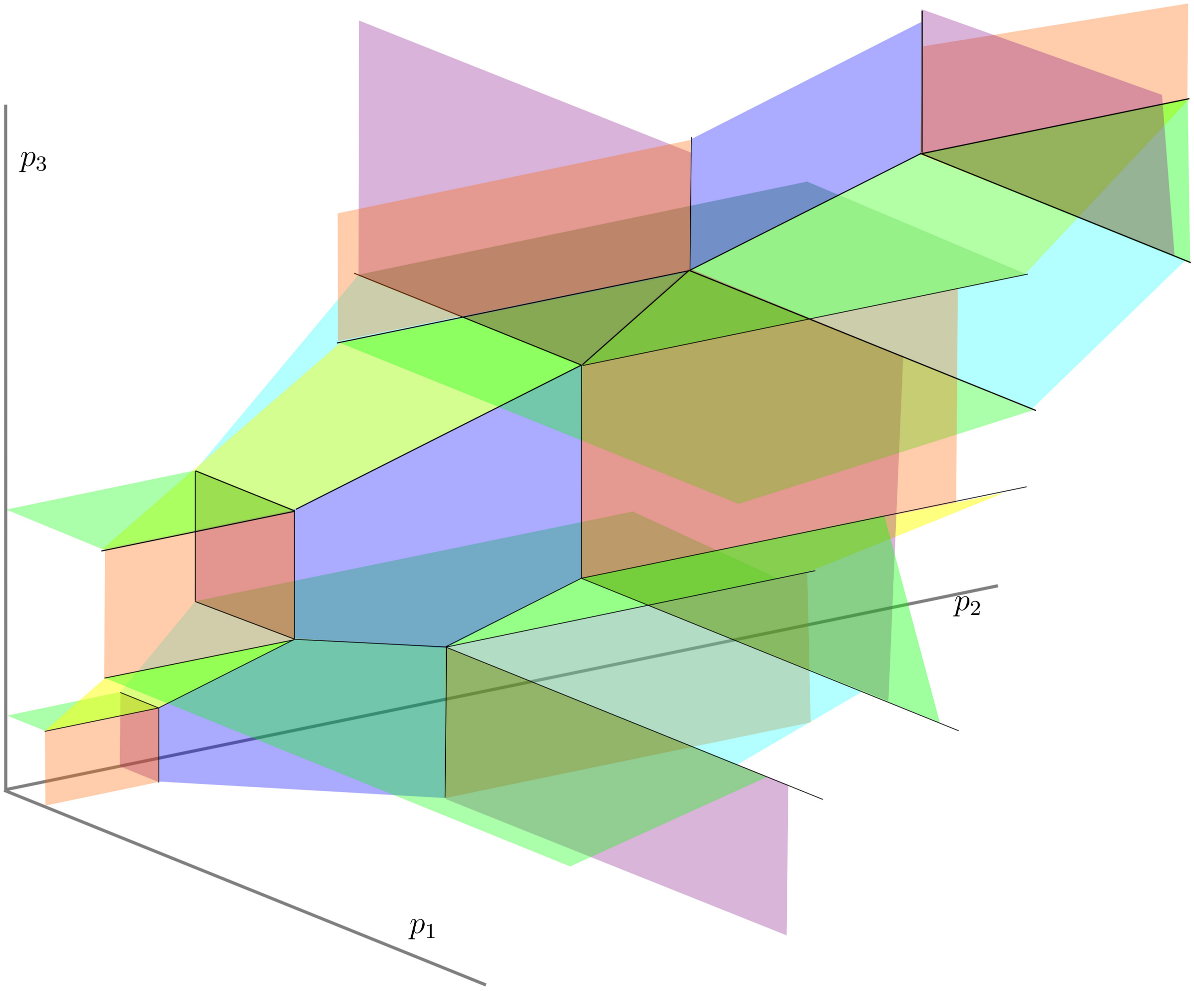


Figure 6: The LIP of a valuation dual to Fig. 4c.

as in Fig. 3, corresponding to the fact that the "lower" cube in Fig. 4c is the same as Fig. 1b. Now imagine interchanging the second and third axes of Fig. 3, obtaining a new LIP. The upper part of Fig. 6 has the same combinatorial type as this new LIP. It is straightforward to infer, again, a general parametric form for any valuation of this combinatorial type.

D Mixed volumes and $M^n_k(\cdot,\cdot)$

The quantity $M^n_k(\cdot,\cdot)$ arises from the "mixed volume" in algebraic geometry (see e.g. Sangwine-Yager, 1993 or Cox et al., 2005, Chapter 7). Recall in Figs. 7c and 7d, the value for $M^2(\{0,1,2\}^2,\{0,1,2\}^2)$ was equal to the sum of the areas shaded in gray. These gray areas are all the 2-cells of aggregate-demand complexes, with the property that one edge comes from the first individual demand complex and one edge comes from the second. We generalise:

Definition D.1 (See e.g. Cox et al. 2005, Defns. 7.6.4, 7.6.5, 7.6.6 and Thm. 7.6.7)**.** Suppose $Q = Q^1 + \cdots + Q^m \subsetneq \mathbb{R}^n$, where $Q^1,\ldots,Q^m$ are polytopes with vertices in $\mathbb{Z}^n$.

1. A *subdivision* of Q is a collection of polytopes $R^1,\ldots,R^s$ such that $Q = R^1 \cup \cdots \cup R^s$ and such that, for $i \neq j$, the intersection $R^i \cap R^j$ is a face of both R^i and R^j.

2. A subdivision $R^1,\ldots,R^s$ of Q is a *mixed subdivision* if each R^i can be written as $R^i = F^1 + \cdots + F^m$, where F^j is a face of Q^j for each j, and where $n = \dim(F^1) + \cdots + \dim(F^m)$, and where if $R^j = F'^1 + \cdots + F'^m$, then $R^i \cap R^j = (F^1 \cap F'^1) + \cdots + (F^m \cap F'^m)$.

3. A cell $R = F^1 + \cdots + F^m$ in a mixed subdivision is a *mixed cell* if $\dim(F^i) \leq 1$ for all i. In particular if $m = n$ then $\dim(F^i) = 1$ for all i.

4. When $m = n$, define the *mixed volume* $MV_n(Q^1,\ldots,Q^n) := \sum_R \mathrm{vol}_n(R)$, where the sum is over all mixed cells R of a mixed subdivision.

To understand these definitions, observe that the maximal cells of a demand complex form a subdivision of the convex hull of its domain. Similarly, the maximal cells of the aggregate-demand complex of m agents gives a subdivision of the convex hull of their aggregate domain. If the intersection between the individual LIPs is transverse, then this is a mixed subdivision. The mixed cells are dual to intersections of facets in their interiors; in Figs. 7c and 7d, these are the gray areas.

In both of Figs. 7c and 7d, the sum of the areas of mixed cells is 2. Indeed, the sum of the volumes of mixed cells is always independent of the choice of mixed subdivision; this result is implicit in our definition of mixed volume above (see Huber and Sturmfels, 1995, Thm. 2.4; the standard definition is stated in their proof). So we can use very simple subdivisions to calculate mixed volumes: see Ex. D.4.

Recall that equilibrium fails for two LIPs, $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, iff it fails at an intersection 0-cell. Suppose cells C_{σ^1}, C_{σ^2} of the respective LIPs meet transversely at such a point. In the demand complexes, we correspondingly have cells σ^1, σ^2, of dimensions $k, n-k$, and such that $\sigma = \sigma^1 + \sigma^2$ is dual to the intersection 0-cell itself. As in Lemma 4.16, equilibrium will fail if the aggregate-demand complex cell $\sigma^1 + \sigma^2$ is "too big". So, as in Sections 5.1.1-5.1.2, we wish to add up the volumes of all aggregate-demand complex cells such as $\sigma^1 + \sigma^2$. And we can do this using mixed volumes.

To calculate a mixed volume we need n polytopes, with each mixed cell being a sum of pieces of dimension 1. But we have *two* polytopes: the convex hulls of the two domains. And we are interested in the sum of aggregate cells like $\sigma^1 + \sigma^2$, but $\dim \sigma^1 + \dim \sigma^2 = n$ (because the intersection is transverse). As Fact D.2 shows, the solution is to take $k := \dim \sigma^1$ copies of the first domain and $n-k$ copies of the second:

Fact D.2 (follows from Huber and Sturmfels, 1995, Thm. 2.4)**.** Suppose the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse. The total volume of aggregate-demand complex cells dual to intersection 0-cells at which an $(n-k)$-cell of $\mathcal{L}_{u^1}$ meets a k-cell of $\mathcal{L}_{u^2}$ is equal to $\frac{1}{k!(n-k)!}MV_n(\mathrm{conv}(A^1),\ldots,\mathrm{conv}(A^1),\mathrm{conv}(A^2),\ldots,\mathrm{conv}(A^2))$, in which we take k copies of $\mathrm{conv}(A^1)$ and $n-k$ copies of $\mathrm{conv}(A^2)$.

The additional factor of $\frac{1}{k!(n-k)!}$ perfectly cancels the factors we used in defining weights of cells–consistent with defining $M^n_k(\cdot,\cdot)$ as a mixed volume in this way.

Lemma D.3 (Cox et al., 2005, Thm 7.4.12.d)**.** *If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite, then $M^n_k(A^1, A^2)$ is the mixed volume of k copies of* $\mathrm{conv}(A^1)$ *with $(n-k)$ copies of* $\mathrm{conv}(A^2)$*, for $k = 1, \ldots, (n-1)$.*

Proof of Facts 5.15. 1 is Cox et al. (2005, Exercise 7.7.b). 2 is an elementary calculation. □

Example D.4. Let $n = 3$ and suppose that A^1 and A^2 are the discrete-convex sets with vertices $\{(0,0,0),(2,0,0),(0,2,0),(2,2,0)\}$ and $\{(0,0,0),(1,0,0),(0,0,2),(1,0,2)\}$ respectively: the domains of the demand complexes shown in Figs. 14a-b.

We calculate $M^3_1(A^1, A^2)$ and $M^3_2(A^1, A^2)$ by considering: agent $1'$, with valuation $u^{1'}(\mathbf{x}) = 0$ for all $\mathbf{x} \in A^1$; and agent $2'$, with valuation $u^{2'}(\mathbf{x}) = x_1 + x_3$ for all $\mathbf{x} \in A^2$. Then $\Sigma_{u^{1'}}$ has a single 2-cell of volume 4 (*not* the demand complexes pictured in Fig. 14). The corresponding 1-cell of $\mathcal{L}_{u^{1'}}$ is in direction $\mathbf{e}^3$ and passes through $\mathbf{0}$. It therefore intersects a weight-2 facet of $\mathcal{L}_{u^{2'}}$ corresponding to the edge of $\mathrm{conv}(A^2)$ from $\mathbf{e}^1$ to $\mathbf{e}^1 + 2\mathbf{e}^3$, and so the demand complex cell corresponding to this intersection 0-cell has volume $4\times 2 = 8$. So by Fact D.2 and Definition D.3 we know $M^3_2(A^1, A^2) = 2!1!\times 8 = 16$.

Similarly, $\Sigma_{u^{2'}}$ has a single 2-cell of volume 2, and the corresponding 1-cell is in direction $\mathbf{e}^2$ and passes through $(1,0,1)$. It therefore intersects a weight-2 facet of $\mathcal{L}_{u^{1'}}$ corresponding to the edge of $\mathrm{conv}(A^2)$ from $\mathbf{0}$ to $2\mathbf{e}^2$, and so the demand complex cell corresponding to this intersection 0-cell has volume $2 \times 2 = 4$. So by Fact D.2 and Defn. D.3 we know $M^3_1(A^1, A^2) = 2!1! \times 4 = 8$.

We conclude that $M^3(A^1, A^2) = 8 + 16 = 24$.

Proof of Thm. 5.22. See Bertand and Bihan, 2013, Thm. 6.1. Alternatively, see that if C_{σ^1} and C_{σ^2} intersect transversely, then it follows from our definitions of cell weights and subgroup indices that $\mathrm{mult}(C_{\sigma^{\{1,2\}}}) = k!(n-k)!\mathrm{vol}_n(\sigma^{\{1,2\}})$, where $k = \dim \sigma^1$. Thm 5.22 now follows from Fact B.2. □