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Who gains from efficient auctions?

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Abstract

I provide conditions under which increasing efficiency by combining competitive auctions of substitute goods increases *both* expected bidder and auctioneer surplus relative to running separate simultaneous auctions. I also provide conditions under which one side's expected surplus falls. I show how the distributional effects depend on the bidders' aggregate bid functions and the auctioneer's costs of production. I estimate these effects using novel bid-level data from Mexico's domestic government bond market.

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1 Introduction

Multi-unit auctions are often held separately and simultaneously for different goods. For example, in the primary market for government debt, independent auctions are held for bonds of different characteristics by the same institution at the same time.

This is inefficient if either the auctioneer or the bidders view the goods as substitutes. Selling goods using *multi-product* multi-unit auctions can increase efficiency. In these auctions, the allocation across goods depends on the submitted bids and the auctioneer's own supply preferences. They are straightforward to implement and have been used in both financial markets and renewable energy procurement and support.[1]

Switching to the efficient multi-product auction naturally increases total surplus relative to running independent auctions. But the effect on the distribution of surplus between bidders and the auctioneer is less clear. Who gains, and who loses? I provide conditions under which the efficient mechanism benefits both sides of the market when participants bid competitively for substitute goods.[2] I also provide conditions under which one side of the market, either the bidders or the auctioneer, disproportionally benefits while the other side's expected surplus falls.

My results are especially simple in the special case of my model in which the goods are symmetric, i.e. the expected inverse demand curves, and auctioneer's marginal cost curves,

[1] In financial markets, a notable use of multi-product auctions is quantitative easing: both the Federal Reserve and the Bank of England have held multi-product auctions, in which differentiated securities are bought or sold at the same time. The Bank of England runs other multi-product auctions, too. Its weekly Indexed Long-Term Repo operations are Product-Mix Auctions, which jointly allocate liquidity to financial institutions against multiple types of collateral. Other examples, which motivate my empirical analysis in Section 7, are from Mexico. Banco de México, the Mexican central bank, conducts swaps of a range of government securities (differing in, e.g., their term, coupons, and whether nominal or real interest rates) within multi-product auctions called "subastas de vasos comunicantes". It also conducts similar swaps, referred to as "permutas de valores gubernamentales", on behalf of the Mexican federal government.

In renewable energy procurement and support, multi-product auctions are increasingly popular to allocate contracts across different technologies (Szabó et al., 2020). Introduced in 2010, the Netherlands' SDE+ programme was one of the first large multi-technology auctions of energy price support in Europe. All technologies except offshore wind compete within one auction, subject to reserve prices specific to each technology. In other cases, only comparable technologies compete. For example, the UK Contracts for Difference scheme divides technologies into groups (in the 4th round, these were "established", e.g. onshore wind and solar; "less established", e.g. advanced conversion technologies; and offshore wind), and auctions are run separately for each group.

[2] The mechanism that I consider (i.e. a Product-Mix Auction) automatically increases efficiency in my model in which each bidder demands just one of the goods, of which she demands a single unit, so the bidder bids her true value. My model applies more generally if there are sufficiently many bidders that competitive bidding is a reasonable assumption in uniform-price auctions. This precludes the oligopolistic effects in multi-market models such as Chen and Duffie (2021), Rostek and Yoon (2021), Wittwer (2021), etc.

are identical across goods, and the shocks to demand are exchangeable. Both the bidders (in aggregate) and the auctioneer gain if the aggregate demand curves are neither extremely convex nor extremely concave. Otherwise, either the bidders or the auctioneer lose. For commonly used distributions, both bidders and the auctioneer gain from switching to the efficient mechanism. A notable exception is that bidders with constant-elasticity aggregate demand lose.

I also estimate these distributional effects in the primary market for Mexican government debt. I use novel bid-level data from auctions of zero-coupon bonds (known as CETES) held by the Mexican central bank to estimate the bidders' and auctioneer's preferences. I find the efficient multi-product auction of 182-day and 364-day bonds increases both bidder and auctioneer surplus relative to running the existing pair of independent auctions, assuming participants bid competitively. The efficient auction of 28-day and 91-day bonds also increases both surpluses. I show how my theoretical model predicts these empirical results.

The closest work to mine is Fabra and Montero (2023). They analyse several mechanisms that allocate multiple units of different goods, assuming linear aggregate bid functions and linear marginal benefits for the auctioneer. They compare running a pair of independent auctions, in which the quantities of the two goods are fixed in advance to maximise expected welfare, with a multi-product auction in which the goods are treated interchangeably.[3] My paper allows arbitrary aggregate bid functions and auctioneer costs of production. Also going beyond Fabra and Montero (2023), I obtain results comparing the effects of the independent auctions to those of the efficient mechanism.[4]

Section 2 describes an auctioneer who wishes to allocate a fixed total quantity across two goods that she views as imperfect substitutes, for example, two bonds with different maturities or two alternative energy sources.[5] The objective is to maximise welfare, as conventionally defined by the sum of total bidder and auctioneer surplus.[6] For each good, there is a large number of risk-neutral bidders with unit demand, who face perfectly correlated shocks to their values.[7] It follows that the efficient auction is a Product-Mix Auction (PMA)

[3]Fabra and Montero (2023) define welfare by a weighted sum of the sum of bidder and auctioneer surplus, and total revenue. I focus on the case in which the weighting on revenue is zero.

[4]Fabra and Montero (2023) illustrate their model's predictions for Spain's 2021 auction of wind and solar energy, which I also discuss in Section 7. I thank Natalia Fabra and Juan-Pablo Montero for kindly sharing their simulation estimates.

[5]Except in the special case in which the goods are symmetric and the production costs are independent across all goods (see Corollary 3), allowing for more than two goods creates additional complexity so is left to future research.

[6]Section 6.1 analyses the case in which the objective is to maximise auctioneer surplus.

[7]With a large number of bidders, idiosyncratic shocks would be unimportant (and in the limit, irrelevant).

as described in Klemperer (2008).[8,9] The PMA finds the competitive equilibrium, assuming bidders submit bids equal to their true values and that the auctioneer expresses her true costs of producing the different goods. In my model, it is indeed rational for all participants to behave "truthfully" in this way, so that the PMA maximises ex post welfare.[10]

The main contribution of this paper is to compare the effects on bidder and auctioneer surpluses of this efficient mechanism to running a pair of separate simultaneous auctions (SSA). In the SSA, the quantity sold of each good is chosen in advance to maximise expected welfare, and so is insensitive to the shocks to bidders' values. All sales mechanisms are uniform-price.[11] The PMA increases total welfare by adjusting its allocation to the shocks, but the gain is not necessarily shared by both sides of the market.

Section 3 shows that we can decompose the difference in expected bidder surplus between the PMA, i.e. the "efficient auction", and the SSA into two effects. The Bidder Uncertainty Effect captures the impact of the randomness of the quantity sold of each good in the efficient auction. With this randomness, more bidders win when the price is lower. So bidders as a group gain if the randomness does not affect the average market-clearing price. This is just a reinterpretation of the classical result that a consumer with downward-sloping demand benefits from a mean-preserving spread in price (Waugh, 1944). Bidders *a fortiori* gain if the randomness reduces the average price. If the average price instead rises, bidders gain only if the benefits of the mean-preserving spread dominate. Specifically, bidders gain (lose) from the randomness if the aggregate bid function is log concave (log convex) (Proposition 1).

The second effect, the Bidder Allocation Effect, captures the impact on expected bidder surplus of changes in the expected allocations of the goods. It is positive (negative) if, from the perspective of the bidders, the efficient auction increases the expected quantity sold of the relatively more (less) profitable good (Proposition 2).

The impact of the efficient auction on the auctioneer is more complex than on the bidders for two reasons. First, the auctioneer's marginal costs of producing the two goods may not

[8] I consider a simpler version of a PMA, in which bidders are not permitted to express their preferences between goods in the form of "paired bids", than Klemperer (2008, 2010, 2018). In other words, I consider a "one-sided" PMA. However, the two mechanisms are equivalent in my setting because the bidders view the two goods as non-substitutable.

[9] In our setting in which bidders are infinitesimal, the PMA is equivalent to the Vickrey auction.

[10] My model also applies if a large number of bidders for each good have identical downward-sloping demand curves, since it is rational for bidders who are small relative to the market to bid truthfully.

[11] In my competitive framework, discriminatory pricing would give identical results in equilibrium providing the bidders observe the shocks to both goods.

be independent so her surpluses from selling the two goods cannot be analysed separately. Second, the marginal winning bidder sets the auction price so, on the margin, bidder surplus is zero but auctioneer surplus may be different from zero.

Section 4 shows that it is therefore helpful to divide the difference in auctioneer surplus between the efficient auction and the SSA into two effects, which distinguish the impact on the price level from the impact on the relative prices. The first effect, the Auctioneer Price Level Effect, is positive (negative) if the efficient auction increases (decreases) the expected price level, as measured by the price of one of the goods. Specifically it is positive (negative) if both the aggregate bid function for that good is convex (concave) and the efficient auction decreases (increases) the expected quantity sold of that good (Proposition 3).

The second effect, the Auctioneer Relative Price Effect, captures the impact of the efficient auction on the relative prices. There are two aspects to this. The first corresponds to the Bidder Uncertainty Effect—the auctioneer gains from the uncertainty that the efficient auction generates in the relative prices. The second reflects the impact of the average difference in relative prices between the efficient auction and the SSA. Overall, the Auctioneer Relative Price Effect is positive (negative) if both the auctioneer's inverse marginal cost of one good relative to another is log concave (log convex) and the efficient auction increases (decreases) the good's expected quantity sold (Proposition 4).

Section 5 considers special cases of the model. In Section 5.1, I focus on the case in which both the bidders' aggregate preferences, and the auctioneer's preferences, are symmetric in the two goods. The distributional effects are particularly clean. If the aggregate bid functions are convex but log concave, the efficient auction increases the expected surplus of both the bidders and the auctioneer. Bidders as a group gain from the uncertainty in the quantities sold in the efficient auction, but this uncertainty also translates into higher expected auctioneer revenue. If the aggregate bid functions are either concave or very convex (i.e. log convex), only one side of the market benefits in expectation. Section 5.2 examines the asymmetric case of the generalised Pareto distribution, special cases of which correspond to the more commonly used functional forms of demand: linear, log linear, and constant elasticity. Section 5.3 analyses the case in which the auctioneer's marginal cost of increasing the quantity produced of one good while reducing the quantity of the other by the same amount is constant.

Section 6 extends the model. When the objective is to maximise auctioneer surplus, the auctioneer of course benefits from the PMA and the conditions for the bidders to gain and to lose are similar to those in the main model (Section 6.1). Section 6.2 shows how the distributional effects may differ with a variable total supply. Section 6.3 analyses an alternative multi-product design, commonly used in quantitative easing.[12] This "reference price auction" (RPA) imposes a fixed price difference between the goods. Because the efficient mechanism is between the extremes of fixing quantities, i.e. an SSA, and fixing relative prices, i.e. an RPA, it is not surprising that the results of the comparison of the efficient auction to the RPA are broadly opposite to that of the SSA.

In Section 7, I estimate the distributional effects using bid-level data from the Mexican primary market for government debt. I group the four bonds that I study into two pairs: 28- and 91-day CETES, and 182- and 364-day CETES. For each pair, I find that the efficient auction increases both average bidder and average auctioneer surplus relative to the existing SSA design. The effects for each pair of bonds can be understood in terms of the model. For the 182- and 364-day pair, the aggregate bid functions are log concave, so the bidders gain from the uncertainty in issuance and the Bidder Uncertainty Effect is positive. This dominates the negative, albeit statistically insignificant, Bidder Allocation Effect. For the 28- and 91-day pair, the positive effect on bidder surplus is driven by a positive Bidder Allocation Effect, which results from the efficient auction increasing the expected allocation of the bond which is relatively more profitable on the margin for the bidders. The Bidder Uncertainty Effect is also positive, but statistically insignificant. For the auctioneer, the Auctioneer Relative Price Effect is zero under the assumptions of our model so the results are purely a function of the Auctioneer Price Level Effect. It is positive for the 28- and 91-day pair because the efficient auction increases the price of the reference good. It is also positive, albeit imprecisely estimated, for the 182- and 364-day bonds.

Section 8 concludes. Proofs of the results are found in Appendix A.

2 Model

An auctioneer allocates a single unit across two divisible goods, $j = \{1, 2\}$. Her total cost of producing q_1 of good 1 and q_2 of good 2 is $C(q_1, q_2)$, which is three-times continuously differentiable. It is convenient to assume her marginal cost of good $j = \{1, 2\}$, $MC_j(q_1, q_2) = \frac{\partial C(q_1,q_2)}{\partial q_j}$, is strictly increasing and that the goods are strictly imperfect substitutes, so that $\frac{\partial^2 C(q_1,q_2)}{\partial q_j \partial q_k} < \frac{\partial^2 C(q_1,q_2)}{\partial q_j^2}, k \neq j$. We do this for convenience to permit invertibility, but the cases in which the marginal cost curves have zero slopes or the goods are perfect substitutes follow

[12] Both the Federal Reserve and Bank of England used this design in their quantitative easing programmes (Bank of England, 2022; Song and Zhu, 2018).

automatically, as shown in Section 5.3.

The auctioneer always sells the entire unit.[13] It is therefore helpful to also define the auctioneer's *relative* marginal cost of producing good 1 (with reference to good 2), $\widetilde{MC}_1(q_1) = \frac{\partial C(q_1, 1-q_1)}{\partial q_1} - \frac{\partial C(q_1, 1-q_1)}{\partial q_2}$. This expresses the net marginal cost of increasing the quantity allocated of good 1 and correspondingly reducing the quantity of good 2, so that one unit is allocated in total. We can define the relative marginal cost of good 2 (with reference to good 1) similarly.

Since marginal costs are strictly increasing, her relative marginal cost is strictly increasing. We can therefore also define a corresponding inverse relative marginal cost curve, $\widetilde{mc}_1(\widetilde{p}_1)$, which specifies the quantity that the auctioneer is willing to sell of good 1 as a function of its relative price (equal to $\widetilde{p}_1 = p_1 - p_2$).[14] I assume $\lim_{\widetilde{p}_1 \to -\infty} \widetilde{mc}_1(\widetilde{p}_1) = 0$ and $\lim_{\widetilde{p}_1 \to \infty} \widetilde{mc}_1(\widetilde{p}_1) = 1$.

Good $j = \{1, 2\}$ is demanded by a measure, greater than one, of infinitesimal, risk-neutral bidders, each with value for an increment dq_j and quasi-linear utility. Bidders view the two goods as non-substitutable; bidders for good 1 have no interest in good 2 and vice versa. Bidders' values are sums of idiosyncratic components, capturing preference heterogeneity among the bidders, and a correlated one, interpreted as a common taste shock to bidders for good j. Their idiosyncratic values are distributed by the twice continuously differentiable and strictly increasing function F_j with density function f_j.[15] The bidders for good j face a shock, θ_j, to their values, which is distributed on $[\underline{\theta}_j, \bar{\theta}_j]$ with zero mean. The vector of shocks is denoted by $\boldsymbol{\theta} = (\theta_1, \theta_2)$.

In auctions with uniform-pricing, a bidder with unit demand has a dominant strategy to submit a bid equal to his value. It follows that bidders in our setting also have this dominant strategy, which I refer to as 'bidding truthfully'. The aggregate bid function for good j — its demand curve — is therefore $d_j(p_j, \theta_j) = (1 - F_j(p_j - \theta_j))$, which is continuous and strictly decreasing in p_j and only depends on the difference between p_j and θ_j. We can also define a corresponding inverse demand curve, $D_j(q_j, \theta_j)$, which is additive in the shock θ_j. I describe $d_j(p_j, \theta_j)$ as log convex (log concave) if it is log convex (log concave) with respect to p_j,

[13] I assume the submitted bids are always sufficiently high relative to the costs of production that the auctioneer never prefers to sell less than the full unit. Section 6.2 allows the total quantity sold to vary.

[14] The inverse relative marginal cost curve for good 2 is similarly defined and can be equivalently expressed by $\widetilde{mc}_2(\widetilde{p}_2) = (1 - \widetilde{mc}_1(-\widetilde{p}_2))$.

[15] With a large number of bidders, idiosyncratic shocks would be unimportant and, in the limit, irrelevant to the analysis: the outcome would be identical and, because bidders are risk-neutral, they collectively value the auction outcome as would a single representative consumer, so the measurement of bidder surplus would be unchanged.

holding θ_j constant, i.e. $\frac{\partial^2 log(d_j(p_j,\theta_j))}{\partial p_j^2} \geq (\leq)\ 0$. Similarly, I describe $D_j(q_j,\theta_j)$ as convex (concave) if $\frac{\partial^2 D_j(q_j,\theta_j)}{\partial q_j^2} \geq (\leq)\ 0$.

Total bidder surplus, denoted $TBS(q_1,\boldsymbol{\theta})$, equals the total gross value to bidders of receiving q_1 of good 1 (and $(1-q_1)$ of good 2), less the total payment to the auctioneer, given the shocks $\boldsymbol{\theta}$. Analogously, total auctioneer surplus, $TAS(q_1,\boldsymbol{\theta})$, equals the total payment received by the auctioneer, minus her total cost of producing the two goods.

The auctioneer can adopt either of two sales mechanisms: a Product-Mix Auction (PMA) or a pair of separate simultaneous auctions (SSA).[16] In this context, the PMA is a multi-product auction in which bidders submit bids for the two goods and the auctioneer expresses her preferences between the goods.[17] It uses the information from the resulting demand and supply curves to find the competitive equilibrium, under the assumption that all bidders bid truthfully and the auctioneer expresses her true costs of producing the goods. Truthful behaviour is indeed rational for all participants as winning bidders of a good pay the highest losing bid for that good. The PMA therefore maximises ex post welfare, defined conventionally by the sum of total bidder surplus and auctioneer surplus. I refer to the PMA as the "efficient auction".

In the SSA, the quantity sold of each good is fixed ex ante and chosen by the auctioneer to maximise expected welfare. Bidders submit the same bids (for each good) as they submit in the efficient auction, because it is optimal to bid truthfully and bidders view the two goods as non-substitutable. The two auction prices equal the highest losing bids on the corresponding goods.

The quantity sold of good 1 under mechanism $m \in \{S,E\}$, where $m=S$ refers to the SSA and $m=E$ to the efficient auction, is q_1^m (and $q_2^m = 1-q_1^m$), where q_1^S is deterministic and q_1^E is a function of $\boldsymbol{\theta}$, and $p_j(q_j^m,\theta_j) = D_j(q_j^m,\theta_j)$ is good j's auction price. The efficient auction finds the quantities such that the relative value of the goods to the marginal winning bidders equals the goods' relative marginal cost, i.e. $D_1(q_1^E(\boldsymbol{\theta}),\theta_1) - D_2(1-q_1^E(\boldsymbol{\theta}),\theta_2) = \widetilde{MC}_1(q_1^E(\boldsymbol{\theta}))$.

[16] I consider an alternative design, in which the price difference between goods is fixed in advance, in Section 6.3.

[17] This is a "one-sided" PMA because only the auctioneer expresses preferences between goods. In the general PMA, both the bidders and the auctioneer are permitted to do so, with bidders submitting "paired bids" (see Klemperer, 2008, 2010, 2018). Because bidders view the two goods as non-substitutable, the one-sided PMA and general PMA are equivalent in my setting.

The fixed quantities in the SSA satisfy this in expectation, $\mathbb{E}\left[D_1(q_1^S,\theta_1) - D_2(1-q_1^S,\theta_2)\right] = \widetilde{MC}_1(q_1^S)$.[18]

The relative price of good 1 is $\widetilde{p}_1(q_1^m,\boldsymbol{\theta}) = p_1(q_1^m,\theta_1) - p_2(1-q_1^m,\theta_2)$. Let $\underline{q}_1 = q_1^E((\underline{\theta}_1,\bar{\theta}_2))$ and $\bar{q}_1 = q_1^E((\bar{\theta}_1,\underline{\theta}_2))$ denote the minimum and maximum quantities sold of good 1 in the efficient auction, and $(\underline{q}_2,\bar{q}_2) = (1-\bar{q}_1, 1-\underline{q}_1)$. I assume that these are interior solutions.

I label the goods so that the efficient auction increases the expected quantity sold of good 1, that is, $\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right] \geq q_1^S$. A sufficient condition for this labelling is that inverse demand for good 1 relative to good 2 is weakly more convex than the relative marginal cost of good 1 (see Appendix B).

3 Bidder surplus

The expected surplus of a winning bidder of good $j = \{1,2\}$ is the difference between his expected value and the good's expected auction price, conditional on him winning. His probability of winning and the conditional expected price differ between the two sales mechanism and this difference depends on the distribution of shocks and the bidder's own value. In general, some bidders gain and some lose from the efficient auction relative to the SSA,[19] so I sum the effects across bidders.[20]

Measuring (total) bidder surplus as a function of quantities, rather than prices as is conventional, is helpful.[21] Bidder surplus for good $j \in \{1,2\}$ is the sum of the values of winning bidders, less their payment to the auctioneer. The difference in the value of the winners is simply the area under the inverse demand curve between the quantities allocated in the efficient auction and SSA. That is, it equals the sum of Areas L_j, M_j, and N_j in Figure 1 for shocks $\boldsymbol{\theta}$.

Similarly, the difference in auctioneer revenue is measured by the area under the marginal

[18] The SSA allocation maximises expected welfare and so solves

$$\max_{q_1^S} \mathbb{E}\left[\int_0^{q_1^S} D_1(q_1,\theta_1)dq_1 + \int_0^{1-q_1^S} D_2(q_2,\theta_2)dq_2 - \int_0^{q_1^S} \widetilde{MC}_1(q_1)dq_1\right]$$

[19] For example, a bidder with a low value always loses in the SSA but wins with positive probability in the efficient auction, and so unambiguously gains. In contrast, a bidder with a high value always wins in the SSA but sometimes loses in the efficient auction. The efficient auction also changes the expected price he pays, conditional on winning. If the demand curve is concave, he gains. If the demand curve is convex, the net effect on his expected surplus depends on the relative importance of these two opposing effects.

[20] This is of course equivalent to the average effect on expected individual bidder surplus.

[21] Bulow and Klemperer (2012) demonstrate the value of measuring surplus in this way.

revenue curve for good j, $MR_j(q_j, \theta_j) = \frac{\partial(D_j(q_j,\theta_j)q_j)}{\partial q_j}$, because this curve describes the change in total payment for good j due to a marginal increase in its quantity sold. This is equal to Area N_j and is equivalent to the difference in the product of price and quantity sold, i.e. (Area M_j + Area N_j) − (Area J_j + Area K_j).

The difference in total bidder surplus for good $j \in \{1, 2\}$ is therefore the area between the inverse demand curve and the marginal revenue curve between the quantities sold in the efficient auction and SSA. For shocks $\boldsymbol{\theta}$, it is the sum of Areas L_j and M_j.

Another way to understand this is that the marginal winning bidder is indifferent to winning because his bid sets the auction price. So bidder surplus on the margin is simply the change in total payment of the inframarginal winners, i.e. $D_j(q_j, \theta_j) - MR_j(q_j, \theta_j) = -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j$. For shocks $\boldsymbol{\theta}$, the difference in total bidder surplus for good $j \in \{1, 2\}$ is therefore

$$\int_{q_j^S}^{q_j^E(\boldsymbol{\theta})} \left(D_j(q_j, \theta_j) - MR_j(q_j, \theta_j)\right)\, dq_j = \int_{q_j^S}^{q_j^E(\boldsymbol{\theta})} -\frac{\partial D_j(q_j, \theta_j)}{\partial q_j} q_j \; dq_j.$$

I decompose the difference in expected total bidder surplus between the efficient auction and SSA into two effects. Let

$$\text{Bidder Uncertainty Effect} = \mathbb{E}\left[\sum_{j=\{1,2\}} \left(\int_{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]}^{q_j^E(\boldsymbol{\theta})} -\frac{\partial D_j(q_j, \theta_j)}{\partial q_j} q_j \; dq_j\right)\right]$$

and

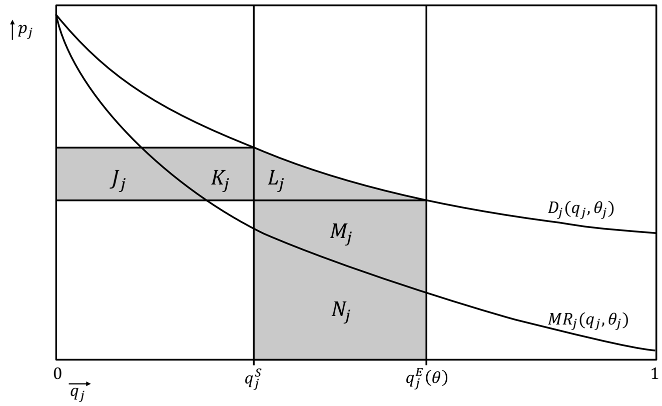
$$\text{Bidder Allocation Effect} = \mathbb{E}\left[\sum_{j=\{1,2\}} \left(\int_{q_j^S}^{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]} -\frac{\partial D_j(q_j, \theta_j)}{\partial q_j} q_j \; dq_j\right)\right]$$

The following lemma immediately follows.

Lemma 1 *The difference in expected total bidder surplus between the efficient auction and SSA,* $\mathbb{E}\left[TBS(q_1^E(\boldsymbol{\theta}), \boldsymbol{\theta}) - TBS(q_1^S, \boldsymbol{\theta})\right]$*, equals the sum of the Bidder Uncertainty Effect and the Bidder Allocation Effect.*

These effects each relate to properties of the bidders' demand curves and the auctioneer's costs of production.

Figure 1: Change in total bidder surplus for good $j \in \{1, 2\}$: two equivalent measurements



Note: The difference in total bidder surplus between selling $q_j^E(\boldsymbol{\theta})$ and q_j^S is measured by the sum of Areas L_j and M_j. This is equivalent to the conventional measure of the change in bidder surplus, which is the sum of Areas J_j, K_j, and L_j. This is evident by comparing two equivalent measures of total payment. The conventional measure of the change in total payment is (Area M_j + Area N_j) less (Area J_j + Area K_j). This is equivalent to the area under the marginal revenue curve, i.e. Area N_j. Hence, Area M_j = (Area J_j + Area K_j).

3.1 Bidder Uncertainty Effect

The Bidder Uncertainty Effect captures the benefit to bidders of uncertainty in the quantities sold of the two goods in the efficient auction.

A marginal increase in the quantity sold of good $j = \{1, 2\}$ reduces the auction price for all winning bidders, and causes an additional bidder to win when he would otherwise lose. Only the first effect is relevant to bidder surplus, as the marginal winner is indifferent between winning and losing, paying a price precisely equal to his value $D_j(q_j, \theta_j)$.

In the simple case of linear demand, bidders in aggregate benefit from uncertainty, just as a consumer with downward-sloping demand benefits from a mean-preserving spread in price in Waugh's (1944) classical result. This is because equally sized changes in the quantity sold of a good cause equally sized changes in its auction price, and more bidders win when the auction price is low. With non-linear demand, the changes in auction price are asymmetric. Uncertainty unambiguously benefits the bidders if the demand curve is concave: when the quantity sold increases, the larger number of winners benefit from a greater reduction in price. But if the demand curve is sufficiently convex, the relatively steep price rise for the smaller number of winners, when the quantity sold declines, dominates the price fall from the increase in quantity. So bidders as a group lose out from the uncertainty.

Specifically, the Bidder Uncertainty Effect is positive if bidder surplus for good $j = \{1, 2\}$, $\int_0^q -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j \; dq_j$, is a convex function of the quantity sold, q. In this case, expected bidder surplus for good j in the efficient auction, i.e. $\mathbb{E}\left[\int_0^{q_j^E(\theta)} -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j \; dq_j\right]$ is greater than bidder surplus evaluated at the expected quantities sold, i.e. $\int_0^{\mathbb{E}\left[q_j^E(\theta)\right]} -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j \; dq_j$ (by Jensen's inequality), so it is immediate from its definition that the Bidder Uncertainty Effect is positive. Simple manipulation of $\int_0^q -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j \; dq_j$ shows that it is a convex function of q if and only if the demand curve is log concave.[22] So:

Proposition 1 *The Bidder Uncertainty Effect is positive (negative) if both demand curves are log concave (log convex).*[23]

A large number of well-known distributions have a log concave density function, f_j, which implies a log concave demand curve, $d_j(p_j, \theta_j)$.[24] Examples include uniform (corresponding to linear demand), normal, logistic, extreme value, and exponential (corresponding to log linear demand).[25] Indeed, there are many arguments for why log concave demand might be a natural assumption. For example, Weyl and Fabinger (2013) argue that, if bidders' willingness to pay for the good were proportional to their income, then log concave demand fits the income distributions commonly observed in many countries.[26] A notable exception to the set of distributions with log concave density is Pareto (corresponding to constant-elasticity demand).

There are two additional ways to understand this result. First, my model also applies to a situation in which all bidders for good j have identical downward-sloping demand curves, as described by the demand curve, $d_j(p_j, \theta_j)$, because it is also rational for bidders in this case

[22]This can also be understood in terms of the fatness of the right tail of the distribution of bidders' values. With a fat tail (corresponding to a log convex demand curve), few bidders value the good extremely highly. If only a small quantity is sold, it will be sold only to these bidders and the auction price—corresponding to the marginal winner's value—will be very high. As more bidders win when larger quantities are sold, the price will decline only gradually as the values of the marginal winners become more similar.

[23]Naturally, it is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j, \theta_j) : d_j(p_j, \theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j = \{1, 2\}$. The effect is strictly positive (negative) if also at least one demand curve is strictly log concave (log convex).

[24]The survival function (in this context, the demand curve) of a continuously differentiable log concave density function is also log concave, but the converse does not necessarily hold for log convex densities (Bagnoli and Bernstrom, 2005).

[25]See Bagnoli and Bernstrom (2005) for an extensive list.

[26]On the other hand, Fabinger and Weyl (2012) suggest that log convex demand might be a reasonable characterisation if bidders have heterogeneous valuations for a good but resale is possible. The ability to resell the good places a floor on bidders' willingness to pay and heterogeneous valuations mean that the demand curve is downward-sloping above this floor.

to behave truthfully when there are sufficiently many of them. In this view, each bidder's surplus is measured by the area between the inverse demand curve and the marginal revenue curve. If a bidder is risk-loving (risk-averse) with respect to quantity, that is, he prefers (dislikes) uncertainty in the quantity sold, his demand curve is log concave (log convex).[27]

Second, the randomness in the quantities sold in the efficient auction is akin to the rationing created by price controls in Bulow and Klemperer (2012). In my context, for any given quantity sold, the price of each good reacts one-for-one to the common shock to bidders' values for that good, because it is equal to the marginal winner's value for that good. And so the bidder surplus does not depend directly on the shocks, and depends only on the winners' idiosyncratic values (i.e. values less the shock). The efficient auction creates uncertainty in the quantities sold, so it is in effect rationing bidders with medium idiosyncratic values. The reason is that the lower part of these medium-value bidders would always lose if the quantities sold were the expected quantities sold in the efficient auction but do sometimes win in the efficient auction, while the higher part of the medium-value bidders would always win if the quantities sold were the expected quantities but do sometimes lose. Bulow and Klemperer (2012) show that if the demand curves are log concave (log convex), bidders gain (lose) in aggregate from rationing if there is no reduction in supply. So it follows that, in our context, bidders gain (lose) from the uncertainty in the quantities sold generated by the efficient auction.

3.2 Bidder Allocation Effect

The Bidder Allocation Effect represents the impact on expected total bidder surplus of the difference between sales mechanisms in the expected quantity sold of each good. The efficient auction increases the expected quantity sold of good 1, so the Bidder Allocation Effect is positive (negative) if good 1 is relatively more (less) profitable on the margin for bidders.

The marginal winning bidder for each good is indifferent between winning and losing, so that only the change in surplus for inframarginal winners is relevant. Good 1 is therefore relatively more profitable if an increase in its quantity sold (and corresponding reduction in the quantity sold of good 2) reduces the average price paid by the inframarginal winners across the two goods.

[27] The coefficient of relative risk aversion for an agent with random wealth w and expected utility $\mathbb{E}\left[u(w)\right]$ is $\frac{-wu''(w)}{u'(w)}$. If the representative bidder's expected utility is the area between the demand curve and the marginal revenue curve, then the coefficient of relative risk aversion for the bidder with random allocation q_j is $r(q_j) = \frac{-q_j}{(D_j(q_j,\theta_j)-MR_j(q_j,\theta_j))}\frac{\partial(D_j(q_j,\theta_j)-MR_j(q_j,\theta_j))}{\partial q_j}$. If the demand curve is log concave (log convex), $r(q_j) \leq 0$ ($r(q_j) \geq 0$) (see proof of Proposition 1).

Sufficient conditions for this to hold for a marginal change at the SSA allocation are that good 1's expected auction price is higher than good 2's (i.e. the marginal cost of good 1 is higher than that of good 2), and that demand for good 1 is less price elastic.[28] The reason is that a marginal increase in the quantity sold of good 1 causes a larger quantity-weighted change in price in this case. If the demand curves are also log concave, bidder surplus for each good is a convex function of the quantity sold (see Section 3.1) so larger changes in the allocation also increase expected bidder surplus. So:

Proposition 2 *The Bidder Allocation Effect is positive (negative) if both demand curves are log concave (log convex),*[29] *good 1's expected price elasticity of demand is smaller (larger) in absolute value than good 2's and the relative marginal cost of good 1 is positive (negative) at the quantities sold in the SSA.*[30]

We will see that Proposition 2 helps us understand why the estimated Bidder Allocation Effect is positive in the case of combining the auctions for 28-day and 91-day Mexican Treasury bonds (see Section 7).

Many frequently used demand curves are log concave (see Section 3.1). Moreover, if the demand curves are log concave, then if one of the other two conditions required for the Bidder Allocation Effect to be positive holds, it is more likely that the other holds. This is because if each demand curve is log concave, the absolute value of its price elasticity is decreasing in its quantity. And so, if good 1 is less price elastic than good 2 at the SSA allocation, the quantity sold of good 1 in the SSA is typically relatively large. In this case, the relative marginal cost of good 1 is typically positive, because it is increasing in the quantity sold of good 1. On the other hand, it seems less likely that the two conditions required for the Bidder Allocation Effect to be negative when demand is log convex would both hold.

Propositions 1 and 2 provide conditions under which we can determine the effect of switching to the efficient auction on expected bidder surplus.

Corollary 1 *The efficient auction increases (decreases) expected bidder surplus relative to the SSA if both demand curves are log concave (log convex), good 1's expected price elasticity*

[28]It is of course also sufficient that both a larger quantity is sold of good 1 than of good 2 and the inverse demand curve for good 1 is steeper.

[29]As for Proposition 1, it is sufficient that this condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j, \theta_j) : d_j(p_j, \theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j = \{1, 2\}$.

[30]The inequality is strict if the efficient auction strictly increases the expected quantity sold of good 1 and one of the three stated conditions holds strictly.

of demand is smaller (larger) in absolute value than good 2's, and the relative marginal cost of good 1 is positive (negative) at the quantities sold in the SSA.

3.3 Market-specific bidder surplus

The impact of the efficient auction on the expected surplus of bidders for good $j \in \{1, 2\}$ can be decomposed into good-specific Bidder Uncertainty and Allocation Effects. By the labelling of the two goods, the good-specific Bidder Allocation Effect is positive (negative) for good 1 (good 2), and the conditions for the good-specific Bidder Uncertainty Effect are a natural corollary to Proposition 1 (see Appendix C).

4 Auctioneer surplus

Auctioneer surplus is the difference between the total revenue raised in the auction and the auctioneer's total cost of producing the goods. The efficient auction increases expected surplus for at least one side of the market. If it reduces expected bidder surplus—as determined by the conditions in Section 3—it therefore increases expected auctioneer surplus. In this section, I provide additional conditions which determine the efficient auction's impact on expected auctioneer surplus. (Analogously, the efficient auction must increase expected bidder surplus if it reduces expected auctioneer surplus.)

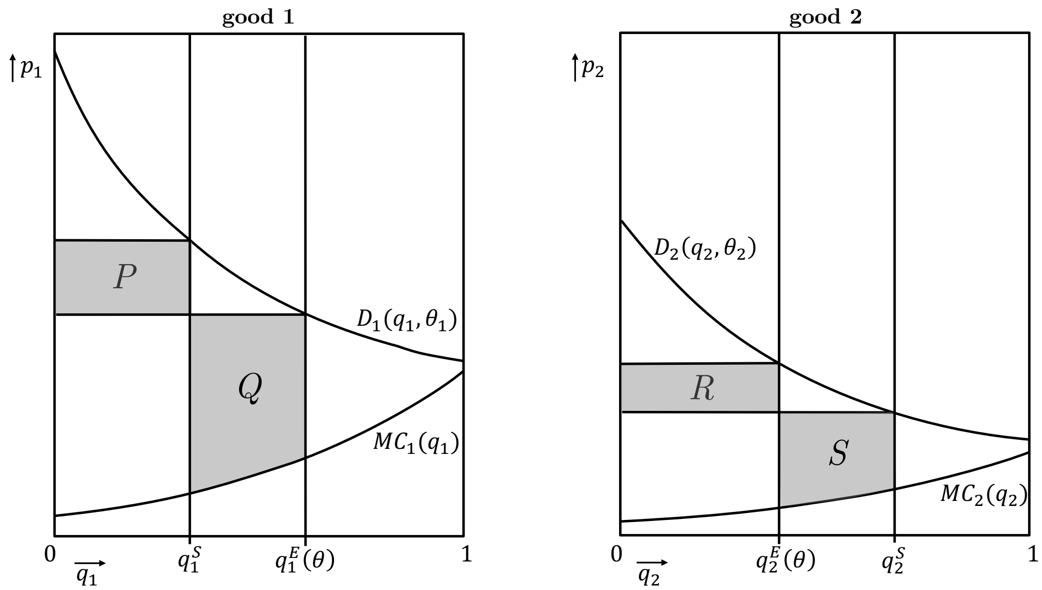
Measuring the impact of the efficient auction on the auctioneer is more complicated than for the bidders for two reasons. First, the auctioneer's marginal cost of each good may be strictly below its auction price because the total supply is fixed. This would create strictly positive surplus on the margin for the auctioneer, whereas the marginal winning bidder is always indifferent between winning and losing, paying an auction price equal to his own bid.

This is relevant even when the auctioneer has independent marginal costs of production, in which case the difference in auctioneer surplus between the efficient auction and SSA can be measured conventionally by considering the two goods separately. This is illustrated for a realisation of shocks for which $q_1^E(\boldsymbol{\theta}) > q_1^S$ (and correspondingly $q_2^E(\boldsymbol{\theta}) < q_2^S$), in Figure 2 by (Area Q − Area P + Area R − Area S).

Second, with more general cost functions, the two goods cannot be analysed separately, because the marginal cost of producing one good is dependent on the quantity produced of the other.[31]

[31] In Section 6.2, I show that auctioneer surplus can be analysed identically to bidder surplus in the case that there are both independent marginal costs (so that the two goods can be analysed separately) and variable total supply (so that the auctioneer's marginal cost for each good equals its auction price).

Figure 2: Conventional measure of auctioneer surplus with independent marginal costs



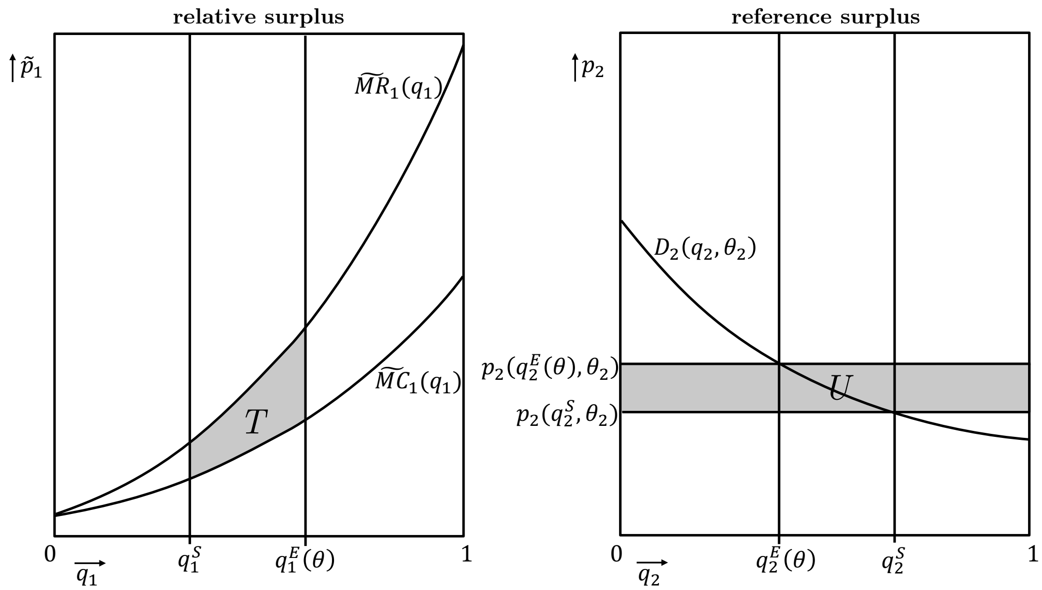
Note: If the auctioneer's costs of producing the two goods are independent, the difference in auctioneer surplus between selling $q_1^E(\boldsymbol{\theta})$ units of good 1 (and correspondingly $q_2^E(\boldsymbol{\theta})$ units of good 2) and selling q_1^S units of good 1 (and correspondingly q_2^S units of good 2), given the realisation of shocks $\boldsymbol{\theta}$, is (Area Q − Area P + Area R − Area S).

To address both of these issues, it is therefore helpful to decompose auctioneer surplus differently from bidder surplus in a way which isolates changes in the price level from changes in the relative price structure of the goods. The auctioneer's total revenue, $p_1(q_1,\theta_1)q_1 + p_2(1-q_1,\theta_2)(1-q_1)$, consists of the revenue that would be generated if she sold the entire unit of a "reference good" at its auction price (and none of the other good) and the additional revenue resulting from a part of the unit being reallocated to the other good. Measured with reference to good 2, the "reference revenue" is $p_2(1-q_1,\theta_2)$ and the "relative revenue" is $\widetilde{p}_1(q_1,\boldsymbol{\theta})q_1$.

We can correspondingly decompose total cost, which is measured by the sum of the auctioneer's "reference cost" of producing the entire unit of good 2, $C(0,1)$, and her "relative cost" of good 1, the area under her relative marginal cost curve, $\int_0^{q_1} \widetilde{MC}_1(q)dq$.

First, the auctioneer's "reference surplus"—the difference between her reference revenue and reference cost—is measured by $(p_2(1-q_1,\theta_2) - C(0,1))$. Given shocks $\boldsymbol{\theta}$ for which $q_1^E(\boldsymbol{\theta}) > q_1^S$, the difference in reference surplus between the efficient auction and SSA is measured by Area U in Figure 3.

Figure 3: Auctioneer surplus measured with reference to good 2



Note: The Auctioneer Relative Price Effect with reference to good 2 is the expected difference in relative surplus between the efficient auction and the SSA, that is, the expected size of Area T. The Auctioneer Price Level Effect with reference to good 2 is the expected difference in reference surplus between the efficient auction and the SSA, that is, the expected size of Area U.

Second, as for bidder surplus, the auctioneer's "relative surplus"—the difference between her relative revenue and relative cost—can be expressed as a function of quantities. The auctioneer's relative cost is measured by the area under the relative marginal cost curve and relative revenue is measured accordingly. That is, the auctioneer's relative marginal revenue curve is given by $\widetilde{MR}_1(q_1) = \frac{\partial(\widetilde{MC}_1(q_1)q_1)}{\partial q_1}$, which is the change in relative revenue due to a marginal increase in the quantity of good 1 (and corresponding reduction in good 2) if the relative price is set equal to the relative marginal cost. The auctioneer's relative revenue is the area under this curve, up to the quantity allocated of q_1, plus an adjustment, $(\widetilde{p}_1(q_1, \boldsymbol{\theta}) - \widetilde{MC}_1(q_1))q_1$, which reflects the difference between the relative price and the auctioneer's relative marginal cost.

And so her relative surplus is equal to

$$\widetilde{p}_1(q_1, \boldsymbol{\theta})q_1 - \int_0^{q_1} \widetilde{MC}_1(q)dq = \int_0^{q_1} \left(\widetilde{MR}_1(q) - \widetilde{MC}_1(q)\right) dq_1 + \left[\widetilde{p}_1(q_1, \boldsymbol{\theta}) - \widetilde{MC}_1(q_1)\right] q_1$$

Given shocks $\boldsymbol{\theta}$ for which $q_1^E(\boldsymbol{\theta}) > q_1^S$, the difference in relative surplus between the efficient auction and SSA is measured by the sum of Area T in Figure 3 and the adjustment $(\theta_2 - \theta_1)q_1^S$.

The adjustment is zero in the efficient auction and zero in expectation in the SSA, and so is irrelevant to the difference in expected surplus between sales mechanisms.

Equivalently, we can decompose total revenue and total cost with reference to good 1 to define the reference and relative surpluses accordingly.[32]

I use these measures to decompose the difference in expected total auctioneer surplus between the efficient auction and SSA into two effects, the Auctioneer Price Level Effect and the Auctioneer Relative Price Effect, which respectively measure the auctioneer's reference surplus and relative surplus with reference to good $j = \{1, 2\}$:

$$\begin{aligned} \text{(Auctioneer Price Level Effect)}_j &= \mathbb{E}\left[p_j(q_j^E(\boldsymbol{\theta}), \theta_j) - p_j(q_j^S, \theta_j)\right] \\ \text{and (Auctioneer Relative Price Effect)}_j &= \mathbb{E}\left[\int_{q_k^S}^{q_k^E(\boldsymbol{\theta})} \left(\widetilde{MR}_k(q_k) - \widetilde{MC}_k(q_k)\right) dq_k\right] \end{aligned}$$

where $k \neq j, k \in \{1, 2\}$.

Lemma 2 *The difference in expected auctioneer surplus between the efficient auction and SSA,* $\mathbb{E}\left[TAS(q_1^E(\boldsymbol{\theta}), \boldsymbol{\theta}) - TAS(q_1^S, \boldsymbol{\theta})\right]$*, equals the sum of the Auctioneer Price Level Effect and the Auctioneer Relative Price Effect, where the decomposition is made with reference to either good.*

These effects each relate to properties of the bidders' demand curves and the auctioneer's costs of production.

4.1 Auctioneer Price Level Effect

The Auctioneer Price Level Effect captures the impact of the efficient auction on expected auctioneer surplus through its impact on the price level, measured by the auction price of the reference good. Both the uncertainty in the quantities sold in the efficient auction and the difference in the expected quantities sold between mechanisms, are relevant.

If the demand curve of the reference good is convex (concave), uncertainty in the quantities sold increases (reduces) its expected auction price, increasing (reducing) expected revenue for the auctioneer and therefore expected auctioneer surplus.

[32] See Appendix D for the relationship between the areas in Figures 2 and 3, and for an illustration of the relative surplus and reference surplus with reference to good 1.

Moreover, by the labelling of the two goods, the expected quantity sold of good 2 (good 1) is smaller (larger) in the efficient auction than the SSA, increasing (reducing) the reference good's expected auction price, i.e. the price level, when measured with reference to good 2 (good 1).

Proposition 3 *The Auctioneer Price Level Effect with reference to good 2 (good 1) is positive (negative) if the demand for good 2 (good 1) is convex (concave).*[33]

4.2 Auctioneer Relative Price Effect

The Auctioneer Relative Price Effect captures the impact of the variation in the relative price structure caused by the efficient auction on the auctioneer's expected surplus.[34]

The efficient auction chooses an allocation at which, given the shocks, the relative price of good 1 equals its relative marginal cost. In the same way that the marginal winning bidder is indifferent between winning and losing, the auctioneer is indifferent on the margin between additional units of the two goods. Moreover, the SSA maximises expected welfare by setting the expected relative price equal to the relative marginal cost. The deviations of the relative price from relative marginal cost in the SSA therefore net out in expectation – the impact of the efficient auction can be measured as if the relative prices in the SSA and the efficient auction were set equal to the relative marginal cost of the quantities sold in the SSA and the relative marginal cost of the quantities sold in the efficient auction, respectively.

Because the efficient auction sets a relative price for good 1 equal to its relative marginal cost, which is an increasing function, more of good 1 is sold in the efficient auction when its relative price is higher. There are two implications of this.

First, uncertainty in the relative price has an effect on the auctioneer which is analogous to the Bidder Uncertainty Effect, with the auctioneer's inverse relative marginal cost curve, $\widetilde{mc}_1(\widetilde{p}_1)$, corresponding to the bidders' demand curve, $d_1(p_1)$. That is, the auctioneer benefits from a mean-preserving spread in the relative price, and therefore also benefits if the randomness *increases* the average relative price, i.e. if the inverse relative marginal cost

[33] Naturally, it is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j, \theta_j) : d_j(p_j, \theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j \in \{1, 2\}$.
The inequality is strict if the efficient auction strictly increases the expected quantity sold of good 1 relative to the SSA and the condition on the convexity of demand holds strictly.

[34] The relative prices of the goods also vary in the SSA, as the price of each good reacts one-for-one to the shock to demand for that good. Because the quantities sold in the SSA are fixed, this variation has no effect on expected auctioneer surplus.

curve is concave. It is only if the inverse relative marginal cost curve is sufficiently convex (specifically, if it is log convex) that the uncertainty causes the average relative price to fall so far that the auctioneer loses from the uncertainty in the relative price.

The second implication is that *ceteris paribus* the auctioneer prefers a higher average relative price. So the larger (smaller) expected quantity sold of good 1 (good 2) in the efficient auction relative to the SSA is associated with a higher (lower) relative marginal cost of good 1 (good 2), increasing (reducing) the auctioneer's relative surplus with reference to good 1 (good 2).

Proposition 4 *The Auctioneer Relative Price Effect with reference to good 2 (good 1) is positive (negative) if the inverse relative marginal cost curve of good 1 (good 2) is log concave (log convex).*[35]

A simple example of a positive Auctioneer Relative Price Effect with reference to good 2 is when the auctioneer's marginal costs are independent, and the slopes of the two inverse marginal cost curves are log concave.[36] As discussed in Section 3.1, many commonly used distributions have log concave densities, including uniform, normal, logistic, extreme value, and exponential, which correspond to log concave slopes if the inverse marginal cost curves are viewed as distribution functions.

Propositions 3 and 4 provide conditions under which we can determine the effect of switching to the efficient auction on expected auctioneer surplus.

Corollary 2 *The efficient auction increases (decreases) expected auctioneer surplus relative to the SSA if the inverse relative marginal cost curve of good 1 (good 2) is log concave (log convex) and the demand for good 2 (good 1) is convex (concave).*

5 Special cases

I first show that the results are particularly clean in the case in which the two goods are symmetric: the impact of the efficient auction on expected bidder and auctioneer surpluses

[35] The effect is strictly positive (negative) if the relative marginal cost curve is strictly log concave (log convex).

It is of course sufficient that the respective conditions hold only for the range of feasible quantities sold, that is $\forall q_1 \in [\underline{q}_1, \bar{q}_1]$ where $q_2 = 1 - q_1$.

[36] With this property, the auctioneer's supply of each good becomes more elastic as the quantity increases. So uncertainty causes the relative price of good 1 to rise disproportionately at larger quantities sold of good 1. Moreover, the larger expected quantity sold of good 1 in the efficient auction increases the average relative price of good 1. So both implications of the fact that the relative price equals the relative marginal cost of good 1 in the efficient auction are positive. See Appendix E for details.

depends on whether the demand curves are either concave (and therefore log concave), log concave but convex, or log convex (and therefore convex). I then show that these results also largely extend to asymmetric goods when the demand curves have standard functional forms. Finally, I discuss the case in which the auctioneer's marginal cost of one good relative to another is constant. This case will be important for our empirical analysis in Section 7.

5.1 The symmetric case

Suppose the expected inverse demand curves are identical, i.e. $\mathbb{E}\left[D_1(q,\theta_1)\right] = \mathbb{E}\left[D_2(q,\theta_2)\right]$ $\forall q = [0,1]$; the shocks are exchangeable, i.e. the distribution of (θ_1,θ_2) is the same as the distribution of (θ_2,θ_1);[37] and the auctioneer's marginal cost curves are identical, i.e. $\frac{\partial C(x,y)}{\partial q_1} = \frac{\partial C(y,x)}{\partial q_2}$ $\forall x,y \in [0,1]$.

This model, which I refer to as the symmetric case, is instructive. We can split the space of possible demand curves into those for which the bidders and auctioneer share the welfare gain of the efficient auction (that is, both expected bidder and auctioneer surplus increase), and the case in which only one side of the market benefits in expectation.

The Bidder Allocation Effect is zero because the expected quantities sold of the two goods are equal both in the SSA and in the efficient auction in the symmetric case (i.e. $q_1^S = \mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right] = \frac{1}{2}$). So the impact of the efficient auction on expected bidder surplus only depends on the benefit to the bidders of uncertainty. It follows immediately from Proposition 1 that if the demand curves are log concave (log convex), the Bidder Uncertainty Effect is positive (negative) so that the efficient auction increases (decreases) expected bidder surplus.

The effects for the auctioneer are also particularly clean because we can show that the auctioneer unambiguously benefits from the impact of the efficient auction on the relative prices. Positive and negative deviations of the quantity sold of an arbitrary good from its expected value impact the good's relative price symmetrically. The efficient auction chooses an allocation for which the relative price is equal to the auctioneer's relative marginal cost, given the shocks. Since relative marginal cost is an increasing function, the auctioneer sells more of the good when its relative price is higher and *ceteris paribus* prefers selling the good at a higher relative price. So the Auctioneer Relative Price Effect is positive.[38]

[37] A special case of this is that the shocks are independent and identically distributed.

[38] This does not follow from Proposition 4. In this case, the inverse relative marginal cost curve may be neither log convex nor log concave across the entire range of feasible quantities sold (and, given the symmetry, it cannot be log convex across the entire range).

Table 1: Impact of the efficient auction relative to the SSA in the symmetric case

	Demand curves (bidder preferences)		
	Log concave & concave	Log concave & convex	Log convex
Expected bidder surplus	Increases	Increases	Decreases
Expected auctioneer surplus	Depends on degree of concavity*	Increases	Increases

*The efficient auction increases expected auctioneer surplus relative to the SSA if the demand curves are not too concave, and otherwise reduces it.

The sign of the Auctioneer Price Level Effect is determined by the conditions of Proposition 3. If the demand curves are convex, the efficient auction therefore unambiguously increases expected auctioneer surplus. But if the demand curves are concave, the auctioneer loses from the efficient auction's impact on the price level but gains from its effect on relative prices. The efficient auction therefore increases expected auctioneer surplus relative to the SSA if the demand curves are either convex or not too concave, and otherwise reduces it.

Proposition 5, summarised in Table 1, partitions the space of demand curves into the sub-cases in which either only one side of the market benefits in expectation or the bidders and auctioneer share the welfare gain of the efficient auction.

Proposition 5 *In the symmetric case,*

(i) the efficient auction increases (decreases) expected bidder surplus relative to the SSA if both demand curves are log concave (log convex).

(ii) the efficient auction increases expected auctioneer surplus relative to the SSA if the demand curves are convex.

In the case in which the auctioneer's marginal cost of production is independent of the quantities produced of the other goods, $\frac{\partial^2 C(q_1,\ldots,q_J)}{\partial q_j \partial q_k} = 0 \; \forall j,k \in J, j \neq k$, this result extends to $J > 2$ goods.

Corollary 3 *In the symmetric case in which* $\frac{\partial^2 C(q_1,\ldots,q_J)}{\partial q_j \partial q_k} = 0 \; \forall j,k \in J, j \neq k$*,*

(i) the efficient auction increases (decreases) expected bidder surplus relative to the SSA if the demand curves are log concave (log convex).

(ii) the efficient auction increases expected auctioneer surplus relative to the SSA if the demand curves are convex.

Of course, these results continue to hold if the goods are close to symmetric. In the case of the Mexican Treasury auctions analysed in Section 7, each pair of bonds is close to symmetric so the Bidder Allocation Effect is small. The Bidder Uncertainty Effect therefore explains the PMA's positive impact on expected bidder surplus, and the Auctioneer Price Level Effect explains the PMA's positive impact on expected auctioneer surplus in the Mexican case.

5.2 Standard functional forms

I illustrate the results for three commonly used demand curves—linear, log linear and constant elasticity—and more generally for demand curves that are derived from the generalised Pareto distribution (GPD). The results suggest that the classification developed in Table 1 for the symmetric case is likely to be a good approximation in asymmetric cases.

Linear If the demand curves for goods 1 and 2 are linear, the Bidder Uncertainty Effect is positive and the Auctioneer Price Level Effect (with reference to either good) is zero. If the auctioneer's relative marginal cost curve with reference to good 2 is also linear, the Auctioneer Relative Price Effect with reference to good 2 is positive and, because the expected quantity sold of each good is the same in the two sales mechanisms, the Bidder Allocation Effect is zero. So in this case the efficient auction increases both expected bidder and auctioneer surplus relative to the SSA, as we anticipate from the symmetric case because the demand curves are log concave (so the bidders gain) but not strictly concave (so the auctioneer also unambiguously gains). The impacts of the PMA on expected bidder surplus and expected auctioneer surplus are proportional to the variance of the difference of the demand shocks, i.e. $Var(\theta_1 - \theta_2)$.[39]

Log linear With log linear demand, i.e. $D_j(q_j, \theta_j) = \mu_j - \sigma_j log\,(q_j) + \theta_j$, where $\sigma_j > 0$, for $j = \{1, 2\}$, uncertainty in the quantities sold has no impact on expected bidder surplus, so the Bidder Uncertainty Effect is zero. The impact of an increase in good j on the payment of inframarginal winners for good j is constant in its quantity sold, equal to σ_j. If $\sigma_1 \geq (\leq)\ \sigma_2$, the Bidder Allocation Effect is positive (negative), and the efficient auction increases

[39] The linear model allows for direct analogy to the result in consumer theory that the surplus of a consumer with linear demand is determined by the mean and variance of the market price. In our setting, bidders' values for a good, and its auction price at a given allocation, change one-for-one with the common taste shock for that good. So expected bidder surplus for good j under mechanism m can be expressed in terms of the mean and variance of the auction price less the shock to good j, $(p_j^m - \theta_j)$. Conditional on the expected 'adjusted' price, expected bidder surplus is increasing in its variance.

(reduces) expected bidder surplus relative to the SSA.[40] The knife-edge with asymmetric goods between the cases in which $\sigma_1 > \sigma_2$ (the bidders gain) and $\sigma_1 < \sigma_2$ (the bidders lose) corresponds exactly to the knife-edge result of the bidders' indifference to the efficient auction in the symmetric case.

On the auctioneer side, the Auctioneer Price Level Effect (with reference to either good) is positive due to the log linear demand. If the auctioneer's inverse relative marginal cost curve, $\widetilde{mc}_1(\widetilde{p}_1)$, is log concave, including log linear, the Auctioneer Relative Price Effect with reference to good 2 is positive, so that the efficient auction increases expected auctioneer surplus relative to the SSA. And if the inverse relative marginal cost curve is not too log convex, the positive Auctioneer Price Level Effect outweighs the negative Auctioneer Relative Price Effect so that the auctioneer still benefits. The results for the auctioneer therefore correspond to the symmetric case if the auctioneer's marginal cost curve is either log concave or not too log convex.

Constant elasticity If the demand curves have constant expected elasticities, equal to $-\varepsilon_j$ for $j = \{1, 2\}$, the demands are log convex so the Bidder Uncertainty Effect is negative. The Bidder Allocation Effect is also negative if $\widetilde{MC}_1(q_1^S) \leq 0$ and demand for good 1 is more elastic than for good 2, that is, $\varepsilon_1 \geq \varepsilon_2$. In this case, the efficient auction reduces expected bidder surplus and consequently must increase expected auctioneer surplus relative to the SSA. So under these conditions we have the same result as in the symmetric case that only the auctioneer benefits from the efficient auction.

The demand curves are convex, so the Auctioneer Price Level Effect with reference to either good is positive. So, exactly as in the log linear case, the efficient auction increases expected auctioneer surplus if the auctioneer's inverse relative marginal cost curve is not too log convex, again corresponding to the symmetric case.

Generalised Pareto distribution Appendix F analyses the GPD class of demand curves more generally. It also suggests that the results of the symmetric case are likely to remain a good approximation in asymmetric cases.

[40] If $\sigma_1 > \sigma_2$, the demand for good 1 is relatively inelastic, so good 1's auction price falls by more than good 2's rises when an incremental unit is reallocated from good 2 to good 1, causing a greater change in bidder surplus for good 1. The efficient auction increases the expected quantity sold of the relatively profitable good.

5.3 Constant relative marginal costs

An important case is when the auctioneer faces a constant marginal cost of increasing the quantity produced of good 1 while correspondingly reducing the quantity produced of good 2, i.e. $\widetilde{MC}_1(q_1)$ is constant. This arises if, for example, the marginal cost of each good is constant, or if the goods are interchangeable to produce, i.e. $C(x, 1-x) = C(1-x, x)\ \forall x \in [0, 1]$.

It follows automatically from the main analysis that the results of Propositions 1, 2 and 3 continue to hold exactly as before. Moreover it follows directly from its definition that the Auctioneer Relative Price Effect is zero. The reason is that it captures the impact on the auctioneer of variation in the relative price, but in the efficient auction the relative price equals the auctioneer's relative marginal cost so is always constant.

Proposition 6 *If the auctioneer's relative marginal cost is constant,*

(i) the efficient auction increases (decreases) expected bidder surplus relative to the SSA under the same conditions as in Corollary 1.

(ii) the efficient auction increases (decreases) expected auctioneer surplus relative to the SSA if the demand for good 2 (good 1) is convex (concave).[41]

The empirical analysis in Section 7 is an example of this case.

6 Extensions

6.1 Profit-maximising auctions

The main analysis is most relevant to auctions held by public sector institutions, including the application studied in Section 7, in which efficiency is a key objective. In other contexts, it is more reasonable to assume that the auctioneer aims to maximise auctioneer surplus.

A PMA which maximises ex post auctioneer surplus is the "profit-maximising auction". The relevant comparator is the SSA in which the fixed quantities sold of the two goods are chosen to maximise expected auctioneer surplus. As in the main model, I assume the auctioneer always sells the entire unit.

[41]With constant relative marginal costs, it follows from the labelling of the two goods that the demand for good 1 is more convex than the demand for good 2 (see Appendix B).

The profit-maximising auction naturally increases expected auctioneer surplus. By defining the Bidder Uncertainty Effect and Bidder Allocation Effect in an analogous way to that in Section 3, it is straightforward to see that the Bidder Uncertainty Effect is positive (negative) if the demand curves are log concave (log convex), as in the main analysis. The objective is irrelevant to whether the bidders gain or lose as a group from the uncertainty in the quantities sold in the PMA.

I relabel the goods so that the profit-maximising auction increases the expected quantity sold of good 1.[42] The Bidder Allocation Effect is therefore positive (negative) if good 1 is relatively more (less) profitable on the margin than good 2 for the bidders. This is the case if reallocating a marginal unit to good 1 from good 2 increases the bidders' gross value by more (less) than it increases the total payment they make to the auctioneer, i.e. auctioneer revenue. But the relevant SSA equates the auctioneer's expected marginal surplus across the two goods: the expected marginal revenue of good 1 relative to good 2 equals the auctioneer's relative marginal cost. So the Bidder Allocation Effect is positive (negative) if the difference in the expected values of the marginal winners is greater (less) than the auctioneer's relative marginal cost.

Proposition 7 *Relative to the SSA in which the fixed shares sold of the two goods are chosen to maximise expected auctioneer surplus,*

(i) the profit-maximising auction increases expected auctioneer surplus.

(ii) the profit-maximising auction increases (decreases) expected bidder surplus if both demand curves are log concave (log convex),[43] *and the expected inverse demand for good 1 relative to good 2 is greater (less) than the relative marginal cost of good 1 at the quantities sold in the SSA.*[44]

It of course follows that, in the symmetric case, the conditions under which the profit-maximising auction increases (decreases) expected bidder surplus are the same as those under which the PMA increases (decreases) expected bidder surplus in the main model (described in Table 1).[45]

[42] A sufficient condition for this labelling is that marginal revenue for good 1 relative to good 2 is weakly more convex than the relative marginal cost of good 1 (see Appendix B).

[43] Naturally, it is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j,\theta_j) : s_j(p_j,\theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j = \{1,2\}$.

[44] As in Proposition 1, it is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j,\theta_j) : d_j(p_j,\theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j = \{1,2\}$. The effect is strictly positive (negative) if at least one demand curve is strictly log concave (log convex).

[45] This is also the case for any objective function that treats the two goods symmetrically because the

6.2 Variable total supply

The total supply available at auction is often fixed in practice.[46] However, a total supply which varies in response to the strength of demand is another source of efficiency gains.

I compare the welfare effects of an efficient auction with variable supply to an SSA with fixed quantities sold of the two goods, assuming that the goods are independent. The efficient auction is equivalent to running two separate simultaneous auctions with variable supply. This allows us to isolate the impact of the efficiency gains specifically arising from variable supply.

In the SSA (with fixed supplies), the quantity sold of each good is fixed ex ante to maximise expected welfare, and in the variable-supply efficient auction, the quantities sold of the goods maximise welfare. And so, $p_j(q_j^S, \theta_j) = D_j(q_j^S, \theta_j), \mathbb{E}\left[D_j(q_j^S, \theta_j)\right] = \mathbb{E}\left[\frac{\partial C(q_1^S, q_2^S)}{\partial q_j}\right]$ and $p_j(q_j^E(\boldsymbol{\theta}), \theta_j) = D_j(q_j^E(\boldsymbol{\theta}), \theta_j) = \frac{\partial C(q_1^E(\boldsymbol{\theta}), q_2^E(\boldsymbol{\theta}))}{\partial q_j}$. I consider the symmetric case of independent goods, that is, $\frac{\partial^2 C(q_1, \ldots, q_J)}{\partial q_j \partial q_k} = 0 \ \forall j, k \in J, j \neq k$, and variable supply.

In this case, the bidder and auctioneer sides of the market are analogous, and the impact on their surpluses can each be decomposed into the effects of the efficient auction on the uncertainty and expectation of the quantities sold. The gain to bidders from the uncertainty in the efficient auction does not depend on the total quantity sold, and as in Proposition 1, bidders benefit (lose) from the uncertainty of the quantities sold in the efficient auction if the demand curves are log concave (log convex). Similarly, the auctioneer benefits (loses) from this uncertainty if her inverse marginal cost curves are log concave (log convex).

The bidders and auctioneer both gain (lose) from an increase in the expected quantity sold of each good. With variable supply, the efficient auction increases (reduces) the expected quantity sold of each good relative to the SSA if the inverse demand curve for each good is more (less) convex than the marginal cost curve.

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objective is then irrelevant to whether bidders benefit from the uncertainty in the quantities sold in the PMA. Fabra and Montero (2023) consider an objective function which is a weighted sum of welfare and revenue for the linear case.

[46]In the multi-product case, the central bank of Mexico permits a fixed nominal value of securities to be traded in its debt exchange auctions and the total quantity of energy procured in renewable energy auctions is often fixed (Szabó et al., 2020). The quantity sold in separate, simultaneous auctions of primary government debt is also typically fixed, including in Mexico's (see Section 7).

Nonetheless, there are exceptions. The quantity of liquidity issued in the Bank of England's ILTR operations was fixed until 2014, and now varies in response to the bids received and the Bank's privately known preferences. In the Federal Reserve's quantitative easing auctions, the total quantity sold was within a prespecified range.

Proposition 8 *In the symmetric case of independent goods and variable supply,*

(i) the efficient auction increases (decreases) expected bidder surplus relative to the SSA if the inverse demand curve for each good is more (less) convex than the marginal cost curve, that is, $\frac{\partial^2 D_j(q_j,\theta_j)}{\partial q_j^2} \geq (\leq) \frac{\partial^2 MC_j(q_j)}{\partial q_j^2}, j = \{1, 2\}$*, and the demand curves are log concave (log convex).*

(ii) the efficient auction increases (decreases) expected auctioneer surplus relative to the SSA if the inverse demand curve for each good is more (less) convex than the marginal cost curve and the inverse marginal cost curves are log concave (log convex).[47,48]

6.3 Alternative mechanism: reference price auction

The reference price auction (RPA) is a standard mechanism which only partially accommodates the imperfect substitutability of the two goods. Good 1 is assigned a fixed reference price (and the reference price of good 2 is normalised to zero). Within a multi-product auction, the total quantity to be sold is fixed at one unit and the acceptance priority of bids is determined by their differences with their respective reference prices. The auction clearing price of each good equals the marginal losing bid for that good.

The RPA is used in both financial and energy markets. The Bank of England and Federal Reserve used RPAs for their quantitative easing purchase schemes (Bank of England, 2022; Song and Zhu, 2018).[49] Both Germany and the UK use RPAs to allocate renewable energy contracts (Fabra and Montero, 2023).

The RPA fixes the relative price of the two goods. If the auctioneer's marginal cost of producing good 1 relative to good 2, $\widetilde{MC}_1(q_1)$, is constant, the optimal relative price is constant so the RPA is identical to the efficient auction. With more general auctioneer preferences, the RPA is inefficient.

In the symmetric case, the optimal reference price for good 1 is zero, and the RPA is equivalent to running one auction in which bids for the two goods are treated interchangeably. For each good, the expected quantity sold is equal across the two mechanisms and only the impact of uncertainty is relevant to their distributional effects.

[47] In the case in which the auctioneer's relative marginal cost is constant (described in Section 5.3), the auctioneer is indifferent between the two designs.

[48] The effects are each strictly positive (negative) if at least one of their corresponding conditions holds strictly.

[49] Armantier, Holt and Plott (2013) report that the US Treasury also considered using an RPA to purchase "toxic assets" (namely mortgage-backed securities) in the $700 billion Troubled Asset Relief Program (TARP) in 2008.

Table 2: Impact of the efficient auction relative to the RPA in the symmetric case

	Demand curves (bidder preferences)		
	Log concave & concave	Log concave & convex	Log convex
Expected bidder surplus	Decreases	Decreases	Increases
Expected auctioneer surplus	Increases	Increases	Ambiguous

The RPA creates *more* uncertainty in the quantities sold than the efficient auction. This is because the quantities sold of the two goods in the RPA adjust to the relative strength of demand, characterised by the bids, but not to the auctioneer's increasing relative marginal cost. So the conditions under which the bidders benefit from the efficient auction relative to the RPA are opposite to the conditions under which they benefit relative to the SSA. The efficient auction increases (decreases) expected bidder surplus if the demand curves are log convex (log concave), as the bidders gain (lose) from the reduced uncertainty.

Because the efficient auction increases the sum of bidder and auctioneer surplus, the efficient auction must increase expected auctioneer surplus if the demand curves are log concave.

Proposition 9, summarised in Table 2, partitions the space of demand curves into the subcases in which one or both sides of the market benefit in expectation from the efficient auction relative to the RPA.

Proposition 9 *In the symmetric case,*

(i) the efficient auction increases (decreases) expected bidder surplus relative to the RPA if the demand curves are log convex (log concave).

(ii) the efficient auction increases expected auctioneer surplus relative to the RPA if the demand curves are log concave and has an ambiguous impact if the demand curves are log convex.[50]

7 Primary market for Mexican government debt

Primary auctions are currently held separately and simultaneously for different Mexican Treasury bonds. I estimate the effects of instead using efficient multi-product auctions to jointly allocate bonds of different maturities.[51]

[50] In the case in which the auctioneer's relative marginal cost is constant (described in Section 5.3), expected bidder surplus and expected auctioneer surplus are each identical in the efficient auction and RPA.

[51] This is a particularly natural alternative as Banco de México currently runs other multi-product debt-exchange auctions. The quantity traded of each bond in "subastas de vasos comunicantes", as well as in

To recover bidders' and the government's preferences over the bonds, I assume bidders' bids correspond to their valuations and use a revealed preference approach to identify the government's preferences from their issuance choices. Equipped with these preferences, I can estimate the differences in average bidder and auctioneer surpluses between the existing mechanism to allocate bonds and a counterfactual Product-Mix Auction.

7.1 Institutional details and data

I analyse the primary market for Mexican Federal Treasury Certificates (CETES). These are zero-coupon bonds with terms to maturity of typically 28, 91, 182 and 364 days, which were the main source of federal government financing in 1978 – 2017 and represented 25% of Mexican government debt in 2001 – 2017 (Cole et al., 2022). They are issued via separate sealed-bid uniform-price auctions, which are held simultaneously on a weekly schedule. The supply issuance of each bond is fixed so the auctions on any given day can be characterised as SSAs.

I collect bid-level data (including both winning and losing bids and bidder identifiers) for every auction from October 2017—the date on which the current uniform-price protocol was introduced—to August 2021. This constitutes 205 observations of the 28-, 91- and 182-day bond auctions, and 98 of the 364-day bond auctions.[52] To my knowledge, this is the first analysis of this period in which the uniform-price protocol was used.[53]

Bids Each bid for a particular bond specifies the quantity that the bidder demands and the discount rate that they are willing to accept. Bidders can submit multiple bids with minimum increments of one basis point and minimum quantity increments of 5,000 Mexican pesos (MEX$). I convert the yields into bid prices so that bid functions (describing the bidder's bid as a function of the quantity demanded) are decreasing in quantity.

Descriptive statistics are provided in Table 3.

"permutas de valores gubernamentales" (conducted on behalf of the Mexican federal government), depends on the bids submitted and the auctioneer's supply preferences.

[52]Auctions of all maturities were held weekly, except auctions of 364-day bonds are held only monthly in the period October 2017 – June 2020.

[53]A discriminatory-pricing rule was used from November 1995 to October 2017 and studied by Castellanos and Oviedo (2008) and Cole et al. (2021). Umlauf (1993) studied a previous (1990) change from a discriminatory- to a uniform-pricing rule.

Table 3: Summary statistics of CETES primary auctions, 3 October 2017 – 31 August 2021

Maturity (days)		28	91	182	364
Number of auctions		205	205	205	98
Total quantity allocated	(mean, million pesos)	6,305	10,205	12,283	10,587
	(std. dev., million pesos)	1,640	2,498	1,492	2,978
Number of bidders	(mean)	20.37	20.28	16.24	15.16
	(std. dev.)	3.42	2.89	3.09	2.74
Number of bids	(mean)	64.80	53.14	45.22	42.80
	(std. dev.)	12.23	10.84	11.01	12.08
Discount rate	(mean, percentage points)	6.52	6.55	6.43	5.65
	(std. dev., percentage points)	1.52	1.51	1.49	1.45
Bid price	(mean, pesos)	9.95	9.84	9.68	9.43
	(std. dev., pesos)	0.01	0.04	0.07	0.14
Cover ratio	(mean)	3.25	3.07	2.68	2.67
	(std. dev.)	0.67	0.67	0.64	0.61
Quantity demanded by bidder	(mean, % of total allocated)	15.95	15.14	16.49	17.63
	(std. dev., % of total allocated)	18.06	17.77	16.88	17.08
Quantity allocated to winner	(mean, % of total allocated)	6.94	7.23	9.28	9.64
	(std. dev., % of total allocated)	9.69	12.35	12.32	12.84

Note: The cover ratio is the aggregate quantity demanded in an auction as a proportion of the total issuance. Discount rates and bid prices are weighted by the quantities demanded at those prices by bidders. The standard deviations of discount rate, bid price, quantity demanded by bidder and quantity allocated to winner are across auctions and bidders.

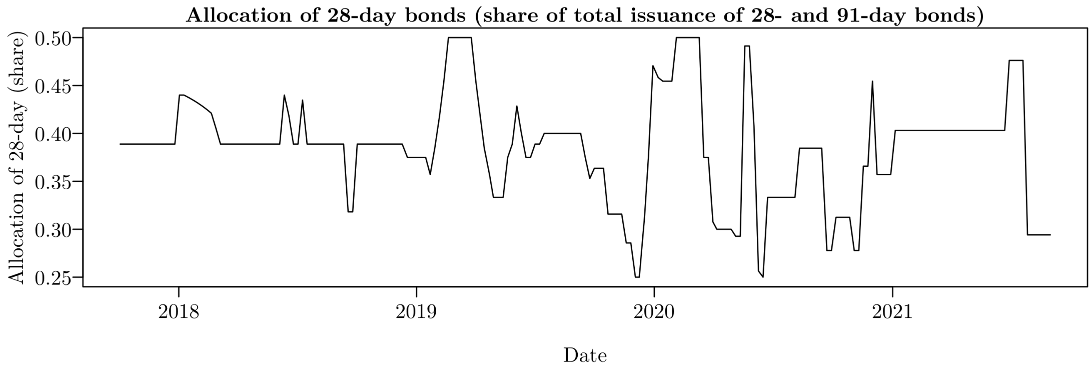
Issuance For each bond, the issuance is fixed at a quantity which is announced typically four days before the auction.[54] Both the total issuance of CETES and the share issued of each bond vary over time (Figure 4).

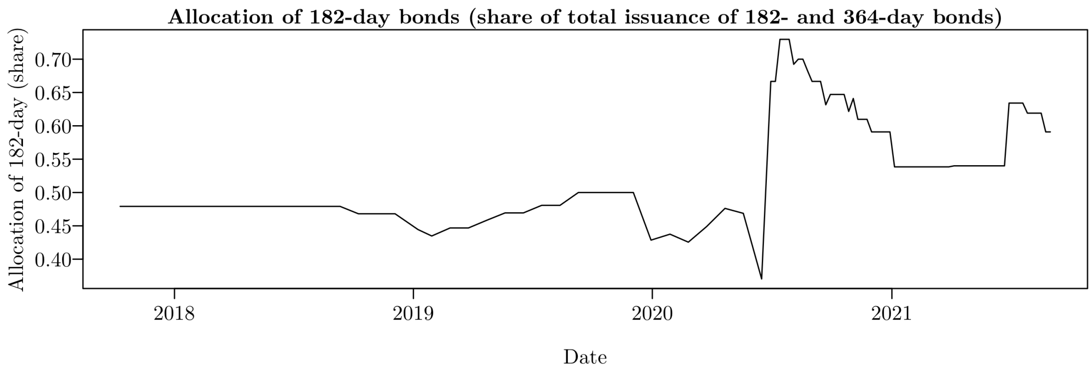
The federal government's annual financing plan lays out its long-term debt portfolio optimisation strategy (Gobierno de México, 2022). In the plan, CETES are grouped into two pairs of bonds—28- and 91-day, and 182- and 364-day bonds—and the total issuance of each pair of CETES is determined within an optimisation model, but the allocation within pair is not. Motivated by this, my counterfactual analysis focuses on the optimal allocation within pair, abstracting from the choice of the total issuance for the pair.

The government's preferences between CETES are unobserved. In line with both the large literature on sovereign debt portfolio optimisation (e.g. Missale and Blanchard, 1994; Greenwood et al., 2015a, b) and the optimal portfolio models implemented in practice in Mexico (Gobierno de México, 2022), as well as in, e.g., Canada (Bolder and Deeley, 2011), Sweden

[54]The auctions are typically held on Tuesdays and the amount to be issued of each bond is announced on the last day of the previous working week.

Figure 4: Issuance in CETES primary auctions, 3 October 2017 – 31 August 2021





(Bergstrom et al., 2002), Turkey (Balibek and Memis, 2012), and the UK (Pick and Anthony, 2006), I characterise the government's objective by a tradeoff between minimising the costs of issuance and maintaining a prudent risk profile.[55] Loosely speaking, shorter-term bonds are favourable from a financial perspective—bidders are willing to pay a higher price—but are unfavourable from a risk perspective because shorter-term bonds need to be rolled over more frequently and this occurs at uncertain interest rates.

Specifically, I assume the government's total cost of issuing a bond is equal to its nominal debt service charges (reflecting its financial cost) less its Macaulay duration (reflecting its benefits in reducing rollover risk), weighted by an unobserved time-varying preference for duration.[56] For zero-coupon CETES, the nominal debt service charge is simply the bond's face value and its Macaulay duration is its term to maturity. The government's surplus is

[55] I abstract from the possibility that the choice of issuance between CETES is determined by cash management objectives—the IMF (2021) reports that active cash management by the Mexican government through CETES issuance was limited during the sample period.

[56] Allen, Kastl and Wittwer (2022) use a similar specification to approximate the difference in the Bank of Canada's surplus from issuing Treasury bills of different maturities. They approximate the issuance cost of one bond relative to another by its initial relative price. More generally, alternative metrics of the cost and risks of issuance may be relevant to the choice over issuance of Treasury bills (see, e.g., Gobierno de México (2022) and Bolder and Deeley (2011)).

the difference between total revenue and total cost, and so on date t is

$$AS_t(q_{1t}, q_{2t}) = \sum_{j=\{1,2\}} \left[p_{jt}(q_{jt}) - v_j + \alpha_t \frac{t_j}{364} \right] q_{jt} \tag{1}$$

where, for bond $j \in \{1, 2\}$ on date t, q_{jt} is its quantity issued, $p_{jt}(q_{jt})$ is its auction price, v_j is its face value, and t_j is its term in days, and α_t is an unknown preference parameter.

This is a simplified version of the preference specification used to allocate bonds in Banco de México's debt-exchange auctions ("subastas de vasos comunicantes") in which multiple securities, including CETES, are exchanged within a multi-product auction. My interpretation of conversations with practitioners at Banco de México is that this is a reasonable approximation for the government's preferences between CETES.

7.2 Method

I group the bonds into two pairs: 28- and 91-day CETES, and 182- and 364-day CETES. For each pair, I compare the bidder and government surpluses under the actual mechanism to a counterfactual Product-Mix Auction (PMA) which jointly allocates the two bonds to maximise welfare (conventionally defined by the sum of total bidder and government surplus) given the same fixed total issuance.[57] The share issued of each of the two bonds is fixed in advance in the original mechanism but varies in the PMA. Surpluses are averaged over the set of dates T on which both bonds are auctioned. The following method refers to one pair of bonds, labelled bonds 1 and 2.

Estimating bidder preferences I assume that the submitted bids correspond to bidders' marginal valuations for the bonds, that bidders do not view the bonds as substitutes.[58] Each bidder submits a bid function, specifying the price they are willing to pay as a function of the quantity they demand. The bid function of bidder i for bond j in auction t is denoted

[57]In principle, we could run a counterfactual auction which jointly allocates all four bonds to maximise welfare. I focus on jointly allocating the bonds within pairs, as they are more plausibly substitutable so that Equation 1 is a more reasonable approximation—indeed, the government's annual financing plan allocates the bonds across but not within the pairs (Gobierno de México, 2022).

[58]Bidding according to their valuations is optimal if bidders are price takers so that they cannot exert unilateral market power by submitting bids below their valuations to push down the auction price. For the 28-day bond, for example, there are on average 20.4 bidders per auction, and the mean and standard deviation of the allocations of winning bidders are 6.9% and 9.7% of the total issuance, respectively, suggesting that this might be a reasonable approximation. Cole et al. (2022) also assume bidders behave competitively in their analysis of auctions of CETES in the period in which a discriminatory pricing rule was used, which precedes my sample period.

Allen, Kastl and Wittwer (2022) find that bills with 3-, 6-, and 12-month terms are weak substitutes for the average bidder in Canadian Treasury auctions.

$b_{ijt}(q_{ijt})$. I assume bidders are risk neutral so their surpluses are the differences between their bids and the auction prices. Bidder i's surplus for good j is the area between their bid function and the auction price for the quantity that they win:

$$\int_0^1 \mathbb{I}\big(b_{ijt}(q_{ijt}) \geq p_j(q_{jt})\big)\Big(b_{ijt}(q_{ijt}) - p_{jt}(q_{jt})\Big)dq_{ijt}$$

where $\mathbb{I}(.)$ is the indicator function; q_{jt} is the issuance of bond $j \in \{1, 2\}$ on date t and $p_j(q_{jt})$ is its auction price. Total bidder surplus on date t, $BS_t(q_{1t}, q_{2t})$, is the sum of these surpluses over goods and winning bidders, i.e.

$$BS_t(q_{1t}, q_{2t}) = \sum_{j=\{1,2\}} \sum_{i \in N_{jt}} \int_0^1 \mathbb{I}\big(b_{ijt}(q_{ijt}) \geq p_j(q_{jt})\big)\Big(b_{ijt}(q_{ijt}) - p_j(q_{jt})\Big)dq_{ijt} \tag{2}$$

where N_{jt} is the set of bidders for bond $j \in \{1, 2\}$ on date $t \in \{1, ..., T\}$.

Estimating government preferences The government's (i.e. auctioneer's) surplus on date t is described by Equation 1. The preference parameters, α_t $\forall t \in T$, are unobserved and are calibrated to match the share of the total issuance which is allocated of bond 1, q^S_{1t} (and corresponding of bond 2, $q^S_{2t} = (1 - q^S_{1t})$). The observed choices are shown in Figure 4.

The calibration method is based on revealed preference: I assume that the allocation (q^S_{1t}, q^S_{2t}), which is fixed in advance of each auction, is chosen by the government to maximise expected welfare, i.e. the expected sum of auctioneer and bidder surplus defined in Equations 1 and 2.[59]

In order to recover the preference parameters from the data, we must estimate the government's beliefs over the distribution of bids that will be submitted in auction t, given the information, denoted $I_{t,D}$, available at the time of the decision.

These beliefs and a particular value of the parameter, α_t, imply an optimal share to allocate of bond 1, $q^S_{1t}(\alpha_t)$, which is the share that maximises expected welfare over $q_{1t} \in [0, 1]$. The preference parameter, α_t, is recovered by matching the observed share to the optimal one.

[59]My interpretation of conversations with staff at Banco de México is that the government's objective for the auctions is not only to ensure issuance is cost-effective, but also to ensure well-functioning markets for the CETES, and that efficiency is an important objective. While positive weight might be placed on both auctioneer surplus and efficiency, I simplify the main analysis by assuming efficiency, i.e. maximising welfare, is the only objective. This allows for clear comparisons with the theoretical model. I also repeat the exercise assuming the fixed shares maximise expected auctioneer surplus and compare the outcomes to the multi-product auction that maximises auctioneer surplus (see Appendix G).

To estimate the expected distribution of bids, I split the bid function of each potential bidder into two components. Bidder i's bid function for bond j in auction t, $b_{ijt}(q_{ijt})$, is the sum of the secondary market price at the time of the auction, s_{jt}, and a "residualised bid function", $\epsilon_{ijt}(q_{ijt})$, i.e. $b_{ijt}(q_{ijt}) = s_{jt} + \epsilon_{ijt}(q_{ijt})$. The residualised bid function captures the bidder's idiosyncratic valuation for the bond as well as factors which impact the valuation of all bidders which are not captured by the secondary market price. The expected bid function of bidder i for bond j in auction t at the time of the issuance decision is therefore

$$\mathbb{E}\left[b_{ijt}(q_{ijt})|I_{t,D}\right] = \mathbb{E}\left[s_{jt}|I_{t,D}\right] + \mathbb{E}\left[\epsilon_{ijt}(q_{ijt})|I_{t,D}\right]$$

where $\epsilon_{ijt}(q_{ijt})$ is the difference between the bidder's bid and the secondary market price, as a function of the quantity he demands.

I estimate the two components of the bid functions separately. First, the expected secondary market price on the day of auction t is estimated by the secondary market price on the day of the decision, which is typically five days prior to the auction.[60,61]

Second, to estimate the auctioneer's beliefs over the distribution of the residualised bids, I apply the resampling method developed by Hortaçsu (2002), Kastl (2011) and Hortaçsu and Kastl (2012). This approach is now standard in the empirical auction literature to estimate bidders' beliefs about the distribution of bids submitted by other bidders; I adapt it to simulate the auctioneer's beliefs.

Each bid function observed in the data represents one realisation of a bidder's marginal value function so the empirical distribution of residualised bids can be used to estimate the population distribution of residualised bids that the auctioneer expects to face in auction t.

To estimate the distribution of residualised bids in auction t, I resample from the the set of bid functions submitted within auction $t-1$, which is the last auction observed before the decision for auction t. Restricting the sample in this way avoids the bias that unobserved heterogeneity across auctions would cause if we pooled the bid functions across auctions. Moreover, the distribution of bid functions in the most recent auction will be most informative of the distribution in the next auction if there is a trend in the distribution over

[60] The decision is made the day before the pair of auctions is announced. At this point, the federal government's decision is communicated to Banco de México, which runs the auction.

[61] This is an approximation, which is equivalent to assuming that the secondary market price follows a random walk. We expect that natural alternative forecasting techniques would produce similar results because data is only available at a daily frequency and the forecast is just five days ahead. I use the secondary yield provided by Valuación Operativa y Referencias de Mercado (Valmer) converted to price.

time.[62] Nonetheless, resampling bids from only one auction reduces the precision of the results and we do require that the bidders' valuation functions are identically distributed for the estimator to be consistent.[63]

For each auction t and bond $j \in \{1, 2\}$, there are N_{jt} potential bidders, each submitting a bid function. I draw N_{jt} bid functions observed in the data for bond j in auction $t-1$ with replacement (with bids equal to zero for non-participating bidders). This simulates one possible realisation of residualised bid functions, equal to the drawn bid functions less the secondary market price on the day of auction $t-1$. Repeating this resampling a large number of times simulates the full distribution of residualised bids the government expects to face for bond j in auction t.

The two estimated components—the estimated secondary market price and estimated distribution of residualised bids—provide an estimate of the government's beliefs over the distribution of bids in auction t at the time of the decision over the share to allocate of each bond. For any given α_t, the estimated optimal share allocated of bond 1, q_{1t}^S, is the share that maximises average welfare given the estimated distribution of bids. The parameter α_t is calibrated by matching the optimal share to the observed one.

Surpluses For each date $t \in T$, the counterfactual PMA selects the shares allocated to the two bonds, (q_{1t}^E, q_{2t}^E), which maximise welfare given the bids submitted on date t and given the calibrated α_t. The differences in surpluses between this counterfactual and the actual mechanism are calculated given the bids submitted on date t. Estimated government and bidder surpluses in auction t, given an allocation (q_{1t}, q_{2t}), are the empirical counterparts to Equations 1 and 2 and are denoted $\hat{AS}_t(q_{1t}, q_{2t})$ and $\hat{BS}_t(q_{1t}, q_{2t})$.

Decomposition The Bidder Uncertainty Effect is estimated by

$$\frac{1}{T}\sum_{t \in T}\left(\hat{BS}_t(q_{1t}^E, q_{2t}^E) - \hat{BS}_t(\bar{q}_{1t}^E, \bar{q}_{2t}^E)\right)$$

[62] The residualised bid functions of a bidder depend on his information set, which includes both independent private signals and correlated signals which are not captured by the secondary market price. The presence of correlated signals would imply within-auction correlation of residualised bid functions. If the residualised bid functions were instead independent across bidders and their distribution was stable across auctions, we could use the full set of bid functions observed in the sample period.

[63] Cassola, Hortaçsu and Kastl (2013) discuss the consistency of the estimator. In Appendix H, I use an alternative resampling procedure, which relaxes this assumption.

and the Bidder Allocation Effect by

$$\frac{1}{T}\sum_{t\in T}\left(\hat{BS}_t(\bar{q}^E_{1t},\bar{q}^E_{2t})-\hat{BS}_t(q^S_{1t},q^S_{2t})\right)$$

where $(\bar{q}^E_{1t},\bar{q}^E_{2t})=(\frac{1}{T}\sum_{t\in T}q^E_{1t},\frac{1}{T}\sum_{t\in T}q^E_{2t})$ (see Section 3).

As in Section 5.3, the auctioneer's marginal cost of good 1 relative to good 2 is constant, so the Auctioneer Relative Price Effect is set to zero, and the estimated Auctioneer Price Level Effect equals the average difference in auctioneer surplus:

$$\frac{1}{T}\sum_{t\in T}\left(\hat{AS}_t(q^E_{1t},q^E_{2t})-\hat{AS}_t(q^S_{1t},q^S_{2t})\right)$$

7.3 Results

Results are shown in Table 4.[64] For each pair of bonds, the tables' upper panels show the issuance shares and bidder surplus in the SSA and the lower panels show the differences in issuance and surpluses between the counterfactual PMA and the SSA.

Since the objective is to maximise total welfare, naturally the PMA is more efficient. For the pair of 182-day and 364-day bonds, the estimated welfare gain is 0.97 basis points of the issuance, i.e. MEX$2.27 million per auction. For the 28- and 91-day pair, the estimated welfare gain is 0.21 basis points (MEX$0.34m) (Table 4).

The PMA also increases both average bidder surplus and average auctioneer surplus for both pairs of bonds. For the 182-day and 364-day pair, the estimated bidder and auctioneer gains are 0.61 basis points of the issuance (or MEX$1.44 million) and 0.36 basis points (MEX$0.84m), respectively. For the 28-day and 91-day pair, the estimated gains are 0.08 basis points (MEX$0.13m) and 0.13 basis points (MEX$0.22m), respectively.

These results can be understood in terms of the predictions of the model. I label the 91- and 364-day bonds by bond 1 and 28- and 182-day bonds by bond 2 so that the counterfactual PMA increases the average allocation of bond 1 in each pair, in line with the model.

The significant, positive Bidder Uncertainty Effect for the 182- and 364-day pair can be understood by the fact that the bid functions are close to linear, and therefore log concave,

[64] I estimate standard errors treating the government preference parameters, α_t $\forall t\in T$, as known as in Backus, Conlon and Sinkinson (2021) and Roussille and Scuderi (2022).

in the neighbourhoods of the quantities issued, suggesting bidders benefit as a group from the uncertainty in the quantities allocated in the PMA (Proposition 1). For the 28- and 91-day pair, the bid functions are also close to linear, and therefore log concave, and the Bidder Uncertainty Effect is also positive, albeit imprecisely estimated.

Similarly, the significant, positive Bidder Allocation Effect for the 28- and 91-day pair can be understood by the fact that demand is more price elastic for bonds with shorter terms to maturity.[65] The price elasticity of demand of bond 1 (the 91-day bond) is therefore smaller in absolute value than that of bond 2's (28-day), and Proposition 2 implies that the Bidder Allocation Effect is positive if the relative marginal cost of good 1 is positive or not too negative at the quantities allocated in the SSA.[66] For the 182- and 364-day pair, our propositions do not predict the Bidder Allocation Effect (which is negative and insignificant).[67]

The impact on auctioneer surplus is explained by the impact on the price level alone as the auctioneer's constant marginal costs imply indifference to the uncertainty in relative prices. The PMA increases auctioneer surplus for the 28- and 91-day pair because it increases the price of the 28-day bond. The effect is positive but imprecisely estimated for the 182- and 364-day pair.

Overall, the welfare gains of the counterfactual PMA are moderate. Under our assumptions that the fixed allocation across the two bonds is determined jointly and optimally, the fact that the decision is made shortly before the auctions take place limits the additional welfare gains of determining the allocation within the auction itself.[68]

For both pairs of Mexican Treasury bonds, the bidders and the auctioneer both gain a significant fraction of the benefits of the efficient auction. However, our model shows this is not necessarily the case in other contexts. In Appendix I, I analyse the 2021 Spanish

[65] In the neighbourhood of the SSA allocation, the elasticity of the 28-day bond (estimated to be $\hat{\epsilon}^D_{1,28} = -12964$) is larger in absolute value than the 91-day bond ($\hat{\epsilon}^D_{2,91} = -7508$), and the elasticity of the 182-day bond ($\hat{\epsilon}^D_{2,182} = -2718$) is larger than the 364-day bond ($\hat{\epsilon}^D_{1,364} = -1614$). Allen, Kastl and Wittwer (2022) also find that demand for bonds is more price elastic for shorter terms in Canadian Treasury bill auctions.

[66] The auctioneer's relative marginal cost of the 91-day bond (bond 1) is negative because its longer term is associated with less rollover risk, but the overall Bidder Allocation Effect is positive because bidders benefit from the larger expected quantity sold of the 91-day bond in the PMA.

[67] Demand for bond 1 (the 364-day bond) is less price elastic than demand for bond 2 (the 182-day bond) and the relative marginal cost of good 1 is sufficiently negative that Proposition 2 does not apply.

[68] In the Bank of England's liquidity auctions, Giese and Grace (2023) find much larger potential welfare gains from the PMA relative to an SSA for which the Bank of England commits to the allocation across goods further in advance. In this case, the PMA increases welfare (conventionally defined by the sum of the bidders' and Bank of England's surpluses) by approximately 50% relative to the SSA.

Table 4: Surplus estimates and model predictions in CETES primary auctions given an objective to maximise total welfare

		28- & 91-day	182- & 364-day
Bond 1	(term, days)	91	364
Bond 2	(term, days)	28	182
Issuance of bond 1 in SSA	(mean, % of total)	39.679	54.631
		(0.556)	(1.153)
	(std. dev., % of total)	6.971	9.366
		(1.479)	(1.670)
Bidder surplus in SSA	(bps of issuance)	0.750	2.838
		(0.038)	(0.246)
	(MEX$ million per auction)	1.213	6.636
		(0.059)	(0.611)
Difference between counterfactual PMA and actual SSA:			
Δ Issuance of bond 1	(mean, % of total)	8.228***	4.284
		(1.920)	(3.216)
	(std. dev., % of total)	23.577***	25.689***
		(1.282)	(1.869)
Δ Welfare	(bps of issuance)	0.211***	0.972***
		(0.035)	(0.181)
	(MEX$ million per auction)	0.342***	2.273***
		(0.056)	(0.430)
Δ Bidder surplus	(bps of issuance)	0.078**	0.614*
		(0.028)	(0.243)
	(MEX$ million per auction)	0.127**	1.437*
		(0.046)	(0.571)
Δ Auctioneer surplus	(bps of issuance)	0.133***	0.358
		(0.031)	(0.238)
	(MEX$ million per auction)	0.215***	0.836
		(0.049)	(0.557)
Bidder Uncertainty Effect	(bps of issuance)	0.024	0.688**
		(0.034)	(0.240)
	(MEX$ million per auction)	0.038	1.610**
		(0.055)	(0.567)
Bidder Allocation Effect	(bps of issuance)	0.055**	-0.074
		(0.018)	(0.093)
	(MEX$ million per auction)	0.088**	-0.173
		(0.030)	(0.218)

Note: bps denote basis points. Surpluses measured in bps are weighted by total quantities issued and surpluses measured in MEX$ are unweighted. Standard errors are in parentheses. The level of welfare and auctioneer surplus in the actual SSA, which depend on the auctioneer's unobserved total benefit from issuing the total supply, are unidentified. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

auctions for wind and solar power and show that the entire benefit of the efficient auction is captured by the bidders.

Appendix G discusses the case in which the auctioneer's objective is to maximise auctioneer surplus. The auctioneer of course benefits from the PMA relative to the SSA. Average bidder surplus falls, but by a much smaller amount than the auctioneer gains.

8 Conclusion

Even when markets are competitive, the welfare gains of a more efficient auction do not necessarily benefit both sides of the market. I provide conditions under which the efficient auction reduces expected bidder surplus, and conditions under which it reduces expected auctioneer surplus, relative to running independent auctions of substitute goods.

However, the conditions required for either side to lose seem extreme. If the model is symmetric in the goods, the efficient auction benefits both sides of the market if the aggregate bid functions are convex and log concave. Bidders only lose if the bid functions are very convex, and the auctioneer only loses if they are very concave. Moreover, the results are even clearer when the auctioneer's objective is to maximise auctioneer surplus, rather than total welfare: the auctioneer of course benefits from the profit-maximising mechanism, and the bidders benefit if the aggregate bid functions are log concave. I show how the results extend if the model is asymmetric in the goods.

The mechanism that is efficient in my model—the Product-Mix Auction (PMA)—probably yields greater benefits relative to running independent auctions than my many-bidder model suggests. First, if the number of bidders is small, the PMA is likely to increase competition and efficiency by reducing large bidders' incentives to strategically bid below their values. Second, with a small number of bidders, idiosyncratic shocks to bidders' values provide an additional source of efficiency gain from the PMA.[69] Finally my model assumes each bidder is only interested in one of the goods. But if the bidders as well as the auctioneer view the goods as imperfect substitutes, the PMA in general further increases efficiency by allowing bidders to make bids that express their relative values for the different goods.[70] So my model does not consider all the potential benefits of the efficient mechanism; I leave analysis of the

[69] The effects on the distribution of surplus between bidders and the auctioneer are non-trivial. Lings (2013) considers the special case of a finite number of identical bidders for interchangeable goods.

[70] In the general PMA (see Klemperer (2008), bidders can submit "paired bids", which express the quantity demanded by the bidder across both goods and a separate price for each of the two goods. The bidder wins at most the quantity he demands, and if he wins he is allocated the good which gives him the highest net surplus (under the assumption that his bids correspond to his valuations).

distribution of these additional benefits to future work.

The magnitude of the benefits also varies across contexts. One of the main determinants of the welfare gains of combining auctions is the amount of uncertainty in the relative strength of demand for the two goods. In the case of Mexican Treasury bonds, the estimated benefits are only moderate because the amount of demand uncertainty is limited in the existing design. (This is because the decision over how to allocate the bonds is made shortly before the auctions take place.) If the uncertainty was greater, we would expect the benefits to be larger.[71] Giese and Grace (2023) find much larger potential welfare gains from the PMA in a context of much more uncertainty; they compare the PMA to running independent auctions for which the allocation across goods is fixed further in advance, in the Bank of England's liquidity auctions. The flexibility of the multi-product auction is therefore most helpful when the economic environment is uncertain.

The empirical work also provides additional support for the view that combining auctions typically benefits both sides of the market. Both the multi-product auction of 28-day and 91-day Mexican Treasury bonds, and that of the 182-day and 364-day bonds, increase both average bidder and auctioneer surplus. This can be explained largely by the fact that the aggregate bid functions are log concave and typically convex in the relevant range.

In conclusion, the fact that both sides of the market are likely to gain, both in theory and in practice, is a strong argument for policymakers to consider combining competitive auctions of substitute goods.

[71]In the linear model (analysed in Section 5.2), the impacts on expected bidder surplus and expected auctioneer surplus are both proportional to the variance of the difference in the demand shocks across the two goods, i.e. the amount of uncertainty about their relative demand. Numerical examples with other demand and marginal cost curves yield similar results.

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Appendices

A Proofs

A.1 Proposition 1

Suppose both demand curves are log concave. (The case of log convex demand curves is analogous.) For good $j \in \{1,2\}$, let $\widehat{D}_j(q_j) = -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j$, which is independent of θ_j because of the additivity of shocks. Evaluated at quantity $q_j = d_j(p_j, \theta_j)$,

$$\widehat{D}_j(d_j(p_j,\theta_j)) = -\frac{d_j(p_j,\theta_j)}{\frac{\partial d_j(p_j,\theta_j)}{\partial p_j}} \implies \frac{\partial \widehat{D}_j(q_j)}{\partial q_j} = \frac{\partial}{\partial p_j}\left(-\frac{d_j(p_j,\theta_j)}{\frac{\partial d_j(p_j,\theta_j)}{\partial p_j}}\right)\frac{\partial D_j(q_j,\theta_j)}{\partial q_j}$$

$$= \frac{\partial^2(log(d_j(p_j,\theta_j)))}{\partial p_j^2}\left(\frac{d_j(p_j,\theta_j)}{\frac{\partial d_j(p_j,\theta_j)}{\partial p_j}}\right)^2 \frac{\partial D_j(q_j,\theta_j)}{\partial q_j}$$

$$\implies \frac{\partial \widehat{D}_j(q_j)}{\partial q_j} \geq 0 \Leftrightarrow \frac{\partial^2(log(d_j(p_j,\theta_j))}{\partial p_j^2} \leq 0$$

because $\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} < 0$, and similarly for the strict inequalities. Applying Jensen's inequality to the function $\int_0^{q_j^E(\boldsymbol{\theta})} \widehat{D}_j(q_j)\, dq_j$ of $q_j^E(\boldsymbol{\theta})$,

$$\text{Bidder Uncertainty Effect} = \mathbb{E}\left[\sum_{j=\{1,2\}}\left(\int_{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]}^{q_j^E(\boldsymbol{\theta})} \widehat{D}_j(q_j)\, dq_j\right)\right] \geq 0$$

A.2 Proposition 2

Observe that

$$\text{Bidder Allocation Effect} = \mathbb{E}\left[\sum_{j=\{1,2\}}\left(\int_{q_j^S}^{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]} -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j\, dq_j\right)\right] = \int_{q_1^S}^{\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]} X(q_1)\, dq_1$$

in which $X(q_1) = \left(-\frac{\partial D_1(q_1,\theta_1)}{\partial q_1} q_1 + \frac{\partial D_2(1-q_1,\theta_2)}{\partial q_2}(1-q_1)\right)$, because the auctioneer sells precisely one unit so that $q_2(q_1) = (1-q_1)$. Evaluating the integrand at the quantities sold in the SSA and writing $\epsilon_j^D(q_j,\theta_j) = \frac{D_j(q_j,\theta_j)}{\frac{\partial D_j(q_j,\theta_j)}{\partial q_j} q_j}$ for the elasticity of demand for good j,

$$X(q_1^S) = -\frac{\mathbb{E}\left[D_1(q_1^S,\theta_1)\right]}{\mathbb{E}\left[\epsilon_1^D(q_1^S,\theta_1)\right]} + \frac{\mathbb{E}\left[D_2(1-q_1^S,\theta_2)\right]}{\mathbb{E}\left[\epsilon_2^D(1-q_1^S,\theta_2)\right]}$$

because the shocks are additive. The pair of quantities sold in the SSA is chosen to maximise expected welfare, so $\mathbb{E}\left[D_2(1-q_1^S,\theta_2)\right] = \mathbb{E}\left[D_1(q_1^S,\theta_1)\right] - \widetilde{MC}_1(q_1^S)$, and therefore

$$X(q_1^S) = \mathbb{E}\left[D_1(q_1^S,\theta_1)\right]\left(\frac{1}{\mathbb{E}\left[\epsilon_2^D(1-q_1^S,\theta_2)\right]} - \frac{1}{\mathbb{E}\left[\epsilon_1^D(q_1^S,\theta_1)\right]}\right) - \frac{\widetilde{MC}_1(q_1^S)}{\mathbb{E}\left[\epsilon_2^D(1-q_1^S,\theta_2)\right]}$$

So if $\mathbb{E}\left[\epsilon_2^D(1-q_1^S,\theta_2)\right] \leq \mathbb{E}\left[\epsilon_1^D(q_1^S,\theta_1)\right]$ and $\widetilde{MC}_1(q_1^S) \geq 0$, then $X(q_1^S) \geq 0$.

If the demand curves for goods $j = \{1,2\}$ are log concave $\forall\{(p_j,\theta_j) : d_j(p_j,\theta_j) \in [\underline{q}_j, \bar{q}_j]\}$, then $\frac{\partial X(q_1)}{\partial q_1} \geq 0$ as shown in Proposition 1, and $q_1^S \leq \mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]$. Combining these three properties, it follows that the Bidder Allocation Effect is positive.

If instead the demand curves for goods $j = \{1,2\}$ are log convex $\forall\{(p_j,\theta_j) : d_j(p_j,\theta_j) \in [\underline{q}_j, \bar{q}_j]\}$, so $\frac{\partial X(q_1)}{\partial q_1} \leq 0$ and $\mathbb{E}\left[\epsilon_2^D(1-q_1^S,\theta_2)\right] \geq \mathbb{E}\left[\epsilon_1^D(q_1^S,\theta_1)\right]$, and $\widetilde{MC}_1(q_1^S) \leq 0$, so $X(q_1^S) \leq 0$, then the Bidder Allocation Effect is negative.

A.3 Lemma 2

I consider the case of good 1; the case of good 2 is analogous. In equilibrium, $p_1(q_1^E(\boldsymbol{\theta}),\theta_1) - p_2(q_2^E(\boldsymbol{\theta}),\theta_2) = \widetilde{MC}_1(q_1^E(\boldsymbol{\theta}))$ and $p_1(q_1^S,\theta_1) - p_2(q_2^S,\theta_2) = \widetilde{MC}_1(q_1^S) + \theta_1 - \theta_2$, and so

$$\begin{aligned}
&\mathbb{E}\left[TAS(q_1^E(\boldsymbol{\theta}),\boldsymbol{\theta}) - TAS(q_1^S,\boldsymbol{\theta})\right] \\
=&\mathbb{E}\Bigg[\left(p_1(q_1^E(\boldsymbol{\theta}),\theta_1)q_1^E(\boldsymbol{\theta}) + p_2(q_2^E(\boldsymbol{\theta}),\theta_2)q_2^E(\boldsymbol{\theta}) - \int_0^{q_1^E(\boldsymbol{\theta})} \widetilde{MC}_1(q_1)\, dq_1 - C(0,1)\right) \\
&- \left(p_1(q_1^S,\theta_1)q_1^S + p_2(q_2^S,\theta_2)q_2^S - \int_0^{q_1^S} \widetilde{MC}_1(q_1)\, dq_1 - C(0,1)\right)\Bigg] \\
=&\mathbb{E}\Bigg[\left(\widetilde{MC}_1(q_1^E(\boldsymbol{\theta}))q_1^E(\boldsymbol{\theta}) - \int_0^{q_1^E(\boldsymbol{\theta})} \widetilde{MC}_1(q_1)\, dq_1\right) - \left(\widetilde{MC}_1(q_1^S)q_1^S - \int_0^{q_1^S} \widetilde{MC}_1(q_1)\, dq_1\right) \\
&+ \left(p_2(q_2^E(\boldsymbol{\theta}),\theta_2) - p_2(q_2^S,\theta_2)\right) - (\theta_1-\theta_2)q_1^S\Bigg] \\
=&\mathbb{E}\left[\int_{q_1^S}^{q_1^E(\boldsymbol{\theta})} \left(\widetilde{MR}_1(q_1) - \widetilde{MC}_1(q_1)\right)\, dq_1\right] + \mathbb{E}\left[p_2(q_2^E(\boldsymbol{\theta}),\theta_2) - p_2(q_2^S,\theta_2)\right]
\end{aligned}$$

A.4 Proposition 3

Let $\bar{D}_j(q_j) = (D_j(q_j,\theta_j) - \theta_j)$, which is a function of q_j only because inverse demand for good j is additive in the shock θ_j. Then,

$$\mathbb{E}\left[p_j(q_j^E(\boldsymbol{\theta}),\theta_j) - p_j(q_j^S,\theta_j)\right] = \left(\mathbb{E}\left[\bar{D}_j(q_j^E(\boldsymbol{\theta}))\right] - \bar{D}_j(\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right])\right) + \left(\bar{D}_j(\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]) - \bar{D}_j(q_j^S)\right)$$

If $d_j(p_j, \theta_j)$ is convex in p_j then, because demand for good j is strictly decreasing, $\bar{D}_j(q_j)$ is convex and $\bar{D}_j(\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]) \leq \ \mathbb{E}\left[\bar{D}_j(q_j^E(\boldsymbol{\theta}))\right]$. The converse is analogous.

By the labelling of the two goods, $q_1^S \leq \mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]$ and correspondingly $q_2^S \geq \mathbb{E}\left[q_2^E(\boldsymbol{\theta})\right]$, so $\bar{D}_2(q_2^S) \leq \ \bar{D}_2(\mathbb{E}\left[q_2^E(\boldsymbol{\theta})\right]) \leq \mathbb{E}\left[\bar{D}_2(q_2^E(\boldsymbol{\theta}))\right]$ if demand for good 2 is convex, and $\bar{D}_1(q_1^S) \geq \bar{D}_1(\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]) \geq \ \mathbb{E}\left[\bar{D}_1(q_1^E(\boldsymbol{\theta}))\right]$ if demand for good 1 is concave.

A.5 Proposition 4

Suppose the inverse relative marginal cost curve with reference to good $j = \{1, 2\}$ is log concave. Evaluated at quantity $q_j = \widetilde{mc}_j(\widetilde{p}_j)$,

$$\widetilde{MC}_j(\widetilde{mc}_j(\widetilde{p}_j)) = \frac{\widetilde{mc}_j(\widetilde{p}_j)}{\frac{\partial \widetilde{mc}_j(\widetilde{p}_j)}{\partial \widetilde{p}_j}} \implies \frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j} = \frac{\partial}{\partial \widetilde{p}_j}\left(\frac{\widetilde{mc}_j(\widetilde{p}_j)}{\frac{\partial \widetilde{mc}_j(\widetilde{p}_j)}{\partial \widetilde{p}_j}}\right)\frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j}$$

$$= -\frac{\partial^2(log(d_j(\widetilde{p}_j, \theta_j)))}{\partial \widetilde{p}_j^2}\left(\frac{\widetilde{mc}_j(p_j)}{\frac{\partial \widetilde{mc}_j(p_j)}{\partial p_j}}\right)^2\frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j}$$

$$\implies \frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j} \geq 0 \Leftrightarrow \frac{\partial^2(log(\widetilde{mc}_j(\widetilde{p}_j)))}{\partial \widetilde{p}_j^2} \leq 0$$

because $\frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j} > 0$, and similarly for the strict inequalities.

Let $h_j(q_j) = \widetilde{MR}_j(q_j) - \widetilde{MC}_j(q_j) = \frac{\partial \widetilde{MC}_j(q_j)}{\partial q_j} q_j$. Relative marginal cost is increasing, so that $h_j(q_j) \geq 0 \ \forall q_j$. By the labelling of the two goods, $\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right] \geq q_1^S$, so

$$\mathbb{E}\left[\int_{q_1^S}^{\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]} h_1(q_1)\ dq_1\right] \geq 0, \ \mathbb{E}\left[\int_{q_2^S}^{\mathbb{E}\left[q_2^E(\boldsymbol{\theta})\right]} h_2(q_2)\ dq_2\right] \leq 0$$

Applying Jensen's inequality to the function $\int_0^{q_1^E(\boldsymbol{\theta})} h_1(q_1)\ dq_1$, the Auctioneer Relative Price Effect with reference to good 2 is positive if $\widetilde{MC}_1(q_1)$ is log concave. Analogously, the Auctioneer Relative Price Effect with reference to good 1 is negative if $\widetilde{MC}_2(q_2)$ is log convex.

Of course, if the inverse relative marginal cost curve of good $j \in \{1, 2\}$ has zero slope, then $h_j(q_j) = 0 \ \forall q_j$ and the Auctioneer Relative Price Effect with reference to good j is zero.

A.6 Proposition 5

The two shocks, (θ_1, θ_2), are exchangeable with zero mean, and so the relative shock to good 1, $(\theta_1 - \theta_2)$, is symmetrically distributed around zero. The auctioneer's and bidders' preferences are each symmetric in the two goods. Using the equilibrium condition for the efficient auction, it follows that $q_j^E(\theta), j = \{1, 2\}$, is also symmetrically distributed, with $q_j^S = \mathbb{E}\left[q_j^E(\theta)\right] = \frac{1}{2}$.

Bidder surplus The conditions follow from Proposition 1 and the fact that the expected quantities sold in the two mechanisms are equal.

Auctioneer surplus Proposition 3 determine the conditions under which the Auctioneer Price Level Effect is positive/negative. To determine the sign of the Auctioneer Relative Price Effect, I suppress the dependence on the vector of shocks, and denote the distribution of $q_k^E, k \in \{1, 2\}$, by $G_k(q_k^E)$. By the symmetry, $\widetilde{MC}_k(1 - q_k^E) = -\widetilde{MC}_k(q_k^E)$, and so

$$
\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{E}\left[q_k^E\right]}^{q_k^E} \left(\widetilde{MR}_k(q_k) - \widetilde{MC}_k(q_k)\right) dq_k\right]& \\
&= \mathbb{E}\left[\widetilde{MC}_k(q_k^E)q_k^E - \int_{\frac{1}{2}}^{q_k^E} \widetilde{MC}_k(q_k)dq_k\right] \\
&= 2\int_{\frac{1}{2}}^{1} \left(\widetilde{MC}_k(q_k^E)\left(q_k^E - \frac{1}{2}\right) - \int_{\frac{1}{2}}^{q_k^E} \widetilde{MC}_k(q_k)dq_k\right) dG_k(q_k^E) \\
&= 2\int_{\frac{1}{2}}^{1} \left(\int_{\frac{1}{2}}^{q_k^E} \left(\widetilde{MC}_k(q_k^E) - \widetilde{MC}_k(q_k)\right) dq_k\right) dG_k(q_k^E) \\
&\geq 0
\end{aligned}
$$

because $\frac{\partial \widetilde{MC}_k(q_k)}{\partial q_k} \geq 0 \ \forall q_k$.

A.7 Corollary 3

For each of the J goods, the same quantity will be sold in the SSA and PMA in expectation, $q_k^S = \mathbb{E}\left[q_k^E(\boldsymbol{\theta})\right] = \frac{1}{J}, k \in J$.

Bidder surplus Lemma 1 and Proposition 1 directly generalise to J goods, and the Bidder Allocation Effect is zero.

Auctioneer surplus Let $MC_k(q_k) = \frac{\partial C(q_1,\ldots,q_J)}{\partial q_k}, k \in J$, and $MC_k(q) = MC(q) \ \forall k, q$, then, for $j \in J$,

$$
\begin{aligned}
&\mathbb{E}\left[TAS(q_1^E(\boldsymbol{\theta}), q_2^E(\boldsymbol{\theta}), ..., q_{J-1}^E(\boldsymbol{\theta}), \boldsymbol{\theta}) - TAS(q_1^S, q_2^S, ..., q_{J-1}^S, \boldsymbol{\theta})\right] \\
=&\mathbb{E}\left[\sum_{k=1}^{J}\left(p_k(q_k^E(\boldsymbol{\theta}), \theta_k)q_k^E(\boldsymbol{\theta}) - p_k(q_k^S, \theta_k)q_k^S - \int_{q_k^S}^{q_k^E(\boldsymbol{\theta})} MC(q_k)\ dq_k\right)\right] \\
=&\sum_{k=1}^{J}\mathbb{E}\left[MC(q_k^E(\boldsymbol{\theta}))q_k^E(\boldsymbol{\theta}) - MC(q_j^E(\boldsymbol{\theta}))\left(\frac{1}{J}\right) - \int_{\frac{1}{J}}^{q_k^E(\boldsymbol{\theta})} MC(q_k)\ dq_k\right] \\
&+\mathbb{E}\left[p_j(q_j^E(\boldsymbol{\theta}), \theta_j) - p_j\left(\frac{1}{J}, \theta_j\right)\right] \\
=&J\mathbb{E}\left[\int_{\frac{1}{J}}^{q_j^E(\boldsymbol{\theta})}\big(MC(q_j^E(\boldsymbol{\theta})) - MC(q_j)\big)\ dq_j\right] + \mathbb{E}\left[p_j(q_j^E(\boldsymbol{\theta}), \theta_j) - p_j\left(\frac{1}{J}, \theta_j\right)\right] \qquad (3)
\end{aligned}
$$

because $p_k(q_k^E(\boldsymbol{\theta}), \theta_k) - p_j(q_j^E(\boldsymbol{\theta}), \theta_j) = MC_k(q_k^E(\boldsymbol{\theta})) - MC_j(q_j^E(\boldsymbol{\theta}))$ and $p_k(q_k^S, \theta_k) - p_j(q_j^S, \theta_j) = MC_k(q_k^S) - MC_j(q_j^S) + \theta_k - \theta_j\ \forall j, k \in J$.

Because $\frac{\partial MC(q_j)}{\partial q_j} \geq 0$, $\int_{\frac{1}{J}}^{q_j^E(\boldsymbol{\theta})}\big(MC(q_j^E(\boldsymbol{\theta})) - MC(q_j)\big)\ dq_j \geq 0\ \forall \boldsymbol{\theta}$ and the first term in Equation 3 is positive. The sign of the second term follows from Proposition 3.

A.8 Proposition 7

The impact on expected auctioneer surplus is immediate. It is clear from Proposition 1 that the Bidder Uncertainty Effect in this case, analogous to that defined in Section 3, is positive (negative) if the demand curves are log concave (log convex).

Turning to the Bidder Allocation Effect in this case, let $X(q_1) = (D_1(q_1, \theta_1) - MR_1(q_1, \theta_1)) - (D_2(1 - q_1, \theta_2) - MR_2(1 - q_1, \theta_2))$. Then,

$$
\begin{aligned}
X(q_1^S) =&\mathbb{E}\left[D_1(q_1^S, \theta_1)\right] - \mathbb{E}\left[D_2(1 - q_1^S, \theta_2)\right] - \mathbb{E}\left[MR_1(q_1^S, \theta_1)\right] - \mathbb{E}\left[MR_2(1 - q_1^S, \theta_2)\right] \\
=&\mathbb{E}\left[D_1(q_1^S, \theta_1)\right] - \mathbb{E}\left[D_2(1 - q_1^S, \theta_2)\right] - \widetilde{MC}_1(q_1^S)
\end{aligned}
$$

where the last line follows from the quantities sold in the SSA maximising expected auctioneer surplus. And so, $X(q_1^S) \geq 0$ if $\mathbb{E}\left[D_1(q_1^S, \theta_1)\right] - \mathbb{E}\left[D_2(1 - q_1^S, \theta_2)\right] - \mathbb{E}\left[MR_1(q_1^S, \theta_1)\right] \geq \widetilde{MC}_1(q_1^S)$. The rest of the proof is identical to that of Proposition 2.

A.9 Proposition 8

Define the Bidder Uncertainty and Allocation Effects as in Section 3 and define the Auctioneer Uncertainty and Allocation Effects similarly,

$$
\text{Auctioneer Uncertainty Effect} = \mathbb{E}\left[\sum_{j=\{1,2\}}\left(\int_{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]}^{q_j^E(\boldsymbol{\theta})}(MAR_j(q_j) - MC_j(q_j))\, dq_j\right)\right]
$$

and

$$\text{Auctioneer Allocation Effect} = \mathbb{E}\Bigg[\sum_{j=\{1,2\}}\left(\int_{q_j^S}^{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]} \left(MAR_j(q_j) - MC_j(q_j)\right) dq_j\right)\Bigg]$$

where $MC_j(q_j) = \frac{\partial C(q_1,q_2)}{\partial q_j}$ and $MAR_j(q_j) = \frac{\partial(MC_j(q_j)q_j)}{\partial q_j}$. Define the inverse marginal cost curve for good j, equal to $mc_j(p_j)$.

It follows immediately that the difference in expected total bidder (auctioneer) surplus between the efficient auction and SSA, $\mathbb{E}\left[TBS(q_1^E(\boldsymbol{\theta}),\boldsymbol{\theta}) - TBS(q_1^S,\boldsymbol{\theta})\right]$ ($\mathbb{E}\left[TAS(q_1^E(\boldsymbol{\theta}),\boldsymbol{\theta}) - TAS(q_1^S,\boldsymbol{\theta})\right]$), equals the sum of the Bidder (Auctioneer) Uncertainty Effect and the Bidder (Auctioneer) Allocation Effect.

It follows from analogy to Proposition 1 that the Auctioneer Uncertainty Effect is positive (negative) if both inverse marginal cost curves are log concave (log convex).[72]

Now consider the Bidder Allocation Effect and Auctioneer Allocation Effects. For $j = \{1,2\}$, let $h_j(q_j) = D_j(q_j,\theta_j) - \theta_j - MC_j(q_j)$, which is a function of q_j only, because inverse demand for good j is additive in the shock θ_j and the goods are independently produced. By the equilibrium conditions, $h_j(q_j^S) = 0$, and $h_j(q_j^E(\boldsymbol{\theta})) = -\theta_j$ so that $\mathbb{E}\left[h_j(q_j^E(\boldsymbol{\theta}))\right] = \mathbb{E}\left[-\theta_j\right] = 0$. If the inverse demand for good j is more convex than its marginal cost, then $h_j(q_j)$ is strictly decreasing and convex, so $h_j\left(\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]\right) \leq 0$ and $\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right] \geq q_j^S$. Conversely, if the inverse demand for good j is less convex than its marginal cost, then $\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right] \leq q_j^S$.

Both $(D_j(q_j,\theta_j) - MR_j(q_j,\theta_j)) = -\frac{\partial D_j(q_j,\theta_j)}{\partial q_j}q_j \geq 0$ and $(MAR_j(q_j,\theta_j) - MC_j(q_j)) = \frac{\partial MC_j(q_j)}{\partial q_j}q_j \geq 0$, and so the result follows.

A.10 Proposition 9

Bidder surplus Let q_1^R be the quantity sold of good 1 in the RPA; $\mathbb{E}(q_1^E(\boldsymbol{\theta})) = \mathbb{E}(q_1^R(\boldsymbol{\theta}))$, so

$$\begin{aligned}
&\mathbb{E}\left[TBS(q_1^E(\boldsymbol{\theta}),\boldsymbol{\theta}) - TBS(q_1^R(\boldsymbol{\theta}),\boldsymbol{\theta})\right] \\
&\qquad = \mathbb{E}\Bigg[\sum_{j=\{1,2\}}\Bigg(\int_{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]}^{q_j^E(\boldsymbol{\theta})} \left(D_j(q_j,\theta_j) - MR_j(q_j,\theta_j)\right) dq_j \\
&\qquad\qquad - \int_{\mathbb{E}\left[q_j^R(\boldsymbol{\theta})\right]}^{q_j^R(\boldsymbol{\theta})} \left(D_j(q_j,\theta_j) - MR_j(q_j,\theta_j)\right) dq_j\Bigg)\Bigg]
\end{aligned}$$

[72]It is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j,\theta_j) : d_j(p_j,\theta_j) \in [\underline{q}_j, \bar{q}_j]\}, j = \{1,2\}$. The effect is strictly positive (negative) if at least one demand curve is strictly log concave (log convex).

Let $\widetilde{\boldsymbol{\theta}} = \theta_2 - \theta_1$ be the relative shock to good 1. If $\widetilde{\boldsymbol{\theta}} \in [0, \bar{\theta}_2 - \underline{\theta}_1]$, $q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})) < \frac{1}{2}\ \forall \theta_1 \in [\underline{\theta}_1, \bar{\theta}_1]$, and $\widetilde{MC}_1'(q_1) \geq 0$, then the quantities sold in the PMA and RPA satisfy

$$\begin{aligned}
&D_1(q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})), \theta_1) - D_2(1 - q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})), \theta_1 + \widetilde{\boldsymbol{\theta}}) \\
&\qquad = \widetilde{MC}_1(q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}}))) \leq \widetilde{MC}_1\left(\frac{1}{2}\right) = 0 \\
&\text{and } 0 = D_1(q_1^R((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})), \theta_1) - D_2(1 - q_1^R((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})), \theta_1 + \widetilde{\boldsymbol{\theta}}) \\
&\qquad \implies q_1^R((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})) \leq q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})) \leq \frac{1}{2} \quad \forall \left\{\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1], \widetilde{\boldsymbol{\theta}} \in [0, \bar{\theta}_2 - \underline{\theta}_1]\right\}
\end{aligned}$$

and, analogously, $\frac{1}{2} \leq q_1^E((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})) \leq q_1^R((\theta_1, \theta_1 + \widetilde{\boldsymbol{\theta}})) \quad \forall \left\{\theta_1 \in [\underline{\theta}_1, \bar{\theta}_1], \widetilde{\boldsymbol{\theta}} \in [\underline{\theta}_2 - \bar{\theta}_1, 0]\right\}$. So $q_1^E(\boldsymbol{\theta})$ second order stochastically dominates $q_1^R(\boldsymbol{\theta})$. By Proposition 1, if the demand curves are log convex (log concave), $(D_j(q_j, \theta_j) - MR_j(q_j, \theta_j)) \leq (\geq)\ 0$, so that $\mathbb{E}\left[TBS(q_1^E(\boldsymbol{\theta}), \boldsymbol{\theta}) - TBS(q_1^R(\boldsymbol{\theta}), \boldsymbol{\theta})\right] \geq (\leq)\ 0$.

Auctioneer surplus The efficient auction increases total expected surplus, so it increases expected auctioneer surplus if it reduces expected bidder surplus.

B Labelling the goods

Lemma 3 *If* $\frac{\partial^2(D_1(q_1,\theta_1) - D_2(1-q_1,\theta_2))}{\partial q_1^2} \geq \frac{\partial^2 \widetilde{MC}_1(q_1)}{\partial q_1^2}\ \forall q_1 \in [\underline{q}_1, \bar{q}_1]$, *the efficient auction increases the expected quantity sold of good 1 relative to the SSA.*[73]

Proof. Let $h(q_1) = D_1(q_1, \theta_1) - \theta_1 - D_2(1 - q_1, \theta_2) + \theta_2 - \widetilde{MC}_1(q_1)$, which is a function of q_1 only, because inverse demand for good $j = \{1, 2\}$ is additive in the shock θ_j. By the equilibrium conditions, $h(q_1^S) = 0$, and $h(q_1^E(\boldsymbol{\theta})) = \theta_2 - \theta_1$ so that $\mathbb{E}\left[h(q_1^E(\boldsymbol{\theta}))\right] = \mathbb{E}\left[\theta_2 - \theta_1\right] = 0$. The function, $h(q_1)$, is strictly decreasing and convex, so $h\left(\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right]\right) \leq 0$, and it is also decreasing, so $\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right] \geq q_1^S$. ■

If the PMA instead maximises auctioneer surplus, as in the profit-maximising auction, the quantity sold of good 1 (and implicitly, of good 2) adjusts so that the marginal revenue of good 1 relative to good 2 equals the auctioneer's relative marginal cost. This implies the following analogous sufficient condition for labelling the goods.

Lemma 4 *If* $\frac{\partial^2(MR_1(q_1,\theta_1) - MR_2(1-q_1,\theta_2))}{\partial q_1^2} \geq \frac{\partial^2 \widetilde{MC}_1(q_1)}{\partial q_1^2}\ \forall q_1 \in [\underline{q}_1, \bar{q}_1]$ *and* $\frac{\partial(MR_j(q_j,\theta_j)}{\partial q_j} < 0$ *for* $j = \{1, 2\}$, *the profit-maximising auction increases the expected quantity sold of good 1 relative to the SSA in which the fixed shares sold of the two goods are chosen to maximise expected auctioneer surplus.*[74]

[73] Of course, the implication also holds strictly, that is, if $\frac{\partial^2(D_1(q_1,\theta_1) - D_2(1-q_1,\theta_2))}{\partial q_1^2} > \frac{\partial^2 \widetilde{MC}_1(q_1)}{\partial q_1^2}\ \forall q_1 \in [\underline{q}_1, \bar{q}_1]$, then $\mathbb{E}\left[q_1^E(\boldsymbol{\theta})\right] > q_1^S$.

[74] The implication also holds strictly.

Proof. Let q_1^P be the quantity sold of good 1 in the profit-maximising auction and and q_1^T be the quantity sold of good 1 in the SSA in which the fixed shares sold of the two goods are chosen to maximise expected auctioneer surplus, respectively. Let $g(q_1) = MR_1(q_1, \theta_1) - \theta_1 - MR_2(1-q_1, \theta_2) + \theta_2 - \widetilde{MC}_1(q_1)$. In equilibrium, $g(q_1^P(\boldsymbol{\theta})) = \theta_2 - \theta_1$ and $g(q_1^T) = 0$. The rest of the proof is the same as that of Lemma 3, and so $\mathbb{E}\left[q_1^P(\boldsymbol{\theta})\right] \geq q_1^T$. ■

C Distribution of bidder surplus across markets

The impact of the efficient auction on the expected surplus of the bidders for a single good can be decomposed into good-specific Bidder Uncertainty and Allocation Effects. For good j, let the Bidder Uncertainty Effect$_j$ $= \mathbb{E}\left[\int_{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]}^{q_j^E(\boldsymbol{\theta})} (D_j(q_j, \theta_j) - MR_j(q_j, \theta_j))\ dq_j\right]$ and the Bidder Allocation Effect$_j$ $= \int_{q_j^S}^{\mathbb{E}\left[q_j^E(\boldsymbol{\theta})\right]} (D_j(q_j, \theta_j) - MR_j(q_j, \theta_j))\ dq_j$. The following decomposition therefore immediately follows.

Lemma 5 *The difference between the efficient auction and SSA in expected surplus of bidders for good $j \in \{1,2\}$, $\mathbb{E}\left[TBS_j(q_j^E(\boldsymbol{\theta}), \theta_j) - TBS_j(q_j^S, \theta_j)\right]$, equals the sum of the good-specific Bidder Uncertainty Effect and Bidder Allocation Effect for good j.*

The Bidder Uncertainty Effect is the sum of the good-specific Bidder Uncertainty Effects, capturing the gain to bidders from the uncertainty in the quantity sold of each good (see Proposition 1).

Corollary 4 *Bidder Uncertainty Effect$_j$, $j \in \{1,2\}$, is positive (negative) if the demand curve for good j is log concave (log convex).*[75]

By the labelling of the two goods, the efficient auction increases the expected quantity sold of good 1, and so Bidder Allocation Effect$_1 \geq 0$ and Bidder Allocation Effect$_2 \leq 0$.

D Auctioneer surplus decomposition

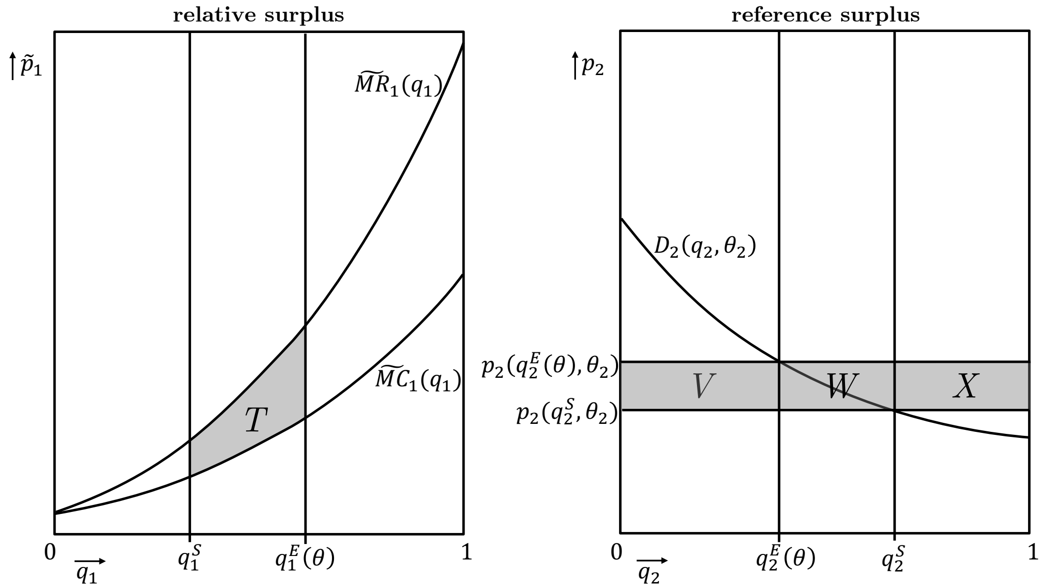
D.1 Relationship between Figures 2 and 3

If the auctioneer has independent marginal costs of production, the areas shaded in Figure 2 correspond to those in Figure 3.

To see the relationship, I denote $\chi(\boldsymbol{\theta}) = (\tilde{MC}_1(q_1^E(\boldsymbol{\theta})) - \tilde{MC}_1(q_1^S))q_1^S$ and I split Area U in Figure 3 into three parts, Areas V, W and X, as illustrated in Figure 5. Comparing Figures

[75] Naturally, it is sufficient that the condition holds only for the range of feasible prices and shocks, that is $\forall\{(p_j, \theta_j) : d_j(p_j, \theta_j) \in [\underline{q}_j, \bar{q}_j]\}$. The effect is strictly positive (negative) if the demand curve is strictly log concave (log convex).

Figure 5: Auctioneer surplus measured with reference to good 2



The Auctioneer Relative Price Effect with reference to good 2 is the expected difference in relative surplus between the efficient auction and the SSA, that is, the expected size of Area T. The Auctioneer Price Level Effect with reference to good 2 is the expected difference in reference surplus between the efficient auction and the SSA, that is, the expected size of (Area V + Area W + Area X).

2 and 5, we have

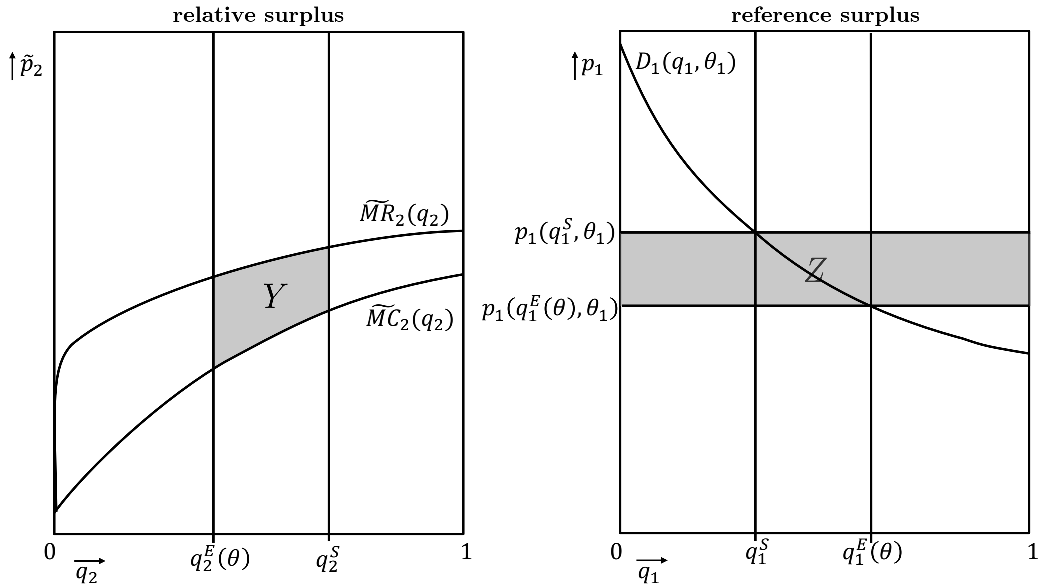
$$\begin{aligned}\text{Area } T + \text{Area } W &= \text{Area } Q - \text{Area } S + \chi(\boldsymbol{\theta})\\ \text{Area } V &= \text{Area } R\\ \text{Area } X &= (\theta_1 - \theta_2)q_1^S - \text{Area } P - \chi(\boldsymbol{\theta})\end{aligned}$$

Given shocks $\boldsymbol{\theta}$ for which $q_1^E(\boldsymbol{\theta}) > q_1^S$, the difference in relative surplus between the efficient auction and SSA is Area L+Area $U+(\theta_2-\theta_1)q_1^S =$ Area T+(Area V+Area W+Area X)$-(\theta_1-\theta_2)q_1^S =$ (Area Q − Area P + Area R − Area S).

D.2 Auctioneer surplus decomposed with reference to good 1

The difference in auctioneer surplus between the efficient auction and SSA can equivalently be measured with reference to good 1. This is illustrated for shocks for which $q_1^E(\boldsymbol{\theta}) > q_1^S$ (and correspondingly $q_2^E(\boldsymbol{\theta}) < q_2^S$) in Figure 6. The difference in relative surplus between the efficient auction and SSA is measured by the adjustment term $(\theta_1 - \theta_2)q_1^S > 0$ less the Area Y. Relative to the SSA, the efficient auction reduces the price level, measured by the price of good 1, reducing the auctioneer's reference surplus by Area Z.

Figure 6: Auctioneer surplus measured with reference to good 1



The Auctioneer Relative Price Effect with reference to good 1 is the expected difference in relative surplus between the efficient auction and the SSA, that is, the expected size of Area Y. The Auctioneer Price Level Effect with reference to good 1 is the expected difference in reference surplus between the efficient auction and the SSA, that is, the expected size of Area Z.

E Example of Proposition 4

Proposition 10 *The Auctioneer Relative Price Effect with reference to good 2 is positive if the two marginal cost curves are independent, that is,* $\frac{\partial^2 C(q_1,q_2)}{\partial q_j \partial q_k} = 0, j,k \in \{1,2\}, j \neq k$, *and the slopes of the inverse marginal cost curves are log concave.*

Proof. Label the inverse marginal cost curves, in the case in which the marginal cost curves are independent, by $mc_j(p_j), j = \{1,2\}$.

Suppose $mc_1'(p_1)$ and $mc_2'(p_2)$ are log concave. We have $q_1 = \widetilde{mc}_1(\widetilde{p}_1)$ and $\widetilde{MC}_1(q_1) = MC_1(q_1) - MC_2(q_2(q_1))$, so

$$MC_1(\widetilde{mc}_1(\widetilde{p}_1)) - MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))) = \widetilde{p}_1$$

Differentiating w.r.t. $\widetilde{p}_1$, noting $MC_j'(q_j) = \frac{1}{mc_j'(MC_j(q_j))}$, $j = \{1,2\}$, and $q_2'(\widetilde{mc}_1(\widetilde{p}_1)) = -1 \ \forall \widetilde{p}_1$,

$$\begin{aligned} &MC_1'(\widetilde{mc}_1(\widetilde{p}_1))\widetilde{mc}_1'(\widetilde{p}_1) - MC_2'(q_2(\widetilde{mc}_1(\widetilde{p}_1)))q_2'(\widetilde{mc}_1(\widetilde{p}_1))\widetilde{mc}_1'(\widetilde{p}_1) = 1 \\ \implies &\left(\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_1'(MC_1(\widetilde{mc}_1(\widetilde{p}_1)))} + \frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_2'(MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))))} \right) \frac{\widetilde{mc}_1'(\widetilde{p}_1)}{\widetilde{mc}_1(\widetilde{p}_1)} = 1 \end{aligned}$$

$\widetilde{mc}_1(\widetilde{p}_1)$ is log concave iff $\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1'(\widetilde{p}_1)}{\widetilde{mc}_1(\widetilde{p}_1)}\right) \leq 0$. Differentiating w.r.t. $\widetilde{p}_1$,

$$\begin{aligned}&\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_1'(MC_1(\widetilde{mc}_1(\widetilde{p}_1)))}+\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_2'(MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))))}\right)\left(\frac{\widetilde{mc}_1'(\widetilde{p}_1)}{\widetilde{mc}_1(\widetilde{p}_1)}\right)\\&+\left(\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_1'(MC_1(\widetilde{mc}_1(\widetilde{p}_1)))}+\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_2'(MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))))}\right)\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1'(\widetilde{p}_1)}{\widetilde{mc}_1(\widetilde{p}_1)}\right)=0\end{aligned} \tag{4}$$

Noting $\widetilde{mc}_1(\widetilde{p}_1) = q_1$,

$$\begin{aligned}&\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_1'(MC_1(\widetilde{mc}_1(\widetilde{p}_1)))}+\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_2'(MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))))}\right)\\&=\frac{\partial}{\partial q_1}\left(\frac{q_1}{mc_1'(MC_1(q_1))}+\frac{q_1}{mc_2'(MC_2(q_2(q_1)))}\right)\frac{\partial \widetilde{mc}_1(\widetilde{p}_1)}{\partial mc_1}\end{aligned}$$

Define $f_1(q_1) = mc_1'(MC_1(q_1)), f_2(q_1) = mc_2'(MC_2(q_2(q_1)))$ and $g_m(q_1) = \frac{q_1}{f_m(q_1)}$, $h_m(q_1) = f_m(q_1) - q_1 f_m'(q_1), m = \{1, 2\}$.

Then $g_m'(q_1) = \frac{1}{(f_m(q_1))^2}\left(f_m(q_1) - q_1 f_m'(q_1)\right)$ and $h_m'(q_1) = -q_1 f_m''(q_1)$. Then $h_m(0) = f_m(0) > 0, m = \{1, 2\}$, because the two marginal cost curves are strictly increasing.

$mc_1'(p_1) = f_1(mc_1(p_1))$ and $mc_2'(p_2) = f(q_1(mc_2(p_2)))$ are log concave,

$$\begin{aligned}\implies &\frac{\partial}{\partial p_1}\left(f_1'(mc_1(p_1))\right) = \frac{\partial}{\partial p_1}\left(\frac{mc_1''(p_1)}{mc_1'(p_1)}\right) \leq 0,\\&\frac{\partial}{\partial p_2}\left(-f_2'(q_1(mc_2(p_2))))\right) = \frac{\partial}{\partial p_2}\left(\frac{mc_2''(p_2)}{mc_2'(p_2)}\right) \leq 0\\\implies &\frac{\partial}{\partial q_1}\left(f_m'(q_1)\right)\frac{\partial mc_m(p_m)}{\partial p_m} = f_m''(q_1) mc_m'(p_m) \leq 0, m = \{1, 2\}\end{aligned}$$

And $mc_m(p_m), m = \{1, 2\}$ are strictly increasing, so $f_m''(q_1) \leq 0, h_m'(q_1) > 0, m = \{1, 2\}$. And so,

$$\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_1'(MC_1(\widetilde{mc}_1(\widetilde{p}_1)))}+\frac{\widetilde{mc}_1(\widetilde{p}_1)}{mc_2'(MC_2(q_2(\widetilde{mc}_1(\widetilde{p}_1))))}\right) \geq 0$$

In addition, inverse relative marginal cost is increasing in $\widetilde{p}_1$, $\widetilde{mc}_1(\widetilde{p}_1) = q_1$ is always non-negative, and the marginal cost curves are strictly increasing, so $\frac{\partial}{\partial \widetilde{p}_1}\left(\frac{\widetilde{mc}_1'(\widetilde{p}_1)}{\widetilde{mc}_1(\widetilde{p}_1)}\right) \leq 0$ follows from Equation 4. The inverse marginal cost curve is log concave, and the rest of the proof follows from Proposition 4. ■

F Generalised Pareto distribution

Suppose bidders' values have a Generalised Pareto distribution. This implies demand curves of the form

$$d_j(p_j, \theta_j) = \left(1 + \frac{\xi_j(p_j - \theta_j - \mu_j)}{\sigma_j}\right)^{-\frac{1}{\xi_j}}, \ j = \{1, 2\}$$

where $\xi_j < 1$, $\sigma_j > 0$.

I consider the cases in which the demand curves are both concave (and therefore log concave); both log concave but both convex; and both log convex (and therefore convex). The results suggest that the classification described in Table 1 is likely to be a good approximation more broadly.

Concave demand If $\{\xi_1, \xi_2\} < -1$, the demand curves are strictly concave. If the inverse relative marginal cost curve, $\widetilde{mc}_1(\widetilde{p}_1)$, is also log convex (or $\widetilde{MC}_1(q_1) = 0$), the efficient auction reduces expected auctioneer surplus relative to the SSA (Propositions 3 and 4). Under these conditions, the efficient auction must increase expected bidder surplus.

If $\frac{\mu_2}{\sigma_2} \geq \frac{1}{\xi_1}$, the expected price elasticity of demand for good 1 is smaller than that for good 2 at all allocations (see below). Switching to the efficient auction also increases expected bidder surplus if $\{\xi_1, \xi_2\} < -1$, $\widetilde{MC}_1(q_1^S) \geq 0$ and $\frac{\mu_2}{\sigma_2} \geq \frac{1}{\xi_1}$ (Propositions 1 and 2).

Log concave and convex demand If $-1 \leq \{\xi_1, \xi_2\} \leq 0$, the demand curves are log concave and convex. If, in addition, $\frac{\mu_2}{\sigma_2} \geq \frac{1}{\xi_1}$ and $\widetilde{MC}_1(q_1^S) \geq 0$, then the efficient auction increases expected bidder surplus relative to the SSA (Propositions 1 and 2), as described above. It increases expected auctioneer surplus if, in addition to the demand curves being convex, $\widetilde{mc}_1(\widetilde{p}_1)$ is log concave (or $\widetilde{MC}_1(q_1) = 0$) (Propositions 3 and 4).

Log convex demand If $0 \leq \{\xi_1, \xi_2\}$, the demand curves are log convex. If $\frac{1}{\xi_2} \leq \frac{\mu_2}{\sigma_2} \leq \frac{1}{\xi_1} \leq \frac{\mu_1}{\sigma_1}$, or, $\frac{\mu_2}{\sigma_2} \leq \frac{1}{\xi_2} \leq \frac{\mu_1}{\sigma_1} \leq \frac{1}{\xi_1}$, then the expected price elasticity of demand for good 1 is larger than that for good 2 at all allocations (see below). If, in addition, $\widetilde{MC}_1(q_1^S) \leq 0$, then the efficient auction reduces expected bidder surplus relative to the SSA and must therefore increase expected auctioneer surplus.

If, alternatively, $0 \leq \{\xi_1, \xi_2\}$ and $\widetilde{mc}_1(\widetilde{p}_1)$ is log concave, switching to the efficient auction increases expected auctioneer surplus.

Expected price elasticities for demand curves in the GPD class The expected price elasticity of demand for good $j \in \{1, 2\}$ is

$$\mathbb{E}\left[\epsilon_j^D(q_j, \theta_j)\right] = \frac{D_j(q_j, \theta_j) - \theta_j}{\frac{\partial D_j(q_j, \theta_j)}{\partial q_j} q_j} = \left(\frac{1}{\xi_j} - \frac{\mu_j}{\sigma_j}\right) q_j^{\xi_j} - \frac{1}{\xi_j}$$

and

$$\frac{\partial \mathbb{E}\left[\epsilon_j^D(q_j,\theta_j)\right]}{\partial q_j} = \xi_j \left(\frac{1}{\xi_j} - \frac{\mu_j}{\sigma_j}\right) q_j^{\xi_j - 1}$$

The expected price elasticities of demand are negative. If $\{\xi_1, \xi_2\} < 0$, this implies $\frac{1}{\xi_j} - \frac{\mu_j}{\sigma_j} \leq 0 \ \forall j = \{1, 2\}$. Under this condition, the expected price elasticities are therefore both increasing in the quantity sold. And so

$$\mathbb{E}\left[\epsilon_1^D(0,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1,\theta_2)\right] \geq 0 \Longrightarrow \mathbb{E}\left[\epsilon_1^D(q_1,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1-q_1,\theta_2)\right] \geq 0 \forall q_1 \in [0,1]$$

We have $\mathbb{E}\left[\epsilon_1^D(0,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1,\theta_2)\right] = -\frac{1}{\xi_1} + \frac{\mu_2}{\sigma_2}$. So, if $\{\xi_1, \xi_2\} < 0$ and $\frac{\mu_2}{\sigma_2} \geq \frac{1}{\xi_1}$, then $\mathbb{E}\left[\epsilon_1^D(q_1,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1-q_1,\theta_2)\right] \geq 0 \forall q_1 \in [0,1]$.

Now turn to the case of $\{\xi_1, \xi_2\} > 0$. If either (i) the expected price elasticities are both increasing in the quantity sold and $\mathbb{E}\left[\epsilon_1^D(1,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(0,\theta_2)\right] \leq 0$ or (ii) the expected price elasticities are both decreasing in the quantity sold and $\mathbb{E}\left[\epsilon_1^D(0,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1,\theta_2)\right] \leq 0$, then $\mathbb{E}\left[\epsilon_1^D(q_1,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1-q_1,\theta_2)\right] \leq 0 \forall q_1 \in [0,1]$. We have

$$\begin{aligned} \mathbb{E}\left[\epsilon_1^D(1,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(0,\theta_2)\right] &= \frac{1}{\xi_2} - \frac{\mu_1}{\sigma_1} \\ \mathbb{E}\left[\epsilon_1^D(0,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1,\theta_2)\right] &= -\frac{1}{\xi_1} + \frac{\mu_2}{\sigma_2} \\ \frac{\partial \mathbb{E}\left[\epsilon_j^D(q_j,\theta_j)\right]}{\partial q_j} & \begin{cases} \geq 0 \forall q_j & \text{if } \left(\frac{1}{\xi_j} - \frac{\mu_j}{\sigma_j}\right) \geq 0 \\ \leq 0 \forall q_j & \text{if } \left(\frac{1}{\xi_j} - \frac{\mu_j}{\sigma_j}\right) \leq 0 \end{cases} \end{aligned}$$

for $j \in \{1, 2\}$. And so, $\mathbb{E}\left[\epsilon_1^D(0,\theta_1)\right] - \mathbb{E}\left[\epsilon_2^D(1,\theta_2)\right] \leq 0$ if (i) $\frac{1}{\xi_1} \geq \frac{\mu_1}{\sigma_1}$, $\frac{1}{\xi_2} \geq \frac{\mu_2}{\sigma_2}$ and $\frac{\mu_1}{\sigma_1} \geq \frac{1}{\xi_2}$ or (ii) $\frac{1}{\xi_1} \leq \frac{\mu_1}{\sigma_1}$, $\frac{1}{\xi_2} \leq \frac{\mu_2}{\sigma_2}$, and $\frac{\mu_2}{\sigma_2} \leq \frac{1}{\xi_1}$. Or equivalently, either

$$\text{(i) } \frac{\mu_2}{\sigma_2} \leq \frac{1}{\xi_2} \leq \frac{\mu_1}{\sigma_1} \leq \frac{1}{\xi_1}; \text{ or } \text{ (ii) } \frac{1}{\xi_2} \leq \frac{\mu_2}{\sigma_2} \leq \frac{1}{\xi_1} \leq \frac{\mu_1}{\sigma_1}$$

G Mexican government debt auctions: maximising auctioneer surplus

Table 5 show results for the case in which the objective is to maximise auctioneer surplus. The PMA naturally benefits the auctioneer but average bidder surplus is lower in the PMA than in the original design, unlike in the main analysis. Nonetheless, the PMA still increases total welfare.

Table 5: Surplus estimates and model predictions in CETES primary auctions given an objective to maximise auctioneer surplus

		28- & 91-day	182- & 364-day
Bond 1	(term, days)	91	364
Bond 2	(term, days)	28	182
Issuance of bond 1 in SSA	(mean, % of total)	39.223	55.455
		(0.639)	(1.283)
	(std. dev., % of total)	7.545	10.136
		(0.598)	(0.933)
Bidder surplus in SSA	(bps of issuance)	0.750	2.754
		(0.036)	(0.268)
	(MEX$ million per auction)	1.213	6.427
		(0.056)	(0.659)
Difference between counterfactual PMA and actual SSA:			
Δ Issuance of bond 1	(mean, % of total)	10.676***	2.316
		(1.954)	(2.972)
	(std. dev., % of total)	23.235***	23.409***
		(1.138)	(1.869)
Δ Welfare	(bps of issuance)	0.243***	1.992***
		(0.039)	(0.475)
	(MEX$ million per auction)	0.393***	4.649***
		(0.064)	(1.127)
Δ Bidder surplus	(bps of issuance)	-0.146***	-0.254
		(0.025)	(0.204)
	(MEX$ million per auction)	-0.237***	-0.593
		(0.039)	(0.475)
Δ Auctioneer surplus	(bps of issuance)	0.389***	2.246***
		(0.033)	(0.383)
	(MEX$ million per auction)	0.630***	5.242***
		(0.054)	(0.909)

Note: bps denote basis points. Surpluses measured in bps are weighted by total quantities issued and surpluses measured in MEX$ are unweighted. Standard errors are in parentheses. The level of welfare and auctioneer surplus in the actual SSA, which depend on the auctioneer's unobserved total benefit from issuing the total supply, are unidentified. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

H Mexican government debt auctions: alternative estimation method

In the main analysis, I estimate the distribution of residualised bids (equal to the difference between bids and the secondary market price) that the government expects to face in auction t by sampling with replacement from the set of bids that bidders submit in auction $t-1$. By pooling bids from just one auction, this approach is consistent even if there is unobserved heterogeneity in the distribution of bids across auctions. Nonetheless, by resampling from the set of individual bidders, I assume that their valuations are identically distributed.[76]

In this appendix, I use an alternative resampling procedure to estimate the differences in surpluses between the counterfactual PMA and the actual mechanism. The approach relaxes the assumption that bidders' valuations are identically distributed, but assumes that the distribution of residualised bids is stable over time.

The set of $T-1$ auctions in the sample, which excludes auction t, represent a set of $T-1$ realisations of the aggregate bid functions for bonds $j = \{1, 2\}$. I simulate one possible realisation of residualised bids by drawing the pair of aggregate bid functions for bonds $j = \{1, 2\}$ from this sample, and subtracting the secondary market prices on the day of the corresponding auction. Repeating this procedure a large number of times simulates the distribution of residualised bids the government expects to face. The method is otherwise identical to the main analysis.

The results using this method for the cases in which the objectives are to maximise welfare and auctioneer surplus, respectively, are shown in Tables 6 and 7.

[76] A third alternative approach would be to group bidders into a finite number of groups and resample within group.

Table 6: Surplus estimates and model predictions in CETES primary auctions given an objective to maximise welfare, under the alternative estimation method

		28- & 91-day	182- & 364-day
Bond 1	(term, days)	28	364
Bond 2	(term, days)	91	182
Issuance of bond 1 in SSA	(mean, % of total)	39.533	53.775
		(0.405)	(0.914)
	(std. dev., % of total)	5.177	7.824
		(0.323)	(0.576)
Bidder surplus in SSA	(bps of issuance)	0.767	2.709
		(0.037)	(0.205)
	(MEX$ million per auction)	1.248	6.362
		(0.058)	(0.512)
Difference between counterfactual PMA and actual SSA:			
Δ Issuance of bond 1	(mean, % of total)	1.735	0.121
		(2.064)	(3.453)
	(std. dev., % of total)	26.599***	29.798***
		(1.099)	(1.789)
Δ Welfare	(bps of issuance)	0.220***	1.027***
		(0.031)	(0.141)
	(MEX$ million per auction)	0.357***	2.411***
		(0.051)	(0.330)
Δ Bidder surplus	(bps of issuance)	0.063*	0.855***
		(0.026)	(0.248)
	(MEX$ million per auction)	0.102*	2.008***
		(0.043)	(0.580)
Δ Auctioneer surplus	(bps of issuance)	0.157***	0.172
		(0.033)	(0.231)
	(MEX$ million per auction)	0.255***	0.403
		(0.054)	(0.543)
Bidder Uncertainty Effect	(bps of issuance)	0.081**	0.923***
		(0.030)	(0.240)
	(MEX$ million per auction)	0.132**	2.168***
		(0.049)	(0.562)
Bidder Allocation Effect	(bps of issuance)	-0.018	-0.068
		(0.013)	(0.039)
	(MEX$ million per auction)	-0.030	-0.160
		(0.021)	(0.093)

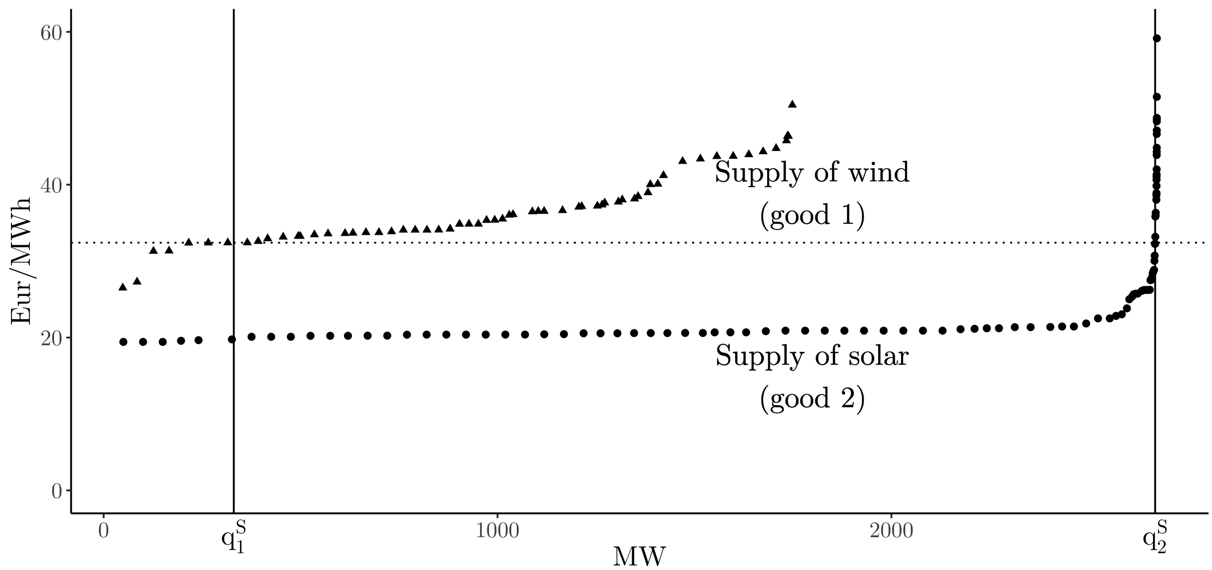
Note: bps denote basis points. Surpluses measured in bps are weighted by total quantities issued and surpluses measured in MEX$ are unweighted. Standard errors are in parentheses. The level of welfare and auctioneer surplus in the actual SSA, which depend on the auctioneer's unobserved total benefit from issuing the total supply, are unidentified. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

Table 7: Surplus estimates and model predictions in CETES primary auctions given an objective to maximise auctioneer surplus, under the alternative estimation method

		28- & 91-day	182- & 364-day
Bond 1	(term, days)	91	364
Bond 2	(term, days)	28	182
Issuance of bond 1 in SSA	(mean, % of total)	39.585	53.893
		(0.427)	(0.893)
	(std. dev., % of total)	5.483	7.643
		(0.320)	(0.524)
Bidder surplus in SSA	(bps of issuance)	0.775	2.674
		(0.039)	(0.211)
	(MEX$ million per auction)	1.261	6.280
		(0.061)	(0.527)
Difference between counterfactual PMA and actual SSA:			
Δ Issuance of bond 1	(mean, % of total)	7.067***	2.115
		(1.875)	(2.464)
	(std. dev., % of total)	23.884***	21.463***
		(1.153)	(1.558)
Δ Welfare	(bps of issuance)	0.193***	0.712***
		(0.029)	(0.140)
	(MEX$ million per auction)	0.314***	1.672***
		(0.048)	(0.329)
Δ Bidder surplus	(bps of issuance)	-0.188***	-0.461**
		(0.023)	(0.150)
	(MEX$ million per auction)	-0.306***	-1.084**
		(0.036)	(0.353)
Δ Auctioneer surplus	(bps of issuance)	0.382***	1.173***
		(0.036)	(0.150)
	(MEX$ million per auction)	0.621***	2.756***
		(0.058)	(0.355)

Note: bps denote basis points. Surpluses measured in bps are weighted by total quantities issued and surpluses measured in MEX$ are unweighted. Standard errors are in parentheses. The level of welfare and auctioneer surplus in the actual SSA, which depend on the auctioneer's unobserved total benefit from issuing the total supply, are unidentified. * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

Figure 7: Aggregate supply curves for wind and solar energy



Supply curves of wind and solar energy, estimated by Fabra and Montero (2023) using data on potential investment projects in Spain with applications for planning permission at the Registry of Renewables Installations in Spain (RIPRE) in January 2019 until March 2020.

I Spanish renewable energy procurement: model predictions

The large number of current and planned auctions for renewable energy contracts across a range of countries (IRENA, 2013) exhibit large variation in design, with a broad trend towards multi-product auctions of multiple technologies (Szabó et al., 2020). In these energy procurement auctions, the auctioneer acts as the buyer and the bidders act as sellers. This is a mirror image of the main model, and the results are of course unchanged. In many cases, the auctions are special cases of PMAs, either with the two technologies treated interchangeably, or with a demand curve expressing the government's preference for diversified procurement.

I illustrate the model's predictions for Fabra and Montero's (2023) analysis of Spanish renewable energy auctions held in January 2021, in which solar and wind producers were permitted to bid for energy contracts within a single auction. The government's total demand was fixed, and in the neighbourhood of the observed quantities procured of the two goods, the two types of energy were treated interchangeably, i.e. the auctioneer viewed the two goods as perfect substitutes.[77]

Using data on 2019 applications for renewable energy projects, Fabra and Montero (2023) estimate the aggregate bid functions, i.e. the supply curves, for solar and wind energy in Spain, which I use to determine the properties of the supply curves relevant to the model

[77] The auction rules specified minimum quotas reserved for each energy type, but the auction outcome triggered neither quota. Note the minimum quota design can be viewed as a PMA in which the demand curve is a step function.

predictions. Figure 7 shows these bid functions in the case of zero shocks.

While the supply curve for wind is approximately linear, the inverse supply curve for solar is convex in the neighbourhood of the observed pair of quantities procured, so wind is labelled good 1 in the model by Lemma 3 in Appendix B. Given the supply estimates, the PMA run by the Spanish government therefore increases the expected quantity procured of wind energy relative to the pair of separate simultaneous auctions that would maximise expected welfare.

Both the Auctioneer Price Level Effect and Auctioneer Relative Price Effect with reference to good 1 (wind) equal zero, given the linearity of the supply curve for wind and the auctioneer's perfect substitute preferences. So the model predicts that the government is indifferent between the PMA and running separate auctions.

The supplies of both goods are log concave, so that bidders prefer the uncertainty in the quantities procured of the two goods in the PMA, and, by Proposition 1, the Bidder Uncertainty Effect is positive.[78]

Because of the auctioneer's indifference, the positive Bidder Uncertainty Effect must dominate so that the PMA increases expected bidder surplus, as it increases expected welfare.

So, in this setting, the model predicts that the entirety of the welfare gain of the PMA translates into higher expected bidder surplus.

[78]The Bidder Allocation Effect is ambiguous. At the quantities procured in the SSA in Fabra and Montero's (2023) simulation (denoted by q_1^S and q_2^S in Figure 7, which are the pair that maximises expected surplus at $\theta_1 - \theta_2 = \mathbb{E}[\theta_1 - \theta_2]$) the supply of wind energy is more price elastic than the supply of solar and the auctioneer's relative demand for good 1 is zero at the quantities procured in the SSA (given the auctioneer views the two goods as perfect substitutes). Proposition 2 therefore does not apply as the supply curves are log concave.