Risk and Liquidity in a System Context

Hyun Song Shin

September 2005
Pricing claims in a system context

Some assets (e.g. loans) are claims against other parties

Value of my claim against $A$ depends on value of $A$’s claims against $B, C,$ etc.

But $B$ or $C$ may have claim against me.

Balance sheet strength, spreads, asset prices fluctuate together

Equity value of financial system as a whole is value of “fundamental assets”
“While many believe that irresponsible borrowing is creating a bubble in housing, this is not necessarily true. At the end of 2004, U.S. households owned $17.2 trillion in housing assets, an increase of 18.1% (or $2.6 trillion) from the third quarter of 2003. Over the same five quarters, mortgage debt (including home equity lines) rose $1.1 trillion to $7.5 trillion. The result: a $1.5 trillion increase in net housing equity over the past 15 months.”

Value of fundamental assets is tide that lifts all boats

Housing $\Rightarrow$ mortgages $\Rightarrow$ CDOs $\Rightarrow$ claims against CDO holders . . .
Balance Sheet Approach

Financial system is a network of interlinked balance sheets

Everything is marked to market

Risk-neutrality in pricing

• no role for risk aversion, but spreads fluctuate due to fluctuations in fundamental asset price

• fluctuations in risk appetite arising from solvency constraints

Spreads can fall when debt rises (“reaching for yield”).

Spreads can rise when debt falls (financial crises).
Related literature

- Balance sheet propagation


Framework

- $n$ entities in financial system
- risky endowments realized at date $T$ with means $\{w_i\}$
- single fundamental asset, price $\nu$
- zero coupon debt of $i$ with face value $\bar{x}_i$ payable at $T$
- risk-free interest rate is zero
Balance Sheets

\( x_i \) is market value of \( i \)'s debt

\( a_i \) is market value of \( i \)'s assets

\( e_i \) is market value of \( i \)'s equity

\[ a_i = e_i + x_i \]

If \( i \) holds proportion \( \pi_{ji} \) of \( j \)'s debt,

\[ a_i = w_i + vy_i + \sum_j \pi_{ji}x_j \]
Merton (1974)

\[ x_i = f_i(a_i, \theta) \]

Figure 1: Market value of total debt \( x_i \) of investor \( i \)
Lemma 1. There exist functions \( \{f_i\} \) such that

\[
x_i = f_i(a_i, \theta)
\]  

(1)

where each \( f_i \) is non-decreasing in \( a_i \), and is bounded above by \( \bar{x}_i \) and

\[
\theta = (v, w, \bar{x})
\]

Lemma 2. The market value of equity is non-decreasing in \( a_i \). That is, the function \( e_i \) defined as

\[
e_i \equiv a_i - \sum_j f_i(a_i, \theta)
\]  

(2)

is non-decreasing in \( a_i \).
System

\[ x_1 = f_1 (a_1 (x), \theta) \]
\[ x_2 = f_2 (a_2 (x), \theta) \]
\[ \vdots \]
\[ x_n = f_n (a_n (x), \theta) \]

where \( x = (x_1, x_2, \ldots, x_n) \).

Solve for fixed point \( x \) in:

\[ x = F (x, \theta) \]
Iterative approach

\[ x^1 = F(0, \theta) \]
\[ x^{t+1} = F(x^t, \theta) \]

“Pessimistic” case

\[ 0 \leq x^1 \leq x^2 \leq x^3 \leq \cdots \]

“Optimistic” case

\[ x^1 = F(\bar{x}, \theta) \]

\[ \bar{x} \geq x^1 \geq x^2 \geq x^3 \geq \cdots \]

Are the limits the same?
Unique solution

**Theorem 3.** There is a unique profile of debt prices $x(\theta)$ that solves $x = F(x, \theta)$.

**Theorem 4.** $x(\theta)$ is increasing in $\theta$.

Result follows from

(i) Tarski’s fixed point theorem

(ii) fact that $\{f_i\}$ are contraction mappings
A complete lattice is partially ordered set \((X, \leq)\) such that each subset \(S \subseteq X\) has both a greatest lower bound \(\inf(S)\) and a least upper bound \(\sup(S)\) in the set \(X\).

In our context, complete lattice with the set \(X\) given by

\[
X \equiv [0, \bar{x}_1] \times [0, \bar{x}_2] \times \cdots \times [0, \bar{x}_n]
\]

and ordering \(\leq\) given by the usual component-wise order.

**Lemma 5.** (Tarski’s Fixed Point Theorem) Let \((X, \leq)\) be a complete lattice and \(F\) be a non-decreasing function on \(X\). Then there are \(x^*\) and \(x_*\) such that \(F(x^*) = x^*\), \(F(x_*) = x_*\), and for any fixed point \(x\), we have \(x_* \leq x \leq x^*\).
Proof. Define the set $S$ as

$$S = \{ x | x \leq F(x) \}$$

(3)

and define $x^*$ as $x^* \equiv \sup S$. For any $x \in S$, $x \leq x^*$. Since $F$ is non-decreasing, $x \leq F(x) \leq F(x^*)$. Thus, $F(x^*)$ is also an upper bound for $S$. But $x^*$ is defined as the least upper bound of $S$. Thus

$$x^* \leq F(x^*)$$

(4)

Applying $F$ to both sides of (4), we have $F(x^*) \leq F(F(x^*))$. But this implies that $F(x^*) \in S$, so that $F(x^*)$ is bounded by $x^*$. That is, $F(x^*) \leq x^*$. Taken together with (4), this means that $F(x^*) = x^*$. Any other fixed point of $F$ must belong to $S$, and so $x^*$ is the largest fixed point. The smallest fixed point $x_*$ is defined as $\inf \{ x | x \geq F(x) \}$, and the argument is exactly analogous.
Argument for Uniqueness

Suppose there are distinct solutions $x, x'$.  

By Tarski, $x \leq x'$ and $x_i < x'_i$ for some $i$  

Equity value of the system under $x$ is strictly lower than under $x'$  

Equity value of the system is value of fundamental assets  

Contradiction.
Solvency Constraints

Value of all assets and liabilities determined by

$$\theta = (v, w, \bar{x})$$

Constraints on equity/debt ratio

$$\frac{a_i - x_i}{x_i} \geq r^*$$

Spreads

$$1 - \frac{x_i}{\bar{x}_i}$$
\[
\bar{x}
\]

\[\theta', \theta\]

solvency region

\[\nu, w\]
Restoring Solvency
Two Scenarios for Spreads

\[ \bar{x} \]

\[ \theta'' \quad \theta \]

\[ \theta' \]

\[ \mathcal{V}, \mathcal{W} \]
Sale $s_i$ to restore solvency

\[
\frac{w_i + v (y_i - s_i) + b_i - (x_i^0 - vs_i)}{x_i^0 - vs_i} \geq r^* \tag{5}
\]

\[
s_i = \min \left\{ y_i, \max \left\{ 0, \frac{(1 + r^*) x_i^0 - w_i - vy_i - b_i}{r^* v} \right\} \right\} \tag{6}
\]
Minimum Leverage Constraint

Constraints on equity/debt ratio

\[ \frac{a_i - x_i}{x_i} \leq r^{**} \]
Scenarios for Spreads and Reaching for Yield
Example of Housing

- Property Price
- Supply of property from old
- Property stock held by young
Applications and Extensions

- Value at risk
- Financial stability
- Correlations in downturns
- “Risk appetite”
- Seniority structure
- Pricing of credit derivatives