Legislative Bargaining
(over discrete bills)*

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Abstract

We provide a theory of legislative bargaining in a setting where the institution-free model of social choice theory yields no equilibrium. We model negotiations over a set of discrete bills (i.e. bills that can only be approved or dismissed) as a repeated game where a status quo is amended until an agreement is reached. Our goal is to understand when logrolling or vote trading agreements occur. In a stylised setting with three voters and two bills, we find that negotiations may lead to three outcomes: (1) the majoritarian one; (2) the logrolling one where two players vote against their position on their least preferred bill in order to gain support for their most preferred bill; and (3) one where the most affected voter by the logrolling outcome, in order to avoid such outcome, agrees to implement the wills of another voter.

JEL Classification: C72, C78, D72

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1 Introduction

Legislative bodies (as well as committees) need to reach joint decisions over multiple bills. Normally, legislators negotiate by giving up on unimportant bills in order to gain support towards those bills they feel strongly about. Regardless the common practice of such arrangements and the empirical evidence provided by Stratmann (1992) we still lack a rigorous theoretical analysis that would explain their observed outcomes.

Whenever there is a set of bills that need to be approved or dismissed, we usually see the outcome of these negotiations through legislators switching or trading their votes. This phenomenon is known in the political science literature as logrolling. Logrolling is defined as the exchanging of political favours, especially the trading of influence or votes among legislators to achieve passage of projects that are of interest to one another. ‘The phrase is likely derived from the old custom of neighbours assisting each other with the moving of logs. If two neighbours had cut a lot of timber which needed to be moved, it made more sense for them to work together to roll the logs – if you’ll help me roll my logs, I’ll help you roll yours.’

The following motivating example perfectly captures the weaknesses of our theory around the bilateral agreements that occur in group decision making over discrete bills. There are three voters that have to decide over the approval or dismissal of two independent bills yielding the following payoffs:

<table>
<thead>
<tr>
<th></th>
<th>first bill</th>
<th>second bill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>approved</td>
<td>dismissed</td>
</tr>
<tr>
<td>voter 1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>voter 2</td>
<td>0</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>voter 3</td>
<td>0</td>
<td>(\frac{1}{4})</td>
</tr>
</tbody>
</table>

The table captures a situation where a majority of voters wishes the dismissal of both bills. However, the first two voters may wish to logroll. That is, they may wish to jointly favour each one’s preferred outcome so that both bills are approved. Nevertheless, this outcome is not stable because the third voter has now incentives to form a coalition with any of the other two voters and implement his preferred outcome in both bills... From a cooperative game theoretical perspective the previous game is characterised by having an empty-core. Precisely, Riker and Brams (1973) show that whenever a logrolling agreement can occur the voters preferences are such that they induce a cycle and no equilibrium ever exists. Their work is circumscribed in the response and lively debate, among logrolling supporters and detractors, that followed the seminal work by Buchanan and Tullock (1962).\(^2\)

\(^1\)http://en.wikipedia.org/wiki/Logrolling
The only attempt in generally analysing the logrolling phenomenon starts from Mueller (1967) and consists on assuming a centralised market for votes. However, these models fail to take into account the institutional arrangements that determine the circumstances under which vote trading occurs.

The existing literature on legislative bargaining has mainly looked at distributive politics or ways to allocate resources among different dimensions. It essentially differs from our setting by assuming a continuous policy space rather than a discrete one where bills can only be approved or dismissed. Bernheim et al (2006) provide a model with a finite policy space that very much resembles our setting apart from the fact that they consider a finite horizon and they allow voters to condition their behaviour on prior actions. Both conditions imply that, under very general conditions, the last proposer manages to implement his most preferred policy. As will be shown below, departing from both their assumptions yields opposite results.

A negotiation is a process of joint decision making. It is communication direct or tacit, between individuals who are trying to forge an agreement for mutual benefit. We want to think of the negotiation that leads to a logrolling outcome as a repeated process where parties can amend the status quo until no further changes are proposed. In this paper, we consider the simplest situation where logrolling can occur, i.e. a generalised version of the example above with 3 voters and 2 bills. Even if our setting seems simple it encompasses most situations where logrolling agreements occur. The impossibility of writing contracts on such agreements, together with the fact that issues are hardly ever voted simultaneously, make complex arrangements unfeasible. Additionally, we can interpret our setting more generally where a faction of voters needs support of at least another faction to pass a set of bills.

Having two bills implies that we have four possible alternatives. Voters engage in a negotiation, modelled as a sequential game where any alternative can (with equal probability) challenge the status quo. Its majoritarian winner becomes the status quo for the next period. Whenever voters do no longer wish to amend the status quo, such alternative is implemented. We also assume that the negotiation can exogenously end at any period; in which case the status quo is implemented. The rate at which this happens is exogenously given and can be arbitrarily small. This modelling feature allows us to take into account the negotiation’s stream of payoffs. Its absence rend the negotiation superfluous – its only relevant aspect is the final outcome and anything can be implemented in equilibrium.

After presenting the model in Section 2, in Section 3 we show that voters only wish

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4 See for instance Shepsle (1979), Baron and Ferejohn (1989) and Duggan and Kalandrakis (2006). There is also some literature on the cooperative aspects of legislative bargaining. For instance, Laver and Shapsle (1990) show the constraints that parties’ preferences impose in the coalitions that can emerge in equilibrium.
5 Young (1991), page 1.
to stop the negotiation at a Condorcet Winner. That is, an alternative that wins all pairwise comparisons whenever voters are farsighted and take into account the future consequences of any present choice.\footnote{See Penn (2006) for a characterisation of sufficient conditions under which dynamically stable voting equilibria (not necessarily Condorcet Winners) exist when voters are farsighted.} We then proceed to characterise all equilibria with Condorcet Winner in Section 4. Our rationale and modelling techniques are based in Roberts (2005) where he proves that Condorcet cycles may be avoided whenever voters implement one of three alternatives over time (a very patient subject that is almost indifferent between his two top alternatives may prefer to always implement his second best alternative rather than cycling around the three available ones). Apart from the higher complexity of our four alternatives model, there is a crucial difference in the way we interpret the repeated interaction among voters: Roberts (2005) considers that at each round (of our negotiation) a social state must be chosen and implemented; instead, we interpret each round of negotiation as an abstraction that captures the way negotiators may propose different alternatives from which only one is implemented.\footnote{In a similar vein, Bernheim and Nataraj (2004) analyse a repeated majoritarian election over a convex policy space from a non-game theoretic perspective. Their focus relies on analysing which sequence of history dependent outcomes can be supported as Dynamic Condorcet Winners and providing a Folk Theorem when agents are patient enough. In our setting, instead, we do not allow for history dependent strategies and we analyse the strategic interactions in a more structured environment (motivated by stylised institutional arrangements).}

There are three possible Condorcet Winners: (1) the alternative that reflects the majoritarian wills; (2) the logrolling one where two parties agree to vote against their own position in their least preferred bill; and (3) one where the most affected voter by the logrolling outcome (voter 3 in the previous example), in order to avoid such outcome, agrees to implement the preferred alternative of another voter. The latter constitutes our main contribution to the literature given that such outcome has always been disregarded when analysing the logrolling phenomenon.\footnote{The threat of an outcome being the driving force of an apparently implausible equilibrium is not new in economics. Within the political economy literature, see for instance Palfrey (1984) where the threat of entry of a third party can lead parties not to converge to the median policy.} We characterise the existence of each outcome in terms of the voters’ relative intensity.

Finally, in Section 5 we discuss the presence of equilibria without CW and the effects of introducing costly negotiation in our setting. Section 6 concludes.

\section{The model}

Three voters \((i = 1, 2, 3)\) need to decide over two bills that can only be approved or dismissed. We assume that there is scope for logrolling. That is, there are two voters that have opposing preferences over two bills which they rank differently (i.e. the bill they most care about does not coincide); moreover, the majoritarian will is such that their
will on their most preferred bill is not implemented. The following table captures the described situation \((m_1, m_2, m_3 \in (0, 1))\).\(^9\)

<table>
<thead>
<tr>
<th></th>
<th>first bill</th>
<th>second bill</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>approved</td>
<td>dismissed</td>
</tr>
<tr>
<td>voter 1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>voter 2</td>
<td>0</td>
<td>(m_2)</td>
</tr>
<tr>
<td>voter 3</td>
<td>0</td>
<td>(m_3)</td>
</tr>
</tbody>
</table>

The payoff matrix encompasses all possible scenarios where there is scope for logrolling. The normalisation of payoffs is done without loss of generality, and \(m_3 \in (0, 1)\) is also done without loss of generality given that the situation \(m_3 > 1\) could easily be rewritten in the above form by relabelling the bills.

We define four possible alternatives in our framework: alternative ‘aa’ corresponds to both bills being approved; alternative ‘dd’ corresponds to dismissing both bills; alternative ‘ad’ corresponds to approving the first bill and dismissing the second one; and, finally, alternative ‘da’ corresponds to approving the second bill and dismissing the first one. The set of alternatives is denoted \(A := \{aa, dd, ad, da\}\) and elements from this set are usually denoted with capital letters \(A, B, C, \ldots\)

The following table captures the voters preferences in terms of the previous alternatives (the obtained utility at each alternative has been linearly transformed and simplified using \(\lambda_i := \frac{m_i}{1 + m_i} \in (0, \frac{1}{2})\)).

<table>
<thead>
<tr>
<th></th>
<th>obtained utility:</th>
</tr>
</thead>
<tbody>
<tr>
<td>voter 1</td>
<td>1 (1 - \lambda_i) (\lambda_i) 0</td>
</tr>
<tr>
<td>voter 2</td>
<td>(da \succ ad \succ da)</td>
</tr>
<tr>
<td>voter 3</td>
<td>(dd \succ ad \succ da \succ aa)</td>
</tr>
<tr>
<td></td>
<td>(ad \succ aa \succ dd \succ da)</td>
</tr>
</tbody>
</table>

The cyclic relation that arises from the use of Majority Rule is depicted in Figure 1. The arrow between any two alternatives indicates which one wins the pairwise comparison.

The negotiation has an infinite horizon. At each round of the negotiation there is a status quo inherited from the previous round. Any alternative different than the status quo is equally likely to become its challenger.\(^10\) The winner is decided by majority voting and becomes the status quo for next period.\(^11\)

\(^9\) All our results are robust to any linear transformation of the payoffs.

\(^{10}\) We cannot trivially drop the assumption that any of the remaining alternatives is selected with equal probability. McKelvey (1976, 1979) and, more recently, Bernheim et al (2006), show the dictatorial power that is endowed to the agenda setter.

\(^{11}\) Few models of legislative bargaining assume the realistic feature that the status quo alternative can
Inspired by most real world scenarios we assume that voters can propose amendments to the status quo and the negotiation only end when no more amendments are proposed. In other words, the negotiation only ends when voters unanimously wish so.

Voter $i$’s decision towards continuing or stopping the negotiation at node $A \in \mathcal{A}$ is denoted $\Sigma_i^A \in \{0, 1\}$, where $\Sigma_i^A = 1$ denotes his wish to stop the negotiation at alternative $A$ and $\Sigma_i^A = 0$ denotes his will to amend the status quo and continue the negotiation. We denote $\Sigma_i$ the set of voter $i$'s stopping decisions, i.e. $\Sigma_i := \{\Sigma_i^A \text{ for } A \in \mathcal{A}\}$. Note that we are only allowing pure, and Markovian (time and history independent) strategies.\footnote{We see the lack of commitment in the political arena as an unambiguous evidence that non-Markovian strategies may be too costly to implement in such scenarios.} Finally, the unanimous rule associated to the stopping decisions can be expressed as the product of the individual decisions: $\Sigma^A := \Sigma_1^A \cdot \Sigma_2^A \cdot \Sigma_3^A$, $\forall A \in \mathcal{A}$.

We assume that the negotiation may end at any period. If so occurs, the status quo at the time is implemented. The rate at which such event happens is exogenously given by the parameter $\beta \in (0, 1)$.

This last assumption captures the fact that the path through which an outcome is reached in a negotiation is not irrelevant. Going through a particular alternative entails a risk by any person that very much opposes such alternative. In the Appendix we show that dropping this condition transforms each alternative into a fictitious one given that the only relevant payoff is the one where the negotiations ends. Moreover, in such scenario any alternative can be supported as an outcome of the negotiation. A strictly positive (but arbitrarily small) $\beta$ allows us to consider the stream of payoffs through which the negotiation evolves and renders its analysis meaningful.

At each round of the negotiation players face a pairwise comparison between the status quo $A \in \mathcal{A}$ and an alternative $B \in \mathcal{A} \setminus A$; a strategy for voter $i$ captures his voting intention evolve over time. Among them are Bernheim et al (2006) and Duggan and Kalandrakis (2006).
in any pairwise comparison and is denoted $V_{i}^{AB} \in \{A, B\}$. The set of all such comparisons is denoted $V := \{V_{i}^{AB} \text{ for } A, B \in \mathcal{A}, A \neq B\}$. $V_{AB}$ denotes the majoritarian will when $A$ is challenged with $B$, i.e. $V_{AB} = A \iff \# \{i : V_{i}^{AB} = A, i = 1, 2, 3\} \geq 2$. Finally, $v_i^A$ denotes the instantaneous payoff that player $i$ gets from alternative $A$. For instance, $v_1^{aa} = 1 - \lambda_1$.

The timing of each round of the negotiation follows.

1. $A \in \mathcal{A}$ is the status quo.
2. Voters cast their vote regarding the stopping decision.
3. If $\Sigma^X = 1$ the negotiation terminates and the status quo is implemented.
4. With probability $\beta$ the negotiation terminates and the status quo is implemented.
5. Any alternative from the set $\mathcal{A} \setminus A$ is equally likely to become the challenger, $B$.
6. Voters cast their vote $V_{i}^{AB}$ and the majoritarian winner becomes the status quo for the next period.

The timing and payoff matrix above define a game with three players whose strategy set is denoted by $\mathcal{V} \times \Sigma = (\mathcal{V}_1 \times \Sigma_1) \times (\mathcal{V}_2 \times \Sigma_2) \times (\mathcal{V}_3 \times \Sigma_3)$. Given everybody’s strategy, the pairwise comparisons between any two alternatives and the stopping decisions do not only depend on the instantaneous utility of both alternatives but on their continuation utilities. In other words, voters are farsighted and take into account the future consequences of any choice. The expected utility at node $A$ for player $i$ is

$$U_i(A) = \begin{cases} v_i^A & \text{if } \Sigma^A = 1 \\ \beta v_i^A + \frac{1-\beta}{3} \left[ U_i(V_{AB}) + U_i(V_{AC}) + U_i(V_{AD}) \right] & \text{if } \Sigma^A = 0 \end{cases}$$

where $B$, $C$, and $D$ denote the three elements from $\mathcal{A} \setminus A$.

We restrict our attention to weakly dominant strategies when characterising the equilibria of our game. We assume that players vote truthfully irrespective of whether their vote is pivotal. In such a way we avoid ill behaved equilibria where, for instance, all players vote in the same manner and no one is ever pivotal. Moreover, we avoid non-credible threats on the voters’ strategies. Finally, we also assume that voters only vote towards continuing the negotiation if they strictly prefer so.

**Definition 1** An equilibrium of the game described above defined by $m_1, m_2, m_3, \beta \in (0, 1)$ is given by a set of pairwise comparisons and continuation decisions for all players such that, for all $i = 1, 2, 3$ the following two conditions hold.
1. \( \forall A, B \in \mathcal{A}, V_i^{AB} = B \text{ if and only if } U_i(A) < U_i(B) \)

2. \( \forall A \in \mathcal{A}, \Sigma_i^A = 0 \text{ if and only if } v_i^A < U_i(A | \Sigma^A = 0) \)

The extra bit of notation introduced in condition 2 explicitly recalls that whenever voter \( i \) switches his stopping decision, he may be affecting the value of \( U_i(A) \). More specifically, whenever \( \Sigma_i^{AB} = 1 \) and voter \( i \) considers what happens if he votes in favour of continuing the negotiation, the expected utility should take into account that \( \Sigma^A = 0 \) not solely in the current period but also in all subsequent ones.

As is common in the literature, we define a *Condorcet Winner* (CW) as an alternative that wins all pairwise comparisons, i.e. \( A \) is a CW if and only if \( V^{AB} = A, \forall B \in \mathcal{A} \setminus A \).

### 3 On the relevance of condorcet winners

CW are specially appealing in our setting because they are the only alternatives where voters unanimously wish to stop the negotiation. Whenever there is (at least) an alternative that wins a pairwise comparison with the status quo, there is (at least) a player that prefers to continue the amendment process. The proposition below formally states such result.

**Proposition 1** An equilibrium of the game described above has a stopping decision only at a CW.

**Proof.** Consider first a status quo \( A \in \mathcal{A} \) that only loses one pairwise comparison against \( B \in \mathcal{A} \) and such that \( \Sigma^A = 1 \). There is a voter \( i \) that is pivotal in the stopping decision at alternative \( A \) and that prefers alternative \( B \) over \( A \), i.e. \( U_i(B) > U_i(A) = v_i^A \). Now we need to consider voter \( i \)'s expected utility in case he defects and votes towards continuing the negotiation: \( U_i(A | \Sigma^A = 0) = \beta v_i^A + \frac{1-\beta}{3} \left[ U_i(B) + 2U_i(A | \Sigma^A = 0) \right] \). The previous inequality implies that: \( U_i(A | \Sigma^A = 0) > v_i^A \). Hence, voter \( i \) cannot agree on stopping at node \( A \), i.e. \( \Sigma^A \neq 1 \).

The case where the status quo loses two pairwise comparisons can be analogously proved considering the voter that agrees on both decisions.

The case where the status quo is a loser (loses its three pairwise comparisons) needs a more detailed analysis. That is because we need to show that even when no player agrees on the three comparisons there is one of them that, on average, prefers to continue the negotiation. The proof of the case where a single player agrees on all three pairwise comparisons is analogous to the one above. We next consider the case where there is no
player that agrees on the three comparisons. In such scenario, continuing the negotiation involves (for every player) two good alternatives and the risk of a bad one. We need the risk of the bad alternative not to overcome the gains of the two good alternatives for (at least) one player. Without loss of generality, the following inequalities hold:

\[
U_1(B), U_1(C) > U_1(A) > U_1(D)
\]

\[
U_2(C), U_2(D) > U_2(A) > U_2(B)
\]

\[
U_3(D), U_3(B) > U_3(A) > U_3(C)
\]

It is necessary for these inequalities to be satisfied that \(A\) is not the least preferred alternative of any player (i.e. \(v^A_i \neq 0\)). Therefore, \(A\) can only be \(dd\). Nevertheless, \(dd\) is the best preferred option of player 3 hence he has no incentives to favour \(B\) and \(D\) over \(A\) as we have assumed.

Finally, the case where the status quo is a CW leads all voters to unanimously prefer to stop the negotiation given that continuing them can never yield a higher utility. ■

The Proposition implies that there exists an endogenous stopping decision if and only if there is a CW. Most importantly, the stopping decision at the CW does not affect the payoff of any voter, i.e. if \(A\) is such that \(V^{AB} = A\) for all \(B \in \mathcal{A} \setminus A\) then \(U_i(A | \Sigma^A = 0) = U_i(A | \Sigma^A = 1) = v^A_i\). These arguments allow us to conclude that the set of equilibria of the games with and without the possibility of endogenously stopping the negotiation are essentially the same. In other words, given the parameters \(m_1, m_2, m_3\) and \(\beta \in (0, 1)\), the set of pairwise comparisons of an equilibrium in any of the two classes of games coincides with the set of pairwise comparisons of the analogous equilibrium in the other class of games. We can now state our main result:

**Proposition 2** As \(\beta \to 0\), the only alternatives that can be implemented in a negotiation are CW.

As the probability of exogenously stopping the negotiation tends to zero, the only alternatives that can be implemented are those where the voters decide to stop the negotiation. The fact that such decision only occurs in CW implies that we just need to characterise such outcomes.

### 4 Equilibria with Condorcet Winner

Proposition 1 allows us to consider a simplified game without the stopping condition. We now analyse such game and characterise the equilibria with CW in terms of the voters’ relative intensity.
The new timing for each round of the negotiation follows:

1. $A \in \mathcal{A}$ is the status quo
2. With probability $\beta$ the negotiation terminates and the status quo is implemented.
3. Any alternative from the set $\mathcal{A} \setminus A$ is equally likely to be the challenger, $B$.
4. Voters cast their vote $V_i^{AB}$ and the majoritarian winner becomes the status quo for the next period.

The strategy set is denoted by $V = V_1 \times V_2 \times V_3$ and the expected utility at node $A$ for player $i$ is

$$U_i(A) = \beta v_i^A + \alpha \left[ U_i(V^{AB}) + U_i(V^{AC}) + U_i(V^{AD}) \right]$$

where $\alpha := (1 - \beta) \frac{1}{3} \in (0, \frac{1}{3})$ and $B$, $C$, and $D$ denote the three elements from $\mathcal{A} \setminus A$. The set of strategies form an equilibrium if and only if for all players $i = 1, 2, 3$ and any two alternatives $A, B \in \mathcal{A}$,

$$V_i^{AB} = B \iff U_i(A) < U_i(B).$$

An equilibrium has at most one CW. The CW can only be accompanied by a cycle among the remaining three alternatives or a ‘loser’ alternative (i.e. an alternative that loses all pairwise comparisons). We analyse each scenario at a time and end this Section by presenting an exhaustive interpretation of the results.\(^\text{13}\)

### 4.1 CW and cycle among the remaining alternatives

Assume there is an equilibrium with a CW at $A \in \mathcal{A}$ and a cycle among alternatives $B, C$ and $D$ ($B \succ D \succ C \succ B$). Figure 2 depicts such configuration. The expected utilities for player $i$ read as follows

$$U_i(A) = \beta v_i^A + \alpha \left[ U_i(A) + U_i(A) + U_i(A) \right] = v_i^A$$

$$U_i(B) = \beta v_i^B + \alpha \left[ U_i(A) + U_i(B) + U_i(C) \right] = \frac{1}{1-\alpha} \left[ \beta v_i^B + \alpha U_i(A) + \alpha U_i(C) \right]$$

$$U_i(C) = \beta v_i^C + \alpha \left[ U_i(A) + U_i(C) + U_i(D) \right] = \frac{1}{1-\alpha} \left[ \beta v_i^C + \alpha U_i(A) + \alpha U_i(D) \right]$$

$$U_i(D) = \beta v_i^D + \alpha \left[ U_i(A) + U_i(B) + U_i(D) \right] = \frac{1}{1-\alpha} \left[ \beta v_i^D + \alpha U_i(A) + \alpha U_i(B) \right].$$

\(^\text{13}\)In what follows we assume that a strict inequality always holds among the expected utilities of any two alternatives. All our results hold true whenever the condition is weakly met. Equilibria that may arise when an equality holds do not survive any usual refinement.

9
Any pairwise comparison needs majority to hold. Hence, given any two pairwise comparisons there always exists a player that votes in the way prescribed by both, i.e. \[ A; B; C; D \] such that \[ A = B, C = D \]. In particular, there always exist a player that votes as prescribed by the equilibrium over any two pairwise comparisons in the cycle (e.g. \( \exists i \in \{1, 2, 3\} : U_i(D) > U_i(C) > U_i(B) \)). This allows us to state the following Lemma.

**Lemma 1** Whenever \( A \) is a CW and \( B, D \) and \( C \) form a cycle such that \( B > D > C > B \), each of the three voters value a different alternative in the cycle higher than the other two. That is,

\[
\begin{align*}
\exists i \in \{1, 2, 3\} : & v_i^D > v_i^C, v_i^B \\
\exists i \in \{1, 2, 3\} : & v_i^B > v_i^D, v_i^C \\
\exists i \in \{1, 2, 3\} : & v_i^C > v_i^B, v_i^D
\end{align*}
\]

**Proof.** As noted above, there exists a player such that \( U_i(D) > U_i(C) > U_i(B) \). We can rewrite the first inequality in terms of the continuation utilities above: \( \beta v_i^D + \alpha U_i(B) > \beta v_i^C + \alpha U_i(D) \). The fact that \( U_i(D) > U_i(B) \) implies that \( v_i^D > v_i^C \). Similarly, \( U_i(D) > U_i(B) \) can be rewritten as \( \beta v_i^D + \alpha U_i(B) > \beta v_i^B + \alpha U_i(C) \) which, once again, implies that \( v_i^D > v_i^B \). The conditions for the remaining two players are deduced analogously from the fact that \( \exists i \in \{1, 2, 3\} : U_i(B) > U_i(D) > U_i(C) \) and \( \exists i \in \{1, 2, 3\} : U_i(C) > U_i(B) > U_i(D) \).

This Lemma allows us to discard \( dd \) as a candidate for CW. If the opposite is true, alternative \( ad \) is the most preferred (among the alternatives in the cycle) by voters 1 and 3. Hence the Lemma is not satisfied. Heuristically, if \( dd \) is a CW, a majority of voters strictly prefer one alternative in the cycle; thus the cycle no longer holds because such alternative wins all its pairwise comparisons.

**Corollary 1** Whenever three elements form a cycle, \( dd \) cannot be a CW.
Simple calculations allow us to avoid the recursive formulation of the expected utilities:

\[ U_i(A) = v_i^A \]
\[ U_i(B) = \frac{\alpha}{1-2\alpha} v_i^A + \frac{1-3\alpha}{(1-\alpha)^3 - \alpha^3} \left( (1-\alpha)^2 v_i^B + \alpha (1-\alpha) v_i^C + \alpha^2 v_i^D \right) \]
\[ U_i(C) = \frac{\alpha}{1-2\alpha} v_i^A + \frac{1-3\alpha}{(1-\alpha)^3 - \alpha^3} \left( (1-\alpha)^2 v_i^C + \alpha (1-\alpha) v_i^D + \alpha^2 v_i^B \right) \]
\[ U_i(D) = \frac{\alpha}{1-2\alpha} v_i^A + \frac{1-3\alpha}{(1-\alpha)^3 - \alpha^3} \left( (1-\alpha)^2 v_i^D + \alpha (1-\alpha) v_i^B + \alpha^2 v_i^C \right) \]

Note that \( da \) and \( ad \) play symmetric roles from the perspective of voters 1 and 2. Each alternative is the most and least preferred alternative by each of the two voters. This implies that voter 3 is pivotal in the pairwise decision among them. Taking this circumstance into account we can show that voter 3 will never vote in favour of a CW at \( ad \) given that he would be accepting to implement his second worst outcome. In other words, when considering the most preferred alternative of any of the remaining voters, he prefers to implement the one of voter 1 because voter 1 supports his views on his most preferred bill. The following Lemma precisely states that the most preferred alternative of voter 2 can never constitute a CW (its proof uses the reformulation of the expected utilities and is left to the Appendix).

**Lemma 2** Whenever three elements form a cycle, \( da \) cannot be a CW.

We are finally left with two candidates for CW in the presence of a cycle. These are the only two alternatives that are ranked on the top two places among (at least) two voters: \( aa \) and \( ad \). Before characterising the parameter values for which such equilibria can be sustained we state a technical Lemma that will prove useful.

**Lemma 3** Whenever \( A \) is a CW and \( B,D \) and \( C \) form a cycle such that \( B \succ D \succ C \succ B \), the player such that \( v_i^D > v_i^C \), \( v_i^B \) needs the following two inequalities to hold:

\[ v_i^D > \frac{v_i^C - \alpha v_i^B}{1-\alpha} \quad \text{and} \quad v_i^D > \frac{v_i^B - (1-\alpha) v_i^C}{\alpha} \]

**Proof.** The two inequalities follow from rewriting the inequalities \( U_i(D) > U_i(C) > U_i(B) \) in terms of the instantaneous utilities. \( \blacksquare \)

### 4.1.1 Condorcet winner at \( aa \)

By Lemma 1, each alternative in the cycle should be ranked first by one of the voters. The fact that the cycle can go either way allows two possible scenarios. In the first one we have that \( ad \succ dd \succ da \succ ad \). The following table summarises all information regarding
the individual pairwise comparisons and their instantaneous utilities.

<table>
<thead>
<tr>
<th>Voter</th>
<th>( U_1(\text{ad}) &gt; U_1(\text{dd}) &gt; U_1(\text{da}) )</th>
<th>( \lambda_1, 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voter 2</td>
<td>( U_2(\text{da}) &gt; U_2(\text{ad}) &gt; U_2(\text{dd}) )</td>
<td>( 1, 0, \lambda_2 )</td>
</tr>
<tr>
<td>Voter 3</td>
<td>( U_3(\text{dd}) &gt; U_3(\text{da}) &gt; U_3(\text{ad}) )</td>
<td>( 1, \lambda_3, 1 - \lambda_3 )</td>
</tr>
</tbody>
</table>

The pairwise comparisons inside the cycle are met if and only if the conditions of Lemma 3 are satisfied. For instance, voter 1’s condition read as follows:

\[
v_1^{ad} > v_1^{dd} - \alpha v_1^{da} \quad \text{and} \quad v_1^{ad} > \frac{(1 - \alpha) v_1^{dd}}{\alpha}.
\]

Once we take into account both inequalities in Lemma 3 for the three players and we also consider the fact that the CW needs to win all pairwise comparisons (i.e. \( U_1(\text{aa}) > U_1(\text{ad}) \) and \( U_2(\text{aa}) > U_2(\text{da}) \)) we get the following conditions for the described configuration to constitute an equilibrium:

\[
\begin{align*}
\lambda_1 &< \frac{\alpha}{1 - \alpha + 2\alpha^2} \\
\lambda_2 &< \alpha \\
\lambda_3 &> \frac{1 - \alpha}{2 - \alpha}
\end{align*}
\]

The bounds bind less the higher \( \alpha \) is. A higher \( \alpha \) implies that the continuation utilities are more important (by having a lower probability of the negotiation to end at any given period) and hence the bounds on the instantaneous utility are relaxed. In the sequel we refer to \( \text{aa} \), the CW in the present case, as the logrolling outcome where the first two voters agree on voting against their position on their least preferred bill favouring in this way the most preferred bill of the other voter. Even though we heuristically explain this outcome as a cooperative one, we are not assuming such behaviour and the outcome arises from agents acting non-cooperatively and taking into account the future rounds of the negotiation and the threat/benefit of seeing any other alternative challenging the status quo.

The CW under the present configuration is the least preferred alternative by voter 3. Thus, the first two voters need to favour the CW over any other alternative (regardless not being the most preferred alternative by any of them). That requires the valuation towards this alternative and their most preferred one to be close enough, i.e. 1 and \( 1 - \lambda_i \) \((i = 1, 2)\) need to be similar. In other words, \( \lambda_1 \) and \( \lambda_2 \) should be bounded above. Note
that, the bound for voter 1 is slightly higher than the one for voter 2. This occurs because
the sense of the cycle implies that the most preferred alternative of the cycle by voter 2
is followed by his second best alternative (valued $\lambda_2$). Instead, voter 1’s most preferred
alternative is followed by his worst preferred alternative (valued 0). This implies that
voter 1’s expected utility at his most preferred alternative of the cycle is lower than the
one of voter 2, thus the requirement for him to favour the CW over any alternative of the
cycle is slightly less demanding.

Finally, the cycle among the alternatives different than the CW requires voter 3 to favour
da over ad. This vote allows him to move towards his most preferred alternative (dd)
but goes against his instantaneous utilities. Therefore, we need the latter effect not to
overcome the former, that is $\lambda_3$ should be close to $1 - \lambda_3$ ($\lambda_3$ is bounded below).

The second possible scenario has the cycle running the opposite way (ad $\succ$ da $\succ$ dd $\succ$
ad). We can analogously find the bounds on the relative intensities that support such equilibrium:

\[
\begin{align*}
\lambda_1 &< \alpha \\
\lambda_2 &< \frac{\alpha}{1 - \alpha + 2\alpha^2} \\
\lambda_3 &> \frac{\alpha}{1 + \alpha}
\end{align*}
\]

The fact that the cycle now runs in the opposite sense than before implies that the bounds
for the first two voters are switched. The bounds for the third voters are relaxed given
that the cycle now runs in the way prescribed by his instantaneous utilities ($U_3 (dd) >$
$U_3 (ad) > U_3 (da)$). Nevertheless, if $\lambda_3$ is too small, he would not want to move towards
his most preferred alternative because the gains from doing so would not overcome the
fact that he is getting closer to his least preferred alternative on the cycle. Hence, we
need a less demanding lower bound on $\lambda_3$ than in the first scenario but a lower bound is
still in place.

4.1.2 Condorcet winner at ad

Once again, we have two possible scenarios where ad is a CW depending on the sense of
the cycle. Whenever the cycle is aa $\succ$ dd $\succ$ da $\succ$ aa, the bounds read as follows:
\[
\begin{cases}
\lambda_1 < \frac{1-a}{2-a} \\
\lambda_2 > \frac{a}{1+a} \\
\lambda_3 < \frac{a}{1-a+2a^2}
\end{cases}
\]

Note that it is now voter 2 the one that has a lower bound. This is because he is now in the position voter 3 was when the CW was \( aa \) – recall that \( aa \) is voter 3’s least preferred alternative and \( ad \) is voter 2’s one. The fact that the cycle runs in the way prescribed by voter 2’s instantaneous utilities leads to the bound that voter 3 had in the second scenario of Section 4.1.1.

Whenever the cycle is \( aa \succ da \succ dd \succ aa \), the bounds read as follows:

\[
\begin{cases}
\lambda_1 < \frac{a}{1+a} \\
\lambda_2 > \frac{1-a}{2-a} \\
\lambda_3 < a
\end{cases}
\]

When we look at any cycle supported in equilibrium, we can realise that any pairwise comparison follows the instantaneous preferences of one voter but needs the future prospects of another voter to be sustained. We use the last equilibrium with CW at \( ad \) and cycle \( aa \succ da \succ dd \succ aa \) to illustrate this reasoning. For instance, the pairwise comparison between \( aa \) and \( da \) is supported by voter 1 in terms of his instantaneous preferences and by voter 3 in terms of his continuation ones. By moving to \( aa \), voter 3 is incurring a cost in terms of his instantaneous preference but gains in terms of moving closer to his most preferred alternative on the cycle. The latter effect dominates the former when \( \lambda_3 \) and 0 are close enough. The players that vote in favour of the pairwise comparisons in terms of the instantaneous preferences are denoted below:

\[ aa \succ_{1} da \succ_{2} dd \succ_{3} aa \]

Instead, the players that vote in favour of the pairwise comparisons in terms of their \( future \) gains are denoted below:

\[ aa \succ_{3} da \succ_{1} dd \succ_{2} aa \]
In the present case we just need to analyse the incentives of voters to favour each alternative in terms of their future gains to understand the bounds on the relative intensities that sustain the equilibrium. We have that $\lambda_1$, $1 - \lambda_2$ and $\lambda_3$ need to be close to zero. Therefore, as has been shown above, $\lambda_1$ and $\lambda_3$ are bounded above and $\lambda_2$ bounded below.

4.2 Condorcet winner and loser

Figure 3 generally captures any equilibrium configuration with a CW and a loser: we assume that there is a CW at $A \in A$, a loser at $C$ and (without loss of generality) $V^{DB} = B$. The expected utilities for player $i$ read as follows:

$$U_i(A) = \beta v_i^A + \alpha [U_i(A) + U_i(A) + U_i(A)]$$
$$U_i(B) = \beta v_i^B + \alpha [U_i(A) + U_i(B) + U_i(B)]$$
$$U_i(C) = \beta v_i^C + \alpha [U_i(A) + U_i(B) + U_i(D)]$$
$$U_i(D) = \beta v_i^D + \alpha [U_i(A) + U_i(B) + U_i(D)]$$

A careful analysis of the expected utilities above allows us to state the following Lemma.

**Lemma 4** Whenever $A$ is a CW, $C$ a loser and $B > D$, there should be a majority of players that satisfies each of the following inequalities: $v_i^A > v_i^B$, $v_i^B > v_i^D$, and $v_i^D > v_i^C$.

**Proof.** The discounted future expected utilities above show that alternatives $C$ and $D$ only differ in their payoff in case that the negotiation terminates today (their continuation payoffs coincide). This implies that the decision between these two alternatives is taken on the basis of their instantaneous utilities. That is, the configuration in Figure 3 constitutes an equilibrium only when two players are such that $v_i^D > v_i^C$. Analogously, the first and
second inequalities are derived from rewriting the discounted future expected utilities:

\[
U_i(A) = \frac{1}{1-\alpha} \left( \beta v_i^A + \alpha U_i(A) \right), \quad U_i(B) = \frac{1}{1-\alpha} \left( \beta v_i^B + \alpha U_i(A) \right)
\]

and

\[
U_i(B) = \frac{1}{1-\alpha} \left( \beta v_i^B + \alpha [U_i(A) + U_i(B)] \right), \quad U_i(D) = \frac{1}{1-\alpha} \left( \beta v_i^D + \alpha [U_i(A) + U_i(B)] \right).
\]

The figure below depicts the majoritarian comparisons that need to hold among the instantaneous utilities. The table next to it considers all possible scenarios where the conditions in the Lemma are satisfied.

Three extra conditions need to be met for any of the previous configurations to constitute an equilibrium; a majority of voters need to prefer alternative \( A \) over \( D \) and \( C \), and alternative \( B \) over \( C \).

These conditions imply that, in the last two scenarios from the table above, \( da \) is preferred over \( dd \). Voter 3 can never favour \( da \) over \( dd \) because he would be moving out from his best preferred alternative and the benefits of avoiding his least preferred alternative (\( aa \)) are not overcome by the reduction in his instantaneous utility (from 1 to \( \lambda_3 \)). Similarly, voter 1 cannot vote in favour of such move given that he would be moving from his second worst alternative to his worst one without improving his future prospects. Consequently, those scenarios cannot constitute an equilibrium. Similarly, the first scenario cannot constitute an equilibrium either. Its proof relies on checking that the previous pairwise comparisons do not have a majoritarian support and is relegated to the Appendix.

We are finally left with only two scenarios (2 and 3) candidates for equilibria.

### 4.2.1 Condorcet winner at \( dd \)

We first analyse scenario (2) where \( dd \) is a CW, \( aa \) a loser and \( ad > da \). The three further conditions that need to be satisfied for such configuration to be an equilibrium are:
1. \( U_i (dd) > U_i (da) \)
2. \( U_i (dd) > U_i (aa) \)
3. \( U_i (ad) > U_i (aa) \).

The three inequalities reflect the instantaneous preferences of voter 3 and are always satisfied by him.

The first inequality cannot be met by voter 2 (it requires moving from his most preferred alternative to a CW that is his second worst one) hence we need voter 1 to satisfy it. Such movement involves an improvement on voter 1’s instantaneous utility but it also involves losing the possibility of selecting his most preferred alternative. Therefore, the former effect needs to be large enough. Simple calculations lead to the following bound: \( \lambda_1 > \frac{\alpha}{1-\alpha} \).

The second inequality cannot be met by voter 1 given that he prefers to stay away from a CW that is his second worst alternative. Voter 2 sees a decrease in his instantaneous utility but also sees the chances of selecting his worst preferred alternative decreased (note that the fact that \( ad \) beats \( da \) in equilibrium makes his least preferred alternative very likely). Therefore, we need the instantaneous decrease in utility (from \( 1 - \lambda_2 \) to \( \lambda_2 \)) to be small enough. The bound requires \( \lambda_2 > \frac{1-2\alpha}{2(1-\alpha)} \).

Finally, the third inequality is trivially met by voter 1 given that he is moving to his most preferred alternative \( (ad) \) and avoiding his least preferred one \( (da) \).

Overall, there exists an equilibrium where \( dd \) is the CW, \( da \) is the loser and \( ad \succ aa \) whenever the following conditions are met:

\[
\begin{align*}
\lambda_1 & > \frac{\alpha}{1+\alpha} \\
\lambda_2 & > \frac{1-2\alpha}{2(1-\alpha)^2}
\end{align*}
\]

### 4.2.2 Condorcet winner at \( ad \)

We finally analyse scenario (3) where \( ad \) is a CW, \( dd \) a loser and \( aa \succ dd \). The three conditions that need to be satisfied for such configuration to be an equilibrium are:

1. \( U_i (ad) > U_i (dd) \)
2. \( U_i(\text{ad}) > U_i(\text{da}) \)

3. \( U_i(\text{aa}) > U_i(\text{da}) \).

Voter 1 satisfies the 3 inequalities.

Voter 2 cannot satisfy the first two inequalities given that they imply moving to a CW that is his least preferred alternative. He satisfies the third one only when his instantaneous utility is low enough (\( \lambda_2 < \frac{\alpha}{1+\alpha} \)).

Voter 3 satisfies the first inequality whenever \( \lambda_3 < \frac{\alpha}{1-\alpha} \) because the reduction in his instantaneous utility (from 1 to \( 1 - \lambda_3 \)) is not overcome by the fact that both his worst preferred alternatives (\( \text{da} \) and \( \text{aa} \)) are avoided. He satisfies the second inequality for any parameter value and can never satisfy the third one.

Overall we can state that there exists an equilibrium where \( \text{ad} \) is the CW, \( \text{dd} \) is the loser and \( \text{aa} \succ \text{da} \) whenever the following conditions are met:

\[
\left\{ \begin{array}{c}
\lambda_2 < \frac{\alpha}{1+\alpha} \\
\lambda_3 < \frac{\alpha}{1-\alpha}
\end{array} \right. 
\]

4.3 Heuristic interpretation of equilibria with condorcet winner

The following table summarises all results above where an equilibrium with a CW exists.

<table>
<thead>
<tr>
<th>Condorcet Winner in:</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{aa} when ( \text{dd} \succ \text{da} \succ \text{ad} \succ \text{dd} ) and</td>
<td>( \lambda_1 &lt; \frac{\alpha}{1-\alpha+2\alpha^2} )</td>
<td>( \lambda_2 &lt; \alpha )</td>
<td>( \lambda_3 &gt; \frac{1-\alpha}{2-\alpha} )</td>
</tr>
<tr>
<td>\text{aa} when ( \text{dd} \succ \text{ad} \succ \text{da} \succ \text{dd} ) and</td>
<td>( \lambda_1 &lt; \alpha )</td>
<td>( \lambda_2 &lt; \frac{\alpha}{1-\alpha+2\alpha^2} )</td>
<td>( \lambda_3 &gt; \frac{\alpha}{1+\alpha} )</td>
</tr>
<tr>
<td>\text{ad} when ( \text{da} \succ \text{aa} \succ \text{dd} \succ \text{da} ) and</td>
<td>( \lambda_1 &lt; \frac{1-\alpha}{2-\alpha} )</td>
<td>( \lambda_2 &gt; \frac{\alpha}{1+\alpha} )</td>
<td>( \lambda_3 &lt; \frac{\alpha}{1-\alpha+2\alpha^2} )</td>
</tr>
<tr>
<td>\text{ad} when ( \text{da} \succ \text{dd} \succ \text{aa} \succ \text{da} ) and</td>
<td>( \lambda_1 &lt; \frac{\alpha}{1+\alpha} )</td>
<td>( \lambda_2 &gt; \frac{1-\alpha}{2-\alpha} )</td>
<td>( \lambda_3 &lt; \alpha )</td>
</tr>
<tr>
<td>\text{ad} when ( \text{da} ) is a loser, ( \text{aa} \succ \text{dd} ) and</td>
<td>( \lambda_1 &lt; \frac{\alpha}{1+\alpha} )</td>
<td>( \lambda_2 &lt; \frac{\alpha}{1+\alpha} )</td>
<td>( \lambda_3 &lt; \frac{\alpha}{1-\alpha} )</td>
</tr>
<tr>
<td>\text{dd} when ( \text{aa} ) is a loser, ( \text{ad} \succ \text{da} ) and</td>
<td>( \lambda_1 &gt; \frac{\alpha}{1+\alpha} )</td>
<td>( \lambda_2 &gt; \frac{1-2\alpha}{2(1-\alpha)^2} )</td>
<td>( \lambda_3 &lt; \frac{\alpha}{1-\alpha} )</td>
</tr>
</tbody>
</table>

Table 1: Equilibria with condorcet winner

A simple analysis allows to state that the majoritarian outcome (\( \text{dd} \)) is implemented whenever voters one and two care very similarly about the two bills that need to be
approved or dismissed.\textsuperscript{14} Note that those parties are precisely the ones that prefer the logrolling outcome to the majoritarian one. When they feel similarly about both bills the logrolling outcome does not terribly improve the welfare achieved by the majoritarian outcome and the latter is implemented in equilibrium.

When the first two voters have very polarised preferences or care much more about their most preferred bill than about the remaining one (i.e. when they most gain from the logrolling outcome) they are able to implement the logrolling outcome (aa) as long as the third player (the most damaged by the logrolling outcome) cares similarly about both bills. The condition on the third voter is required for the cyclic relation (among the alternatives different than the CW) to be sustained in equilibrium.

Finally, there is a third alternative that can be implemented by the voters’ negotiation –an outcome that the logrolling literature has failed to notice. Once we introduce the scope for legislators to negotiate away from the initial majoritarian outcome the literature has only considered the outcome in which two parties agree on voting in favour of each other’s preferred bill.\textsuperscript{15} The sole presence of this outcome gives incentives to the third voter to propose the implementation of the preferred outcome of any of the other two voters. In that way, the third voter avoids the result on both bills being switched by favouring the wishes of one of the voters. Given our specification of the payoffs, voter 3 implements the wills of the voter that shares his same views on his most preferred bill (i.e. voter 1). Note that the third voter gets no reward in exchange of implementing voter 1’s preferred alternative. His only benefit rests on avoiding his worst preferred alternative.

This last outcome is only supported when the third voter and (at least) another voter have polarised preferences. In other words, the last outcome is supported when the third voter loses very little from selecting this outcome rather than the majoritarian one, and the threat of the logrolling outcome is highly present. It turns out that the third voter prefers to implement his second best alternative (which valuation is very close to the one of his most preferred alternative) rather than risking the implementation of his worst preferred one.

Once again, we need to emphasize that even though we provide an almost cooperative heuristic of the equilibria, these arise from a non-cooperative behaviour among voters where they not only consider the instantaneous benefits from choosing an alternative but also its future consequences in the negotiation. Finally, note also that the sets for which each equilibrium is sustained are not disjoint, i.e., given some parameter values, we may not have unicity of equilibria with CW.\textsuperscript{16}

\textsuperscript{14}Recall that \( \lambda_i = \frac{m_i}{1 + m_i} \). Therefore, a high value of \( \lambda_i \) implies a high value of \( m_i \) (i.e. both bills are similarly valued).

\textsuperscript{15}The literature has always understood that a logrolling situation exists if \( xPy \) and \( vPw \), but \( ywPxy \); where \( P \) stand for social preference as defined by the voting rule employed. (Stratmann 1992, pg. 1163).

\textsuperscript{16}For instance, the parameter values \( m_1 = 1/2, m_2 = 2/3 \) and \( m_3 = 1/4 \) support equilibria with CW at ad and dd whenever \( \beta \) is small enough.
5 Discussion

Proposition 2 states which are the outcomes of the negotiation when a CW exists. Nevertheless, equilibria with no CW may exist. In that case, the negotiation never stop endogenously and voters keep on proposing amendments to the status quo without reaching a decision. The implemented alternative only depends on the exogenous stopping decision. Characterising outcomes where the driving force is the exogenous stopping decision may seem a bit ad-hoc and impairs the explanatory power of our theory. Besides, when $\beta$ tends to zero the negotiation would never end and one could think of an extended model where voters had aversion towards non-agreement and would only start the negotiation when they foresee that it has an ending.

At a first sight, it seems that we can easily avoid equilibria without CW by introducing costly negotiations. These could take the form of a discount factor or an (arbitrarily) small cost associated to each round of the negotiation. The former case would erode all benefits of the negotiation in case that no agreement is reached and the latter would imply having unbounded non-agreement costs. Despite the costs that impatience (in either form) attaches to an equilibrium without CW, we cannot introduce them into our model without giving much more structure to it. The reason being that such feature introduces new equilibria and silences most of our predictions. That occurs because a stopping decision in a setting where there is some costs of negotiation implies a discrete increase in the utility derived at that alternative (given that no more discounting or costs of negotiation ever apply); therefore, whenever $\beta$ tends to zero, the instantaneous payoffs in the nodes where no stopping decision occurs tend to be less relevant and the discrete jump in the alternative where such decision occurs leads to a folk theorem. That is, we can always find a $\beta$ small enough such that any alternative is implemented in equilibrium (i.e. this alternative is the unique CW and a stopping decision occurs there). The rationale behind these equilibria is very similar to the one when $\beta = 0$ (see the Appendix).

We do not see our lack of time dimension as a real limitation when modelling negotiations. It implies thinking that each round of the negotiation requires only infinitesimal time to complete. Although literally absurd, the abstraction of imagining each separate decision stage taking an infinitesimal amount of time is not an unreasonable model of a deliberative process in which committee members offer proposals and reach tentative decisions continually in the course of conversation.

Finally note, that we do not asses any optimality analysis given that such analysis is sub-

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17 For instance, the set of pairwise comparisons where all voters support the pairwise comparisons corresponding to their instantaneous utilities constitutes an equilibrium for a large set of parameter values.

18 See Eliaz, Ray and Razin (2004) where the aversion towards disagreement is explicitly modelled in a model with incomplete information where agents need to decide over two alternatives.

19 Austen-Smith and Banks (2004), pg. 196.
ject to the particular normalisation of payoffs we assumed at the beginning of the paper. Our theory provides a benchmark that predicts the possible outcomes of the legislative bargaining in terms of the relative intensity over the bills. Its welfare implications can be computed on a case by case basis once we know the absolute valuation of each voter towards each bill.

6 Conclusion

We have provided a theory of legislative bargaining in a setting where the institution-free model of social choice theory yields no equilibrium. We model negotiations or deliberation as a repeated game where agents challenge the status quo and consider its majoritarian winner. While we do not believe that negotiations literally happen the way we have modelled them, we think that our modelling techniques should not only be applied to the logrolling phenomenon but could generally be extended to any situation where a group of agents need to decide over various bill that need to be approved or dismissed. The model allows multiple extensions such as considering settings with incomplete information, with various outcomes per bill or with alternative voting rules.

Our model allowed us to characterise the outcomes of the negotiation in terms of the voters’ relative intensity. We have shown that the majoritarian and logrolling outcomes are not the only outcomes that can be implemented in equilibrium. The threat of the latter introduces a third outcome where the most affected voter by the logrolling outcome agrees on implementing the wills of one of the logrolling parties. We plan to test the relevance of our predictions by extending the work of Stratmann (1992). We envisage that whenever voters face many bills, the third equilibrium where a voter agrees on implementing another voter’s will may have a counterpart in terms of a future bill. It remains to be proven if voters need to pay back such support in future bills or whether our results hold and the threat of the logrolling outcome is sufficient to gain such support.

References


See Austen-Smith and Feddersen (2002) and references therein for an analysis of deliberation over two alternatives and its effect on different voting rules.


7 Appendix

**Proof of Lemma 2.** The three alternatives in the cycle should be ranked first by one player. The fact that the cycle can go in either way leaves us with two possible scenarios:

<table>
<thead>
<tr>
<th>Voter</th>
<th>$v_i^X$</th>
<th>$U_i(ad) &gt; U_i(dd) &gt; U_i(aa)$</th>
<th>$1, 1 - \lambda_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voter 1</td>
<td>$U_i(ad) &gt; U_i(dd) &gt; U_i(aa)$</td>
<td>$1, 1 - \lambda_1, 1 - \lambda_2$</td>
<td></td>
</tr>
<tr>
<td>Voter 2</td>
<td>$U_i(aa) &gt; U_i(ad) &gt; U_i(dd)$</td>
<td>$1 - \lambda_2, 0, 1 - \lambda_3$</td>
<td></td>
</tr>
<tr>
<td>Voter 3</td>
<td>$U_i(dd) &gt; U_i(aa) &gt; U_i(ad)$</td>
<td>$1, 0, 1 - \lambda_3$</td>
<td></td>
</tr>
</tbody>
</table>

Given that $da$ is the least preferred option by voter 1, the other two voters always need to vote in favour of $da$. For instance, we need $U_3(da) > U_3(dd)$. This equality cannot be met in either case when we take into account that $\alpha \in (0, \frac{1}{2})$ and $\lambda_3 \in (0, \frac{1}{2})$.

**Proof that $aa$ cannot be a CW in the presence of a loser.** We have shown above that the only situation under which $aa$ can be a CW in the presence of a loser is the situation where $da$ is the loser and $dd > ad$. Three inequalities need to be satisfied: $U_i(aa) > U_i(ad)$, $U_i(aa) > U_i(da)$, and $U_i(dd) > U_i(da)$.

The first inequality cannot be met by voter 3 because he cannot be willing to move to a CW that is his worst preferred alternative (i.e. ensuring a 0 payoff). Voter 2 is very much willing to move out from his worst preferred alternative and hence satisfies the inequality. Finally, voter 1 is moving from his best preferred alternative to his second best one and he is avoiding his third preferred alternative. Therefore, the loss on his instantaneous utility should be compensated by his future gains. In other words, $\lambda_1$ should be low enough with respect to his future gains (captured by $\alpha$). The formal expression of such inequality reads as follows:

\[ U_1(aa) > U_1(ad) \Leftrightarrow \lambda_1 < \alpha \]
\[
\begin{align*}
\text{where } & \quad \left\{ \begin{array}{l}
U_1 (ad) = \frac{\beta}{(1-\alpha)} \left( \frac{\alpha}{1-3\alpha} \frac{1-\alpha}{1-2\alpha} (1 - \lambda_1) + \frac{\alpha}{1-2\alpha} \lambda_1 + 1 \right) , \\
U_1 (aa) = 1 - \lambda_1
\end{array} \right.
\end{align*}
\]

The third inequality is satisfied by voter 3 given that he is moving to his most preferred alternative and given that his future prospects are not affected (there is always the same probability of getting to the CW). Instead, it is not satisfied by voter 2 because he is moving out from his best preferred alternative and the benefits of avoiding his least preferred alternative (ad) are not overcome by the reduction in his instantaneous utility (from 1 to \(\lambda_2\)). Therefore, we need \(U_1 (dd) > U_1 (da)\) to be satisfied. Note that by moving to \(dd\), voter 1 improves his instantaneous utility but loses the chance of going through his most preferred alternative. Thus, his instantaneous gain (\(\lambda_1\)) should be large enough.

\[
\begin{align*}
U_1 (dd) > U_1 (da) & \Leftrightarrow \lambda_1 > \alpha \\
\text{where } & \quad \left\{ \begin{array}{l}
U_1 (dd) = \frac{1}{1-2\alpha} \left( \alpha (1 - \lambda_1) + \lambda_1 \right) \\
U_1 (da) = \frac{\alpha \beta}{1-2\alpha} \left( \frac{1}{1-3\alpha} (1 - \lambda_1) + \frac{\alpha}{1-\alpha} \lambda_1 + \frac{1-2\alpha}{1-\alpha} \right).
\end{array} \right.
\end{align*}
\]

The conditions on the relative intensity of voter 1 cannot be simultaneously met thus the first scenario cannot constitute an equilibrium.

**The relevance of the parameter \(\beta\)**

Assume \(\beta = 0\) and agents can decide whether to stop the decisions or not – i.e. \(\Sigma_A = 1(0)\) denote that the negotiation ends (continue) at node \(A\). The expected utilities read as follows:

\[
\begin{align*}
U_i (A) = \begin{cases} 
& v_i^A \quad \text{if } \Sigma_A = 1 \\
& \frac{\delta}{3} \left[ U_i (V^{AB}) + U_i (V^{AC}) + U_i (V^{AD}) \right] \quad \text{if } \Sigma_A = 0
\end{cases}
\end{align*}
\]

Imagine there is only one stopping decision. Proposition 1 implies that the alternative where the negotiation stops is a CW. The remaining pairwise comparisons can go either way given that voters are indifferent between any of the alternatives where the stopping conditions does not apply. Therefore, any alternative can be a CW when \(\beta = 0\).

Note that this kind of setting does not tell us much about negotiations in multiple issue situations. Intermediate nodes are simply *fictitious* nodes that allow to reach the CW. We could argue that this is not a plausible feature of real world negotiations where voters fear some paths towards their most preferred alternative because they entail the risk of ending up in a much worst alternative than the initial one. Enriching the model with an exogenous probability of ending the negotiation (\(\beta > 0\)) does not only give an explicit role to the negotiation but also allows our theory to have predictive and explanatory power over such phenomenon.