Exercises

These are the Oxford University MPhil in Economics examination questions that I set in the period 1995–2003, for a short course roughly corresponding to the "graduate course outline" at the back of the book.

Questions 1, 9, and 11 are somewhat easier than the rest, though for introductory examples the reader is referred to Appendix 1.D. Solutions are at the end of the book.

Revenue Equivalence Theorem—Theory

1. An auctioneer of a single object faces n risk-neutral bidders with private valuations for the object that are independently drawn from a distribution $F(\cdot)$ with density $f(\cdot)$.

(i) State, and sketch a proof of the revenue equivalence theorem for this situation.

(ii) Write down how players bid in a Japanese (i.e., English, or ascending, auction), and hence derive their expected payments conditional on winning, when their values are independently drawn from a uniform distribution on $[0, \bar{v}]$.

(iii) Consider an "all-pay" auction (i.e., a simultaneous sealed-bid auction in which the high bidder wins the object, but every bidder pays her bid). Use the revenue equivalence theorem, together with your solution to part (ii), to solve for the bidding functions for the "all-pay" auction when bidders' values are independently drawn from a uniform distribution on $[0, \bar{v}]$.

(iv) Write down the differential equation for a player's bid as a function of her value in the symmetric equilibrium of an "all-pay" auction when bidders' values are independently drawn from the distribution $F(\cdot)$. Solve the differential equation for the special case in which $F(\cdot)$ is uniform on $[0, \bar{v}]$. (Your solution should be consistent with (iii)!)

[Oxford, 1st Year Micro, 2000]

2. Consider an auctioneer with a single unit facing two risk-neutral buyers with independent private valuations. Each buyer is equally likely to be a "type H" who has value $v_H = 1$ or a "type L" with value $v_L = 0$.

(i) Consider the auction form in which the seller begins by offering a price $a \in (0, \frac{2}{3})$ and buyers simultaneously accept or reject. If one buyer accepts, she receives the unit at price a. If both buyers accept price a, then the unit is allocated by a fair lottery between both players at price a. If both buyers reject price a then the unit is allocated by a fair lottery between both players at price 0. Show that there is an equilibrium in which type Hs always accept. What are the seller's equilibrium profits, and the expected surpluses of different types of buyers?

(ii) Why does the revenue equivalence theorem not apply when comparing auction forms of the type described in part (i) with different values of a?

(iii) Write down incentive compatibility constraints relating the amounts of surplus that types H and L can receive in any mechanism which always assigns the unit to one of the buyers, and always does so efficiently (and gives the two buyers the same probability of winning if they are of the same type). Hence deduce a mechanism that maximizes the seller's profits in this class of mechanisms.

(iv) Assume the seller need not assign the unit to either buyer. What mechanism maximizes her profits?

[Oxford, Economic Theory, 1999]

3. Each of six risk-neutral bidders has a privately known value independently drawn from a uniform distribution on $[0, \overline{v}]$ for a single unit. (No bidder desires more than 1 unit.) An auctioneer has two identical units available. Denote by v_i^j the ith highest remaining valuation when j bidders remain in the auction.

(i) State, and sketch a proof of a version of the revenue equivalence theorem that applies to this context.

(ii) Two successive first-price sealed-bid auctions are held for one unit each. (In each auction only the winning bid is revealed.) In equilibrium, what is the bid in the *second* auction of a bidder with value v who failed to win in the first auction?

(iii) What is the bid in the *first* auction of a bidder with value v?

(iv) Now assume n objects are sold by successive sealed-bid auctions to $(n + 4)$ bidders. Derive whether prices are generally higher or lower in later auctions than in earlier auctions.

(v) Briefly discuss how your answer to (iv) might be sensitive to the assumptions of the problem.

[Oxford, Economic Theory, 2000]

Revenue Equivalence Theorem—Applications

4. In 1991 US Vice-President Quayle proposed that the loser in a lawsuit be required to transfer an amount equal to her own legal expenses to the winner. Quayle claimed this would reduce the amount spent on legal services. (Under current US rules each party pays its own costs.) We will model this by assuming each party $i = 1, 2$ has a privately known value v_i, independently drawn from $F(v)$(with $F(\underline{v}) = 0$) for winning a lawsuit, the two parties independently and simultaneously decide how much to spend on legal services, and the party that spends the most money wins.

(i) Obtain an expression for the amount each player spends under current US rules, by using revenue equivalence with an ascending auction for the prize of winning the lawsuit. What additional assumption(s), if any, did you have to make to use the revenue equivalence theorem?

(ii) Without doing any more calculations, use our model to evaluate Quayle's claim. What additional assumption(s), if any, have you made?

(iii) In European legal systems the loser usually pays a fraction of the winner's actual expenses. Without doing any more calculations do you think this rule will increase or reduce expected legal expenses?

(iv) Use the revenue equivalence theorem to obtain a differential equation for the amount $l(v)$ each party spends under Quayle's rules. Show that

$$l(v) = \frac{v^2}{3}\frac{3 - v}{(2 - v)^2}$$

satisfies your equation when $F(v) = v$.

(v) Very briefly, how satisfactory is the model?

[Oxford, Economic Theory, 1998]

5. It is sometimes said that firms dislike sealed-bid auctions because they can be very embarrassing for managers who find they have paid a lot more than runners-up.

(i) Consider n firms whose values, v_i, are independently drawn from the uniform distribution on $[0, 1]$. i's managers' utility from winning with bid b_i equals

$$u_i = v_i - b_i - k\left(b_i - \max_{j \neq i} b_j\right)$$

(in which the term proportional to k reflects the embarrassment of winning) and a non-winner's utility is zero. Write down the first-order condition that bidding must satisfy, and hence show that there is a linear equilibrium.

(ii) Compute the seller's expected profit, and the bidders' expected utilities from the auction.

(iii) How do the bidders' expected utilities depend on k? How can you explain this result? Hence explain how the seller's expected profit depends on k.

(iv) How do you expect your results would change if losers also suffered embarrassment costs based on the difference between their bids and the winner's bid? [You are not expected to solve such a model.]

(v) Using, but without limiting yourself to, the models discussed in (i) and (iv) briefly discuss why firms often lobby against government proposals to use sealed-bid auctions.

[Oxford, Economic Theory, 2002]

6. This problem explores the institution of "buy prices".

(i) Two bidders compete in a standard ascending auction for a single prize for which they have private values independently drawn from the uniform distribution on $[0, 1]$. What is the expected price, conditional on winning, that a bidder with value x will pay?

(ii) Now assume that prior to the auction the seller announces that at any point during the auction either bidder can immediately end the auction by announcing her willingness to pay the fixed "buy price" of $b \geq 0.5$, in which case that bidder wins the prize and pays the buy price b. Assume that there is an equilibrium in which a bidder with value $x \geq b$ bids the buy price when the bidding has reached $p(x)$, with $p'(x) < 0$. What is the expected price, conditional on winning, that a bidder with value x will pay, as a function of x and $p(x)$?

(iii) Assuming the bidders are risk-neutral, explain why the expected price that a bidder with value x pays conditional on winning is the same in parts (i) and (ii). Now use this to solve for $p(x)$.
[*Hint*: You should get a quadratic equation, one root of which is $p = x$. This root is not the solution, since by assumption $p'(x) < 0$.]

(iv) If the seller is risk-averse (and the bidders are risk-neutral) does she prefer the buy-price auction to (a) a pure ascending auction? and (b) a first-price sealed-bid auction? Why?

(v) If the bidders are risk-averse, do you conjecture expected revenue is higher or lower than in (a) a pure ascending auction? and (b) a first-price sealed-bid auction? Why?

[Oxford, Economic Theory, 2003]

Marginal Revenues

7. (i) Consider a monopolist with a single unit facing two markets with demands $p = 1 - q$ and $p = 2 - q$, respectively, between which she can price discriminate. What are her optimal prices and sales in each market?

(ii) Consider an auctioneer with a single unit facing two risk-neutral buyers with independent valuations uniformly distributed on $[0, 1]$ and $[1, 2]$, respectively. Construct an optimal auction for him. What are the probabilities with which he sells to each buyer?

(iii) By reference to parts (i) and (ii) explain the similarities between, and the differences between, the theory of optimal auctions and the theory of price discrimination.

[Oxford, Economic Theory, 1996]

8. (i) Sketch why the expected revenue from an ascending auction equals the expected marginal revenue of the winning bidder. Under what assumptions does the result extend to any auction form?

(ii) Consider an auction with three risk-neutral bidders. Bidder A has a value of 10. Bidder B has a value that is drawn from a uniform distribution between 0 and 30. Bidder C has a value that is drawn from a uniform distribution between 0 and 50, independently of B's. The seller has a value of 0 and can pre-commit to any allocation mechanism she wishes. Construct the optimal auction.

(iii) Now consider an auction in which each of three risk-neutral bidders observes a private signal independently drawn from a uniform distribution on [0, 1], and the value to any of the bidders is equal to the maximum of the three signals.

(a) What expected price does a standard ascending auction in which all three bidders participate yield?
(b) If one bidder, chosen at random, had been excluded from the auction, what would the expected price have been?
(c) If the auctioneer had been able to transact with only one of the bidders, chosen at random, what is the highest take-it-or-leave-it price she could have set while guaranteeing acceptance?
(d) Comment on your findings in (a), (b), (c).

[Oxford, Economic Theory, 2001]

Common Values

9. Risk neutral bidders $i = 1, 2$ each receive a private signal z_i independently drawn from the uniform distribution on [0, 1]. Bidder i's value for an object is $v_i = 3z_i + z_j$; $i, j = 1, 2$; $i \neq j$.

(i) A single item is sold by English (i.e., Japanese, or ascending) auction. Compute equilibrium bidding strategies.

(ii) A single object is sold by Dutch (i.e., descending) auction.

(a) Use your solution in (i) together with the revenue equivalence theorem to deduce equilibrium bidding strategies.
(b) Solve for the equilibrium strategies directly, by writing out *i's* surplus when her signal is z_i and she bids as if her type was $\tilde{z}_i$ and then obtaining a differential equation from the first-order condition.

(iii) Would the seller prefer a Japanese or a Dutch auction

(a) If the buyers are risk-averse (but the seller is risk-neutral)?
(b) If the seller is risk-averse (but the buyers are risk-neutral as before)?

[Oxford, Economic Theory, 1997]

10. Three risk-neutral bidders $i = 1, 2, 3$ each receive a private signal t_i that is independently drawn from the uniform distribution on $[0, 1]$. Each of two objects has an actual value equal to the average of the three signals.

(i) State a version of the revenue equivalence theorem that applies to this context.

(ii) Describe bidders' strategies in the symmetric equilibrium of the English auction (i.e., an ascending auction in which the two remaining bidders win at the price at which the other quits).

(iii) Compute the expected surplus of a bidder with signal t in the above equilibrium.

(iv) Deduce equilibrium strategies in a sealed-bid discriminatory auction (i.e., each of the two highest bidders wins an object and pays her actual bid).

[Oxford, Economic Theory, 1995]

Risk Aversion

11. Two players $i = 1, 2$ have private values v_i that are independently drawn from the uniform distribution on $[0, 1]$, for a single object.

(i) Assume players' utilities u_i are given by $u_i = v_i - t_i$ when the player receives the object for payment t_i and $u_i = 0$ otherwise.

(a) How do players bid in an ascending auction? What are expected revenues?

(b) Solve for the symmetric equilibrium bidding functions $b(v_i)$ in a first-price sealed-bid auction by obtaining the first-order condition that optimal bidding must satisfy and solving the resulting differential equation. What are expected revenues?

(c) Comment on the comparison of the results of (a) and (b).

(ii) Now assume $u_i = \sqrt{v_i - t_i}$ when the player receives the object. Repeat (a), (b) and (c) above, and provide intuition.

[Oxford, 1st Year Micro, 2002]

Essay Questions

12. "As a mechanism for allocating resources, auctions are efficient and fair." Discuss.

[Oxford, 1st Year Micro, 1999]

13. Are first-price or second-price auctions better?

[Oxford, Economic Theory, 1997]

14. Sketch a proof of the revenue equivalence theorem. Explain why the theorem fails, and how the seller's revenue comparison between first- and second-price auctions is affected, if (i) bidders are risk-averse, and (ii) bidders' information is affiliated.

[Oxford, Economic Theory, 1996]

15. Does behavior in common-value auctions differ importantly from that in private-value auctions?

[Oxford, Economic Theory, 1999]

16. Is the commonly made assumption that bidders' private-information is independent an important one in auction theory? Why? Is it a good assumption?

[Oxford, Economic Theory, 2000]

17. How does the correlation of bidders' information signals affect the design of an auction?

[Oxford, Economic Theory, 2003]

18. When several similar objects are auctioned sequentially, should we expect later prices to be on average higher, or on average lower, than earlier prices?

[Oxford, Economic Theory, 2001]

19. Does it matter that most auction theory has been developed in a single-unit context?

[Oxford, Economic Theory, 2002]

20. What are the main strengths and limitations of existing auction theory for practical applications?

[Oxford, Economic Theory, 1998]