SUPPLEMENT TO "UNDERSTANDING PREFERENCES: "DEMAND TYPES", AND THE EXISTENCE OF EQUILIBRIUM WITH INDIVISIBILITIES"

(*Econometrica*, Vol. 87, No. 3, May 2019, 867–932)

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APPENDIX C: STRONG SUBSTITUTES VALUATIONS IN THREE DIMENSIONS

HERE WE DEVELOP VALUATIONS FOR STRONG SUBSTITUTES, in three dimensions, by using the Valuation-Complex Equivalence Theorem (Theorem 2.14), together with duality between the price complex and the LIP (Proposition 2.20).

Start with a domain of $\{0,1\}^3$. We consider possible demand complexes for strong substitutes valuations. Such a demand complex is a subdivision of the cube $[0,1]^3$, such that the edges are all strong substitutes vectors: they all are in directions $\mathbf{e}^i$ or $\mathbf{e}^i - \mathbf{e}^j$, where $i, j = 1, 2, 3$. Thus, the full set of allowed 1-cells is the collection of red and black edges shown in Figure S.1(a). A candidate subdivision is given in Figure S.1(b). Three 2-cells, distinguished by being red, blue, and yellow, separate the cube into four 3-cells. We label these 3-cells W, X, Y, and Z, as shown. *If* this is the demand complex of a valuation, then that valuation is for strong substitutes. But recall that *not* every such subdivision is necessarily a demand complex of some valuation. To ascertain whether it is, we will develop the dual complex in price space. If we can develop a balanced weighted rational polyhedral complex in this way, then we can apply Theorem 2.14 to obtain a valuation.[1]

We thus plot out the $(n-1)$-cells of the dual in price space, that is, the facets. First identify that the demand complex 3-cell for the lowest quantities ("W") corresponds to a price 0-cell ($\mathbf{p}_W$) for "high" prices. There are 1-cells terminating in $\mathbf{p}_W$ coming in from even higher prices in each of the coordinate directions, corresponding to the three 2-cells of W that are in the boundary of the cube. Between each pair of these 1-cells is a facet; each is dual to one of the three edges in Figure S.1(b) that lies along a coordinate axis. Thus, we obtain Figure S.2(a).

Similar consideration of the red 2-cell in Figure S.1(b) allows us to develop our picture further: see Figure S.2(b). The edges of this red 2-cell correspond to three further facets, all meeting along the 1-cell dual to the red 2-cell itself. This 1-cell runs from $\mathbf{p}_W$ (corresponding to the 3-cell W in Figure S.1(b)) to a new point $\mathbf{p}_X$ (corresponding to the 3-cell X in Figure S.1(b)).

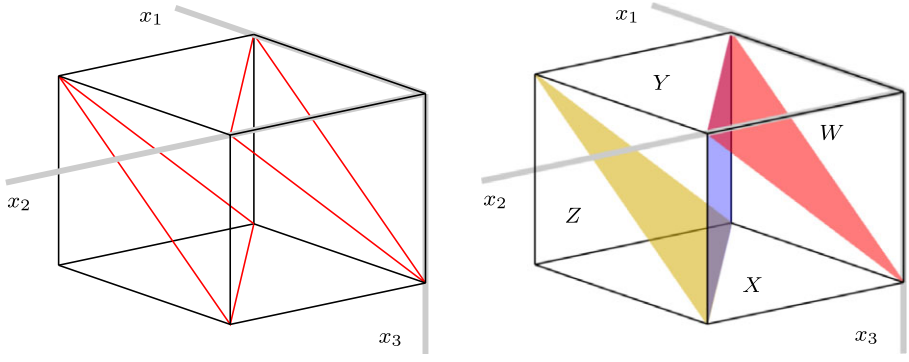
The final result is Figure S.3: a two-dimensional rational polyhedral complex in $\mathbb{R}^3$.

We give weight 1 to every facet of Figure S.3, as it is dual to an edge of "length" 1 in Figure S.1(b). This weighted LIP is balanced. To see this, consider the full set of facets meeting any 1-cell of Figure S.3. This configuration is dual to a 2-cell of Figure S.1(b), taken together with its edges. The vector sum of the edges, going once around the 2-cell, must

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[1] If we could find a valuation directly, such that Figure S.1 is the demand complex, then we would not need to proceed to price space. This does not seem so easy.

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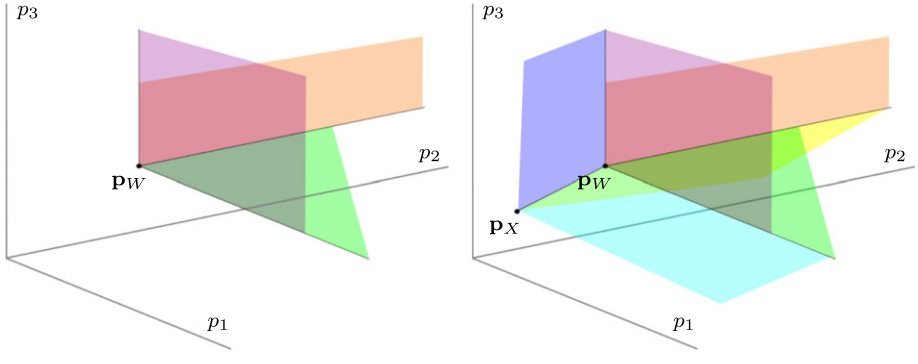
(a) Possible 1-cells in the demand complex for a strong substitutes valuation on $\{0,1\}^3$. Black lines are edges of the cube $[0,1]^3$, and must be 1-cells of the demand complex. The red lines may also be 1-cells of the demand complex.

(b) A polyhedral complex subdivision of $[0,1]^3$, with edges as shown in Figure S.1(a). Letters W, X, Y, Z label the four 3-cells.

FIGURE S.1.—Developing a candidate demand complex.

be zero. But the edges are equal to the normal vectors to the facets (Proposition 2.20). Thus, an oriented weighted sum of the normal vectors to the facets in Figure S.3 is also zero.

Thus, we may apply Theorem 2.14: there exists a valuation u whose LIP is depicted in Figure S.3. Indeed, since we did not yet specify any precise coordinate information, Figure S.3 represents an entire combinatorial type of valuations (Definition 2.22). Moreover, we can see more combinatorial types immediately. Recall that we developed Figure S.3 on the basis of one subdivision of $[0,1]^3$ (namely, Figure S.1(b)) that was consistent with



(a) Some cells of the complex dual to Figure S.1(b): 0-cell $\mathbf{p}_W$ is dual to the 3-cell W in Figure S.1(b); and each facet shown here is dual to an edge lying along a coordinate axis in Figure S.1(b).

(b) Development of Figure S.2(a) with additional cells: 0-cell $\mathbf{p}_X$ is dual to the 3-cell X in Figure S.1(b); and the three facets shown, that meet it, are dual to the edges of the red 2-cell in Figure S.1(b).

FIGURE S.2.—First steps in developing the rational polyhedral complex in price space, dual to Figure S.1(b). Facets are shown cut back, so that others lying "behind them" can be seen. The complete complex is shown in Figure S.3.

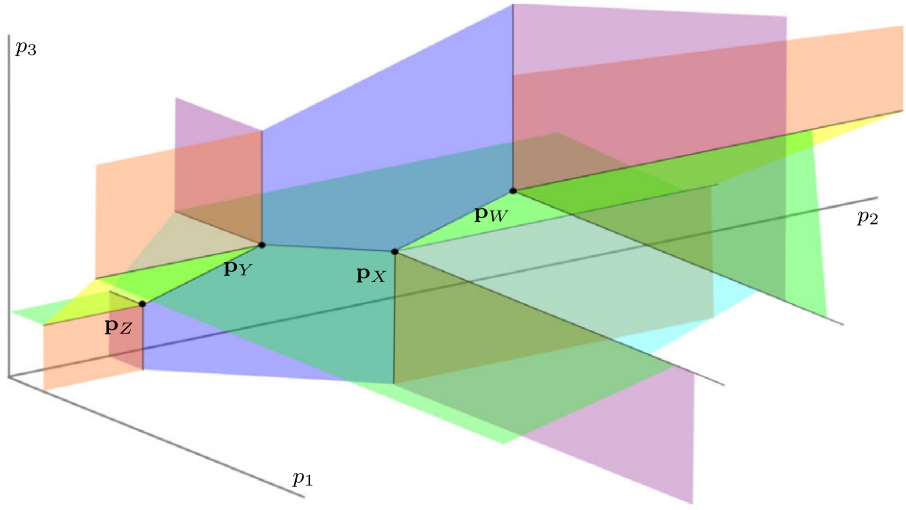
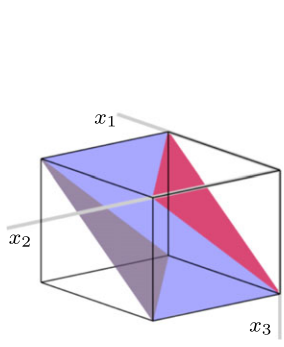


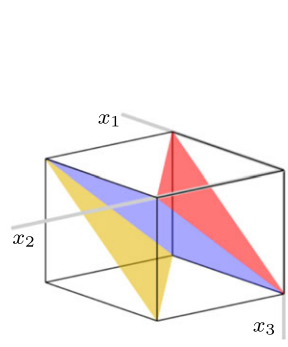
FIGURE S.3.—A rational polyhedral complex in price space, dual to Figure S.1(b).

Figure S.1(a). We can flip Figure S.1(b) over, interchanging the second and third coordinates, and obtain Figure S.4(a). This is, of course, also consistent with Figure S.1(a). Now the blue face has normal $(1, 0, 1)$; in Figure S.1(b), its normal vector was $(1, 1, 0)$. The dual LIP would then, of course, be the image of Figure S.3 under the same transformation. The final option, given in Figure S.4(b), is when the normal vector to the blue facet is $(0, 1, 1)$. We obtain the dual LIP for this case by interchanging the first and third coordinates in Figure S.3.

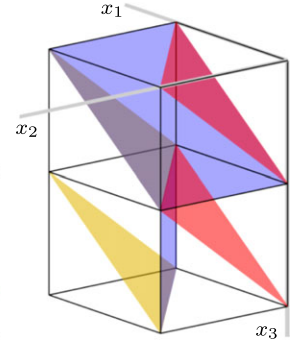
There are, of course, strong substitutes valuations on $\{0, 1\}^3$ with simpler demand complexes; the trivial subdivision, in which $[0, 1]^3$ is itself a 3-cell, also represents a valuation for strong substitutes. But we can recover this demand complex from Figure S.1(b) by merging adjacent 3-cells. Doing so is equivalent, in price space, to bringing together two



(a) Another maximal polyhedral complex subdivision of $[0, 1]^3$, whose edges are as shown in Figure S.1(a).



(b) A third maximal polyhedral complex subdivision of $[0, 1]^3$, whose edges are as shown in Figure S.1(a).



(c) Demand complex for a valuation in which two units of good 3 are available, composed of Figures S.1(b) and S.4(a).

FIGURE S.4.—Additional demand complexes for strong substitutes.

0-cells at the end-points of a 1-cell. If we do so, then this 1-cell collapses into the 0-cells that we are bringing together. The facets adjoining the 1-cell similarly collapse onto 1-cells in their boundaries. This is the same limiting process as we described (in two dimensions) in Example B.2.

In fact, any strong substitutes valuation on $\{0,1\}^3$ may be obtained in this way: it either is of the combinatorial type of Figure S.3; or is a transformation of this which interchanges the coordinate axes; or is the limit of one of these cases, in which some or all of the 0-cells have been brought together.

We now find, explicitly, a general form for any valuation of the combinatorial type shown in Figure S.3. First, we give coordinates to the labeled 0-cells, in such a way that forces consistency with geometry of the complex. That is, first set $\mathbf{p}_W = (a, b, c)$. Then there must exist $\alpha > 0$ such that $\mathbf{p}_X = (a - \alpha, b - \alpha, c - \alpha)$, because we know that the 1-cell connecting these points is in direction $(1, 1, 1)$ (it is dual to the red 1-cell in Figure S.1(b)). Similarly, there exists $\beta > 0$ such that $\mathbf{p}_Y = \mathbf{p}_X - \beta(1, 1, 0)$ for some $\beta > 0$, since the 1-cell connecting $\mathbf{p}_X$ and $\mathbf{p}_Y$ is dual to the blue 2-cell in Figure S.1(b). So $\mathbf{p}_Y = (a - \alpha - \beta, b - \alpha - \beta, c - \alpha)$. Finally, since $\mathbf{p}_X$ lies below $\mathbf{p}_Y$ in direction $(1, 1, 1)$ again, there exists $\gamma > 0$ such that $\mathbf{p}_Z = (a - \alpha - \beta - \gamma, b - \alpha - \beta - \gamma, c - \alpha - \gamma)$. Note that any $a, b, c \in \mathbb{R}$ and any $\alpha, \beta, \gamma > 0$ are consistent with Figure S.3.

Now that we know the coordinates of the facets, we may infer the valuation itself by following a simple rule. The rule is: $u(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} = u(\mathbf{y}) - \mathbf{p} \cdot \mathbf{y}$, where $\mathbf{p}$ is in a facet, and bundles $\mathbf{x}, \mathbf{y}$ are demanded on either side of that facet. See Figure S.5. For any $a, b, c \in \mathbb{R}$ and any $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$, this gives a strong substitutes valuation of the same combinatorial type as Figure S.3. Conversely, any valuation of the combinatorial type Figure S.3 may be presented in this form.

We can go further: the process described above, of collapsing together two 0-cells which are the end-points of the same 1-cell, is the geometric counterpart of just letting one of α, β, γ relax to 0. So Figure S.5 also presents a strong substitutes valuation if we assume only that $\alpha, \beta, \gamma \geq 0$, and by doing so we obtain further combinatorial types of valuations. For example, if $\alpha = \beta = \gamma = 0$, then the valuation is additively separable, the demand complex is the trivial case (one 3-cell consisting of $[0,1]^3$), and the LIP consists of three planes intersecting at (a, b, c). Additional cases correspond to only one or two of these parameters being zero.

Since the remaining combinatorial types are obtained by transforming Figures S.1 and S.3 by just interchanging the coordinate axes, a task it is straightforward to replicate in Figure S.5, we conclude that we have in this way obtained *all* strong substitutes valuations for at most one unit of three goods.

Moreover, we may consider the agent's values for additional units in the same way. Extend the example to make a second unit of good 3 available, and assume that the demand complex breaks down as one cube on top of another. We can keep our existing analysis and apply the same technique to the second cube. Let the demand complex now be that shown in Figure S.4(c). The LIP is given in Figure S.6. The lower part is the same as in Figure S.3, corresponding to the fact that the "lower" cube in Figure S.4(c) is the same

x_1		$u(\mathbf{x})$;
1	*0*	$x_3 = 0$
a	0	*0* $\quad x_2$
$a + b - \alpha - \beta$	b	*1*

x_1		$u(\mathbf{x})$;
1	*0*	$x_3 = 1$
$a + c - \alpha$	c	*0* $\quad x_2$
$a + b + c - 2\alpha - \beta - \gamma$	$b + c - \alpha$	*1*

FIGURE S.5.—The strong substitutes valuation of Figure S.3, given in terms of parameters $a, b, c \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$.

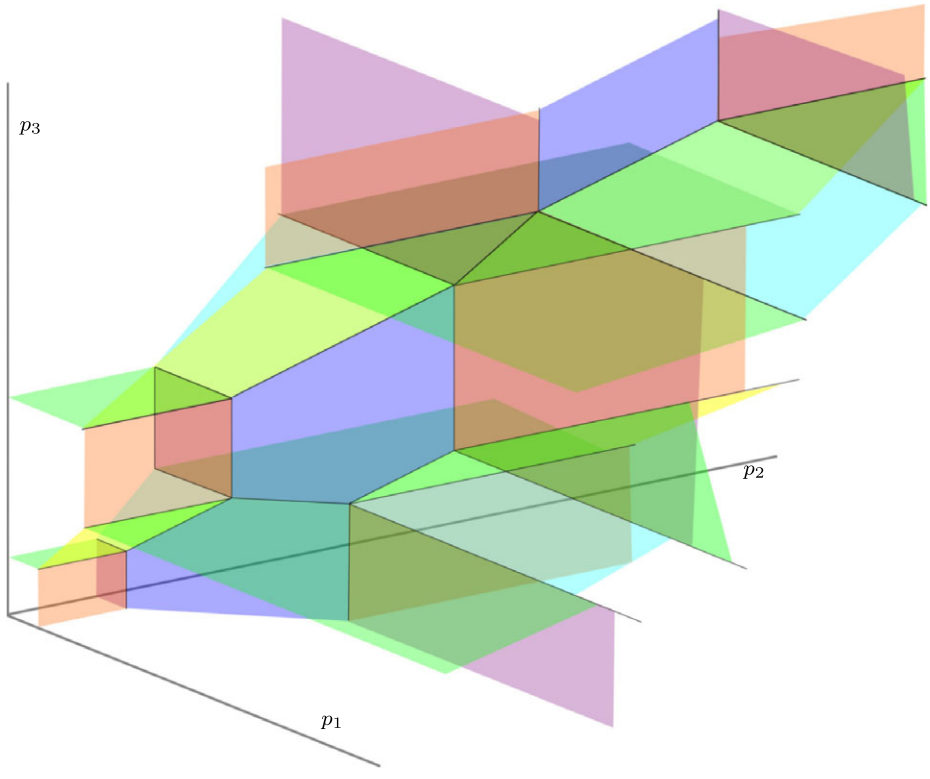


FIGURE S.6.—The LIP of a valuation dual to Figure S.4(c).

as Figure S.1(b). Now imagine interchanging the second and third axes of Figure S.3, obtaining a new LIP. The upper part of Figure S.6 has the same combinatorial type as this new LIP. It is straightforward to infer, again, a general parametric form for any valuation of this combinatorial type.

APPENDIX D: MIXED VOLUMES AND $M_k^n(\cdot,\cdot)$

The quantity $M_k^n(\cdot,\cdot)$ arises from the "mixed volume" in algebraic geometry (see, e.g., Sangwine-Yager (1993) or Cox, Little, and O'Shea (2005, Chapter 7)). Recall in Figures 7(c) and 7(d), the value for $M^2(\{0,1,2\}^2,\{0,1,2\}^2)$ was equal to the sum of the striped areas. These striped areas are all the 2-cells of aggregate-demand complexes, with the property that one edge comes from the first individual demand complex and one edge comes from the second. We generalize as follows:

DEFINITION D.1—See, e.g., Cox, Little, and O'Shea (2005, Definitions 7.6.4, 7.6.5, 7.6.6 and Theorem 7.6.7): Suppose $Q = Q^1 + \cdots + Q^m \subsetneq \mathbb{R}^n$, where $Q^1,\ldots,Q^m$ are polytopes with vertices in $\mathbb{Z}^n$, and suppose that $\dim Q = n$.

1. A *subdivision* of Q is a collection of polytopes $R_1,\ldots,R_s$ such that $Q = R_1 \cup \cdots \cup R_s$ and such that, for $i \neq k$, the intersection $R_i \cap R_k$ is a face of both R_i and R_k.

2. A subdivision $R_1,\ldots,R_s$ of Q is a *mixed subdivision* if each R_i can be written as $R_i = F_i^1 + \cdots + F_i^m$, where F^j is a face of Q^j for each j, and where $\dim(F^1) + \cdots + \dim(F^m) = n$, and where if $R_k = F_k^1 + \cdots + F_k^m$, then $R_i \cap R_k = (F_i^1 \cap F_k^1) + \cdots + (F_i^m \cap F_k^m)$.

3. A cell $R = F^1 + \cdots + F^m$ in a mixed subdivision is a *mixed cell* if $\dim(F^j) \leq 1$ for all j. In particular, if $m = n$, then this condition requires $\dim(F^j) = 1$ for all j.

4. When $m = n$, define the *mixed volume* $MV_n(Q^1, \ldots, Q^n) := \sum_R \text{vol}_n(R)$, where the sum is over all mixed cells R of a mixed subdivision.

To understand these definitions, observe that the maximal cells of a demand complex form a subdivision of the convex hull of its domain. Similarly, the maximal cells of the aggregate-demand complex of m agents gives a subdivision of the convex hull of their aggregate domain. If the intersection between the individual LIPs is transverse, then this is a mixed subdivision. The mixed cells are dual to intersections of facets in their interiors; in Figures 7(c) and 7(d), these are the striped areas.

In both of Figures 7(c) and 7(d), the sum of the areas of mixed cells is 2. Indeed, the sum of the volumes of mixed cells is always independent of the choice of mixed subdivision; this result is implicit in our definition of mixed volume above (see Cox, Little, and O'Shea, (2005 Definition 4.11 and Theorem 7.6.7)). So we can use very simple subdivisions to calculate mixed volumes: see Example D.4.

Recall that equilibrium fails for two LIPs, $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$, iff it fails at an intersection 0-cell. Suppose cells C_{σ^1}, C_{σ^2} of the respective LIPs meet transversely at such a point. In the demand complexes, we correspondingly have cells σ^1, σ^2, of dimensions k, $n-k$, and such that $\sigma = \sigma^1 + \sigma^2$ is dual to the intersection 0-cell itself. As in Lemma 4.16, equilibrium will fail if the aggregate-demand complex cell $\sigma^1 + \sigma^2$ is "too big." So, as in Sections 5.1.1–5.1.2, we wish to add up the volumes of all aggregate-demand complex cells such as $\sigma^1 + \sigma^2$. And we can do this using mixed volumes.

To calculate a mixed volume, we need n polytopes, with each mixed cell being a sum of pieces of dimension 1. But we have *two* polytopes: the convex hulls of the two domains. And we are interested in the sum of aggregate cells like $\sigma^1 + \sigma^2$, but $\dim \sigma^1 + \dim \sigma^2 = n$ (because the intersection is transverse). As Fact D.2 shows, the solution is to take $k := \dim \sigma^1$ copies of the first domain and $n-k$ copies of the second:

FACT D.2—(Follows from Huber and Sturmfels (1995) Theorem 2.4): *Suppose the intersection of $\mathcal{L}_{u^1}$ and $\mathcal{L}_{u^2}$ is transverse. The total volume of aggregate-demand complex cells dual to intersection* 0*-cells at which an $(n-k)$-cell of $\mathcal{L}_{u^1}$ meets a k-cell of $\mathcal{L}_{u^2}$ is equal to $\frac{1}{k!(n-k)!} MV_n(\text{conv}(A^1), \ldots, \text{conv}(A^1), \text{conv}(A^2), \ldots, \text{conv}(A^2))$, in which we take k copies of $\text{conv}(A^1)$ and $n-k$ copies of $\text{conv}(A^2)$.*

The additional factor of $\frac{1}{k!(n-k)!}$ perfectly cancels the factors we used in defining weights of cells—consistent with defining $M_k^n(\cdot, \cdot)$ as a mixed volume in this way.

LEMMA D.3—Cox, Little, and O'Shea (2005, Theorem 7.4.12.d): *If $A^1, A^2 \subsetneq \mathbb{Z}^n$ are finite, then $M_k^n(A^1, A^2)$ is the mixed volume of k copies of $\text{conv}(A^1)$ with $(n-k)$ copies of $\text{conv}(A^2)$, for $k = 1, \ldots, (n-1)$.*

EXAMPLE D.4: Let $n = 3$ and suppose that A^1 and A^2 are the discrete-convex sets with vertices $\{(0,0,0), (2,0,0), (0,2,0), (2,2,0)\}$ and $\{(0,0,0), (1,0,0), (0,0,2), (1,0,2)\}$, respectively: the domains of the demand complexes shown in Figures 14(a)–(b).

We calculate $M_1^3(A^1, A^2)$ and $M_2^3(A^1, A^2)$ by considering: agent 1^*, with valuation $u^{1*}(\mathbf{x}) = 0$ for all $\mathbf{x} \in A^1$; and agent 2^*, with valuation $u^{2*}(\mathbf{x}) = x_1 + x_3$ for all $\mathbf{x} \in A^2$. Then $\Sigma_{u^{1*}}$ has a single 2-cell of volume 4 (the convex hull of the whole domain, *not* the demand complex pictured in Figure 14). The corresponding unique 1-cell C^1 of $\mathcal{L}_{u^{1*}}$ is

in direction $\mathbf{e}^3$ and passes through $(0,0,0)$. The LIP has four facets, corresponding to the four edges of $\text{conv}(A^1)$, and we identify in particular a weight-2 facet F^1 of $\mathcal{L}_{u^{1*}}$ with normal $\mathbf{e}^2$, corresponding to the edge of $\text{conv}(A^1)$ from $(0,0,0)$ to $(0,2,0)$; it is not hard to see that $F^1 = \{\mathbf{p} \in \mathbb{R}^3 : p_1 \geq 0,\ \ p_2 = 0\}$. The remaining facets of $\mathcal{L}_{u^1}$ either all have non-positive first coordinate, or have normal vector $\mathbf{e}^1$.

Similarly, $\Sigma_{u^{2*}}$ has a single 2-cell of volume 2, and the corresponding unique 1-cell C^2 is in direction $\mathbf{e}^2$ and passes through $(1,0,1)$. It also has four facets, and we label as F^2 the weight-2 facet corresponding to the edge of $\text{conv}(A^2)$ from $(1,0,0)$ to $(1,0,2)$. Thus $F^2 = \{\mathbf{p} \in \mathbb{R}^3 : p_1 \leq 1,\ \ p_3 = 0\}$. The remaining facets of $\mathcal{L}_{u^2}$ either all have first coordinate greater than or equal to 1, or have normal vector $\mathbf{e}^3$.

From our descriptions above it is clear that $C^1 \cap \mathcal{L}_{u^{2*}} = C^1 \cap F^2$. The demand complex cell corresponding to this intersection 0-cell has volume $4 \times 2 = 8$. So by Fact D.2 and Lemma D.3, we know $M_2^3(A^1, A^2) = 2!1! \times 8 = 16$.

Similarly, $C^2 \cap \mathcal{L}_{u^{1*}} = C^2 \cap F^1$. The demand complex cell corresponding to this intersection 0-cell has volume $2 \times 2 = 4$. So by Fact D.2 and Definition D.3, we know $M_1^3(A^1, A^2) = 2!1! \times 4 = 8$.

We conclude that $M^3(A^1, A^2) = 8 + 16 = 24$.

PROOF OF THEOREM 5.22: See Bertrand and Bihan (2013), Theorem 6.1. Alternatively, see that if C_{σ^1} and C_{σ^2} intersect transversely, then it follows from our definitions of cell weights and subgroup indices that $\text{mult}(C_{\sigma^{\{1,2\}}}) = k!(n-k)!\,\text{vol}_n(\sigma^{\{1,2\}})$, where $k = \dim \sigma^1$. Theorem 5.22 now follows from Fact D.2. The non-transverse case follows: choose a small shift $\boldsymbol{\epsilon}\mathbf{v}$ making the whole intersection transverse. *Q.E.D.*

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Co-editor Dirk Bergemann handled this manuscript.

Manuscript received 31 July, 2015; final version accepted 16 December, 2018; available online 8 January, 2019.