Setting:

- **Two periods** - now (t = 0) and next period (t = 1). Buy assets for certain prices now and receive uncertain return next period.
- **Two states of nature** next period so gross returns in the two states are represented by 2 x 1 vectors.
- Prices of assets and vectors of returns are denoted by \( p \) and \( v \), respectively.
- A portfolio consists of a set of asset holdings. The price of a portfolio is just \( \sum p_i h_i \) where \( h_i \) is the holding of asset \( i \).
- The cashflow or payoff of the portfolio, which is received in period 1, is uncertain. Suppose there are two assets with payoffs:

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
v_2 &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{align*}
\]

Then the payoff vector of the portfolio with \( h_1 = 3 \) units of asset 1 and \( h_2 = 10 \) units of asset 2 is:

\[
h_1 v_1 + h_2 v_2 = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 10 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}
\]

- With **free portfolio formation**, there are no restrictions on how you form a portfolio. In particular, there are no short selling restrictions.
- We also assume that there are no frictions such as taxes and transaction costs.
Law of One Price and Arbitrages

- Want to obtain a set of criteria for consistently / fairly pricing risky assets. No arbitrage pricing, i.e. ruling out arbitrage opportunities, allows us to do that.
- An arbitrage opportunity is a zero cost, risk free profit opportunity.
- With free portfolio formation, linear pricing / the law of one price (LOP) must hold:

\[ p(\alpha v) = \alpha p(v) \]
\[ p(v_1 \pm v_2) = p(v_1) \pm p(v_2) \]
\[ p(\alpha v_1 \pm \beta v_2) = \alpha p(v_1) \pm \beta p(v_2) \]
i.e. the pricing function is linear. The notation \( p(v) \) denotes the price \( p \) at time 0 of an asset with risky cash flows \( v \) at time 1.
- Sometimes linear pricing / LOP referred to as the value additivity rule.
- If the LOP did not hold, arbitrage opportunities would exist and these arbitrage opportunities would not be consistent with an equilibrium of some non-satiated agent.
- Some examples of arbitrage opportunities:

\begin{align*}
(1) \quad & v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad p_1 = p_2 = p_3 = 1 \\
(2) \quad & v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad p_1 = \frac{1}{2}, \quad p_2 = \frac{3}{4} \\
(3) \quad & v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad p_1 = p_2 = 1 \\
(4) \quad & v_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 12 \\ 8 \end{pmatrix}, \quad p_1 = 1, \quad p_2 = 10 \\
\end{align*}

- Law of one price does not hold in examples (1) and (2). In example (1), \( v_3 = v_1 + v_2 \) yet \( p_3 \neq p_1 + p_2 \). In example (2), \( v_2 = 2v_1 \) yet \( p_2 \neq 2p_1 \).
- In example (3), the law of one price holds since \( v_2 \) is not proportional to \( v_1 \).
However the portfolio consisting of -1 unit (short sale) of asset one and 1 unit (buy) of asset 2 is a **weak arbitrage**. This portfolio has zero cost yet the payoff in at least one state is positive and zero in every other state:

\[-1 \cdot p_1 + 1 \cdot p_2 = -1.1 + 1.1 = 0\]

\[-1v_1 + 1v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\]

- A weak arbitrage is a zero cost portfolio that never results in a loss and sometimes result in a gain.
- In example (4), the portfolio consisting of 10 units of asset 1 and -1 units (short sale) of asset 2 is a **strong arbitrage**. This portfolio has zero cost and generates a positive payoff in every state of the world:

\[10 \cdot p_1 - 1 \cdot p_2 = 0\]

\[10v_1 - 1v_2 = \begin{pmatrix} 8 \\ 12 \end{pmatrix}\]

- These examples show that linear pricing / LOP is **necessary** but **not sufficient** to rule out arbitrage opportunities. Also need to rule strong and weak arbitrages. See diagram.
The Fundamental Theorem of Finance

The theorem says that the following conditions are equivalent:

(i) No arbitrage opportunities;
(ii) The equilibrium of a non-satiated agent;
(iii) The existence of strictly positive state prices $q$ or Arrow-Debreu prices such that the current price of a random cash flow next period equals the sum of the cashflows in each state times the state prices.

- When there are two states, the first state price is the current price of the certain payoff of 1 in state 1 and 0 in state 2 next period i.e. the payoff $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second state price is the current price of the payoff $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ next period.
- For example, if the state prices $q_1 = 1\frac{1}{4}$ and $q_2 = 1\frac{1}{2}$ then the price of the random cash flow $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$ is just $1\frac{1}{4} \cdot -1 + 1\frac{1}{2} \cdot 8 = 10\frac{3}{4}$.

Discussion:

- The fundamental theorem is very important because it is the basis of “no arbitrage pricing”.
- Equilibrium of some non-satiated agent $\Rightarrow$ existence of fair / correct, “no arbitrage” price.
- No arbitrage opportunities $\Rightarrow$ law of one price + existence of state prices etc.
- For example, risk neutral / no arbitrage pricing used to price contingent claims such as options and futures.
- Do not need to model supply and demand or preferences!
- Also used in corporate finance to derive classic Modigliani-Miller result on use of debt versus equity.
- Can use arbitrage pricing to price new assets with attainable / marketable payoffs i.e. payoffs which can be replicated by some portfolio of existing assets.
• Attainable payoffs are payoffs which are in the space spanned by the \( v_i \) which means that they can be expressed as a linear combination of the \( v_i \).

• If the asset's payoff is not attainable, you can still derive upper and lower bounds for its price. These bounds can be useful.

• State prices need not be unique. For example, in incomplete markets (markets with more states of the world than assets etc.) with no arbitrage, state prices are unique. However, all sets of state prices generate the same price for an asset whose payoffs are attainable.

-o0O0o-